C. R. Acad. Sci. Paris, Ser. I 354 (2016) 532–537

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Probability theory

First- and second-order expansions in the central limit theorem for a branching random walk

Développements du premier et du second ordre dans le théorème central limite pour une marche aléatoire avec branchement

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A R T I C L E I N F O

Article history:
Received 2 September 2015
Accepted after revision 25 January 2016

Presented by Jean-Pierre Kahane

A B S T R A C T

We give the first- and second-order asymptotic expansions for the central limit theorem about the distribution of particles in a branching random walk on the real line. In particular, our first-order expansion reveals the exact convergence rate in the central limit theorem; it extends and improves a known result for the branching Wiener process.

R É S U M É

Nous donnons les développements asymptotiques d'ordres un et deux dans le théorème central limite sur la distribution des particules dans une marche aléatoire avec branchement sur la droite réelle. En particulier, le développement asymptotique d'ordre un révèle la vitesse exacte de convergence du théorème central limite, ce qui étend et améliore un résultat connu pour le processus de Wiener avec branchement.

Version française abrégée

Considérons une marche aléatoire avec branchement sur la droite réelle $\mathbb{R}$. Le modèle est défini comme suit. Au temps $n=0$, une particule initiale $\varnothing$ est située en $S_\varnothing = 0 \in \mathbb{R}$. Au temps $n=1$, elle est remplacée par $N = N_\varnothing$ nouvelles particules $\varnothing i = i$ de la génération 1, localisées en $L_i = L_{\varnothing i}, 1 \le i \le N$. En général, chaque particule $u = u_1 u_2 \cdots u_n$ de génération n, située en S_u, est remplacée au temps $n+1$ par N_u nouvelles particules ui de génération $n+1$, localisée en $S_{ui} = S_u + L_{ui}, 1 \le i \le N_u$, où les variables aléatoires N_u et L_u sont respectivement de loi $\{p_i\}_{i\in\mathbb{N}}$ sur $\mathbb{N}$ et de loi $G(\cdot)$ sur $\mathbb{R}$. Toutes les variables aléatoires N_u et L_u, indexées par les suites finies u, sont mutuellement indépendantes.

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http://dx.doi.org/10.1016/j.crma.2016.01.021
1631-073X/

On désigne par $\mathbb{T}_n$ l'ensemble des individus de la n^e génération, représentés par des suites u d'entiers positifs de longueur $|u| = n$; comme d'habitude, la particule initiale est représentée par la suite nulle $\varnothing$ (de longueur 0) ; si $u \in \mathbb{T}_n$, alors $ui \in \mathbb{T}_{n+1}$ si et seulement si $1 \le i \le N_u$.

Soit

$$Z_n = \sum_{u \in \mathbb{T}_n} \delta_{S_u}$$

la mesure de comptage à l'instant n : pour une partie A de $\mathbb{R}$, $Z_n(A)$ est le nombre des particules de la génération n localisées dans A. Alors $\{Z_n(\mathbb{R})\}$ est un processus de Galton–Watson. Il est bien connu que la suite

$$W_n = \frac{Z_n(\mathbb{R})}{m^n} \quad \text{avec } m = \sum_{i=0}^{\infty} i p_i, \quad n \ge 0,$$

forme une martingale non négative par rapport à la filtration naturelle. Supposons que

$$m > 1 \quad \text{et} \quad \sum_{i=1}^{\infty} i (\ln i)^{1+\lambda} p_i < \infty, \tag{1}$$

où la valeur de $\lambda > 0$ sera précisée dans les hypothèses des théorèmes. Sous (1), le processus de Galton–Watson $\{Z_n(\mathbb{R})\}$ est *surcritique* et la suite $\{W_n\}$ converge presque sûrement (p.s.) vers sa limite W, avec $\mathbb{E}W = 1$ par le théorème de Kesten–Stigum (voir [2]).

Nous sommes intéressés par des propriétés asymptotiques de la mesure de comptage Z_n, qui décrit la configuration du système de particules à l'instant n. Sous l' hypothèse (1) pour un $\lambda > 0$ et la condition $\int |x|^2 dG(x) < \infty$, Kaplan et Asmussen [11] ont montré le théorème central limite suivant : pour tout $t \in \mathbb{R}$,

$$\frac{1}{m^n} Z_n((-\infty, nl + \sqrt{n}\sigma t]) \xrightarrow[p.s.]{n\to\infty} \Phi(t) W, \tag{2}$$

où

$$l = \int x \, \mathrm{d}G(x), \quad \sigma^2 = \int (x-l)^2 \, \mathrm{d}G(x), \quad \Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2} \mathrm{d}x.$$

Nous allons approfondir ce théorème en considéront des développement limités. En particulier, nous donnons la vitesse exacte de la convergence dans (2). Les résultats dépendent de deux martingales $\{N_{1,n}\}$ et $\{N_{2,n}\}$, qui concernent les déviations des positions S_u autour de leur moyennes, d'ordre premier et second, centrées et normalisées :

$$N_{1,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} (S_u - nl) \quad \text{et} \quad N_{2,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left[(S_u - nl)^2 - n\sigma^2 \right].$$

Pour la convergence des martingales, nous aurons besoin de la condition de moment :

$$\int |x|^\eta \, \mathrm{d}G(x) < \infty, \quad \eta > 0. \tag{3}$$

Proposition 0.1. *Les suites $\{N_{1,n}\}$ et $\{N_{2,n}\}$ sont des martingales par rapport à la filtration naturelle. Les deux martingales convergent p.s. sous de conditions simples :*

(i) *si (1) a lieu pour un $\lambda > 1$ et (3) a lieu pour un $\eta > 2$, alors $V_1 := \lim_{n\to\infty} N_{1,n}$ existe p.s. dans $\mathbb{R}$;*
(ii) *si (1) a lieu pour un $\lambda > 2$ et (3) a lieu pour un $\eta > 4$, alors $V_2 := \lim_{n\to\infty} N_{2,n}$ existe p.s. dans $\mathbb{R}$.*

Pour le dévoplopement asymptotique, nous aurons besoin de la condition de Cramér :

$$\limsup_{|t|\to\infty} \left| \int \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G(x) \right| < 1. \tag{4}$$

Le théorème suivant concerne le comportement asymptotique d'ordre 1 dans le théorème central limite.

Théorème 0.2. *On suppose les conditions de moment (1) et (3) pour certains $\lambda > 8$ et $\eta > 12$, et la conition de Cramér (4). Alors p.s. nous avons : pour tout $t \in \mathbb{R}$, lorsque $n \to \infty$,*

$$\frac{1}{m^n} Z_n\big((-\infty, nl + \sqrt{n}\sigma t]\big) = \Phi(t)W + \frac{1}{\sqrt{n}}\mathcal{V}(t) + o(\frac{1}{\sqrt{n}}),$$

où $\mathcal{V}(t)$ est la variable aléatoire définie par (11).

Ce théorème révèle la vitesse exacte de convergence dans le théorème central limite. Il étend et améliore un résultat connu pour le processus de Wiener avec branchement [4]. Il a été obtenu par Kabluchko [10, Theorem 5 and Remark 2] sous la condition du second moment pour la loi $\{p_i\}_{i\in\mathbb{N}}$.

Sous des conditions un peu plus fortes, nous pouvons obtenir le dévelopment asymptotique d'ordre 2 :

Théorème 0.3. *On suppose les conditions de moment (1) et (3) pour certains $\lambda > 18$ et $\eta > 24$, et la conition de Cramér (4). Alors p.s., nous avons : pour tout $t \in \mathbb{R}$, lorsque $n \to \infty$,*

$$\frac{1}{m^n} Z_n((-\infty, nl + \sqrt{n}\sigma t]) = \Phi(t)W + \frac{1}{\sqrt{n}}\mathcal{V}(t) + \frac{1}{n}\mathcal{R}(t) + o\big(\frac{1}{n}\big),$$

où $\mathcal{V}(t)$ et $\mathcal{R}(t)$ sont les variables aléatoires définies par (11) et (13).

Les résultats présentés dans cette note peuvent être généralisés au cas des marches aléatoires avec branchement en milieux aléatoires. Pour les preuves complètes et plus de détails, nous renvoyons les lecteurs aux références [5,6].

1. Introduction and results

In this note, we shall present recent progress on first- and second-order asymptotic expansions for the distribution of particles in a branching random walk under suitable conditions. The reader may refer to [5,6] for complete proofs and more related results, where the results are obtained under a general framework, i.e. for a branching random walk with a random environment in time.

Consider a branching random walk on the real line $\mathbb{R}$. Initially, an ancestor particle $\varnothing$ is located at $S_\varnothing = 0$. At time 1, the initial particle $\varnothing$ is replaced by $N = N_\varnothing$ new particles $\varnothing i = i$ of generation 1, with displacement $L_{\varnothing i} = L_i$, so that each particle $\varnothing i = i$ moves to $S_{\varnothing i} = S_\varnothing + L_{\varnothing i}$, that is, $S_i = L_i$, $1 \le i \le N$. In general, at time $n+1$, each particle $u = u_1 u_2 \cdots u_n$ of generation n is replaced by N_u new particles of generation $n+1$, with displacements $L_{u1}, L_{u2}, \cdots, L_{uN_u}$, so that each particle ui has position $S_{ui} = S_u + L_{ui}$, $1 \le i \le N_u$. Each N_u is an integer-valued random variable with common distribution $\{p_i\}_{i\in\mathbb{N}}$, and each L_u is a real-valued random variable with common distribution $G(\cdot)$; all the random variables N_u, L_u, indexed by finite sequences of integers u, are independent of each other.

Let $\mathbb{T}$ be the genealogical tree with $\{N_u\}$ as defining elements. By definition, we have: (a) $\varnothing \in \mathbb{T}$; (b) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$; (c) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \le i \le N_u$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n, where $|u|$ denotes the length of the sequence u.

Denote by $Z_n = \sum_{u\in\mathbb{T}_n} \delta_{S_u}$ the counting measure which counts the number of particles of generation n situated in a given set: for $B \subset \mathbb{R}$,

$$Z_n(B) = \sum_{u\in\mathbb{T}_n} \mathbf{1}_B(S_u).$$

Then $\{Z_n(\mathbb{R})\}$ forms a Galton–Watson process. It is well known that

$$W_n = \frac{Z_n(\mathbb{R})}{m^n} \quad \text{with} \quad m = \sum_{i=0}^{\infty} i p_i,$$

is a martingale with respect to the natural filtration. We shall always suppose that

$$m > 1 \quad \text{and} \quad \sum_{i=1}^{\infty} i(\ln i)^{1+\lambda} p_i < \infty, \tag{5}$$

where the value of $\lambda > 0$ will be specified later. Under (5), the process is supercritical and the martingale $\{W_n\}$ converges a.s. to a non-zero limit W with $\mathbb{E}W = 1$ by the famous Kesten–Stigum theorem (c.f. [2]); moreover, the event $\{W > 0\}$ coincides with $\{Z_n \to \infty\}$ a.s.

We are interested in the asymptotic behavior of the counting measure Z_n, which describes the configuration of the particle system at time n. The question of central limit theorem for Z_n was first posed by Harris [8]. Under the near-minimal conditions (5) for some $\lambda > 0$ and $\int x^2 \, \mathrm{d}G(x) < \infty$, Kaplan and Asmussen [11] proved that for each fixed $t \in \mathbb{R}$, we have a.s.,

$$\frac{1}{m^n} Z_n((-\infty,\ nl + \sqrt{n}\sigma t]) \xrightarrow{n\to\infty} \Phi(t)W, \tag{6}$$

where

$$l = \int x \, \mathrm{d}G(x), \quad \sigma^2 = \int (x-l)^2 \, \mathrm{d}G(x), \quad \Phi(t) = \int_{-\infty}^{t} \phi(x)\mathrm{d}x \text{ with } \phi(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}. \tag{7}$$

This is a central limit theorem, since it reveals that a.s. the random measures Z_n with suitable norming converge to the standard normal law $N(0,1)$. Indeed, the fact that (6) *holds a.s. for each fixed* t implies that a.s. (6) *holds for all rational* t; by the monotonicity of the distributional function $x \mapsto Z_n(-\infty, x]$ and the continuity of Φ, it follows that a.s. (6) *holds for all real* t (that is, the event that (6) holds for all real t has probability 1).

The result (6) has been extended in various forms (see, e.g., [3,7,9]). Inspired by the classical Edgeworth expansion theory of central limit theorems for sums of independent random variables (see, e.g., [12]), we wish to improve (6) by describing its asymptotic expansions.

In this note, we aim to give the first- and second-order expansions in the above central limit theorem when Cramér's condition about the characteristic function holds: that is

$$\limsup_{|t|\to\infty} \left| \int \mathrm{e}^{itx} \, \mathrm{d}G(x) \right| < 1. \tag{8}$$

This condition holds for example if the distribution G has an absolutely continuous component (in Lebesgue's decomposition), but does not hold if G is a lattice distribution.

Our results will depend on the following two martingales, which concern the deviations of S_u from their means, of order 1 and 2, centered and normalized:

$$N_{1,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} (S_u - nl) \quad \text{and} \quad N_{2,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left[(S_u - nl)^2 - n\sigma^2 \right].$$

For the convergence of these martingales, we will need a moment condition of the form

$$\int |x|^\eta \, \mathrm{d}G(x) < \infty, \quad \eta > 0. \tag{9}$$

Proposition 1.1. *The sequences* $\{N_{1,n}\}$ *et* $\{N_{2,n}\}$ *are martingales with respect to the natural filtration*

$$\mathcal{D}_0 = \{\emptyset, \Omega\}, \quad \mathcal{D}_n = \sigma(N_u, L_{ui} : i \geq 1, |u| < n) \text{ for } n \geq 1.$$

These martingales converge a.s. under simple moment conditions:

a) *if* (5) *holds for some* $\lambda > 1$ *and* (9) *holds for some* $\eta > 2$, *then* $V_1 := \lim_{n\to\infty} N_{1,n}$ *exists a.s. in* $\mathbb{R}$;
b) *if* (5) *holds for some* $\lambda > 2$ *and* (9) *holds for some* $\eta > 4$, *then* $V_2 := \lim_{n\to\infty} N_{2,n}$ *exists a.s. in* $\mathbb{R}$.

Our first result is a first-order expansion in the central limit theorem. As usual, for random variables A_n and B_n, we write $A_n = o(B_n)$ a.s. if $\lim_{n\to\infty} \frac{A_n}{B_n} = 0$ a.s. In the following theorems, the numbers 8, 12, 18 and 24 in the hypotheses are for technical reasons; the best constants are not yet known, and seem to be delicate.

Theorem 1.2. *Assume* (5) *for some* $\lambda > 8$, (9) *for some* $\eta > 12$, *and* (8). *Then a.s. we have: for all* $t \in \mathbb{R}$, *as* $n \to \infty$,

$$\frac{1}{m^n} Z_n\big((-\infty,\ nl + \sqrt{n}\sigma t]\big) = \Phi(t)W + \frac{1}{\sqrt{n}} \mathcal{V}(t) + o\Big(\frac{1}{\sqrt{n}}\Big), \tag{10}$$

where

$$\mathcal{V}(t) = -\frac{1}{\sigma}\phi(t)V_1 + \frac{\sigma^{(3)}}{6\sigma^3}(1-t^2)\phi(t)W \text{ with } \sigma^{(3)} = \int (x-l)^3 \, \mathrm{d}G(x). \tag{11}$$

This result reveals the exact convergence rate in (6). In particular, it improves Chen's Theorem 3.1 in [4] for branching Wiener process by weakening the moment conditions for the offspring laws and displacement laws therein; moreover, it extends that to the general case including non-Gaussian displacement under weaker moments conditions. When the second moment condition $\sum_{i=1}^{\infty} i^2 p_i < \infty$ is assumed, Theorem 1.2 was obtained by Kabluchko [10, Theorem 5 and Remark 2].

Under slightly stronger moments conditions, we can obtain the second-order asymptotic expansion.

Theorem 1.3. *Assume (5) for some $\lambda > 18$, (9) for some $\eta > 24$, and (8). Then a.s., we have: for all $t \in \mathbb{R}$, as $n \to \infty$,*

$$\frac{1}{m^n} Z_n\big((-\infty,\ nl + \sqrt{n}\sigma t]\big) = \Phi(t)W + \frac{1}{\sqrt{n}}\mathcal{V}(t) + \frac{1}{n}\mathcal{R}(t) + o\Big(\frac{1}{n}\Big), \tag{12}$$

where

$$\mathcal{R}(t) = -\frac{1}{2\sigma^2} t\phi(t)V_2 - \frac{\sigma^{(3)}}{6\sigma^4} H_3(t)\phi(t)V_1 - \Big[\frac{(\sigma^{(3)})^2}{72\sigma^6} H_5(t) + \frac{(\sigma^{(4)} - 3\sigma^4)}{24\sigma^4} H_3(t)\Big]\phi(t)W, \tag{13}$$

with $\sigma^{(\nu)} = \int (x-l)^\nu \, \mathrm{d}G(x)$ for $\nu = 3, 4$ and $H_m(\cdot)$ is the Chebyshev–Hermite polynomial of degree m:

$$H_m(t) = m! \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k t^{m-2k}}{k!(m-2k)!2^k} \quad (\text{so that } H_3(t) = t^3 - 3t, \quad H_5(t) = t^5 - 10t^3 + 15t).$$

From [1, Theorem 2], we know that for each $\lambda > 0$, the condition (5) implies that

$$W_n - W = o(n^{-\lambda}) \quad \text{a.s.}$$

With this we obtain the following variants of Theorems 1.2 and 1.3:

Corollary 1.4. *Assume (5) for some $\lambda > 8$, (9) for some $\eta > 12$, and (8). Then a.s. on $\{W > 0\}$, we have: for $t \in \mathbb{R}$, as $n \to \infty$,*

$$\frac{1}{Z_n(\mathbb{R})} Z_n\big((-\infty,\ nl + \sqrt{n}\sigma t]\big) = \Phi(t) + \frac{1}{\sqrt{n}}\frac{\mathcal{V}(t)}{W} + o\Big(\frac{1}{\sqrt{n}}\Big).$$

Corollary 1.5. *Assume (5) for some $\lambda > 18$, (9) for some $\eta > 24$, and (8). Then a.s. on $\{W > 0\}$, we have: for all $t \in \mathbb{R}$, as $n \to \infty$,*

$$\frac{1}{Z_n(\mathbb{R})} Z_n\big((-\infty,\ nl + \sqrt{n}\sigma t]\big) = \Phi(t) + \frac{1}{\sqrt{n}}\frac{\mathcal{V}(t)}{W} + \frac{1}{n}\frac{\mathcal{R}(t)}{W} + o\Big(\frac{1}{n}\Big).$$

2. Sketch of proofs

In the proofs of the main results, we follow the route in [11,4]: a key decomposition plays an important role. We need some notation to introduce the decomposition.

Let $\mathbb{T}(u)$ be the shifted tree of $\mathbb{T}$ at u with defining elements $\{N_{uv}\}$ satisfying: 1) $\varnothing \in \mathbb{T}(u)$, 2) $vi \in \mathbb{T}(u) \Rightarrow v \in \mathbb{T}(u)$ and 3) if $v \in \mathbb{T}(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \le i \le N_{uv}$. Set $\mathbb{T}_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$.

For $u \in (\mathbb{N}^*)^k (k \ge 0)$ and $n \ge 1$, write for $B \subset \mathbb{R}$,

$$Z_n(u, B) = \sum_{v \in \mathbb{T}_n(u)} \mathbf{1}_B(S_{uv} - S_u).$$

Define

$$t_n = nl + \sqrt{n}\sigma t, \qquad W_n(u,t) = \frac{1}{m^n} Z_n(u, (-\infty, t]), \quad \text{for} \quad n \ge 1, \quad t \in \mathbb{R}.$$

Let β be a real number chosen suitably (take $\max\{\frac{2}{\lambda}, \frac{3}{\eta}\} < \beta < \frac{1}{4}$ and $\max\{\frac{3}{\lambda}, \frac{4}{\eta}\} < \beta < \frac{1}{6}$ in the proofs of Theorems 1.2 and 1.3 respectively) and set $k_n = \lfloor n^\beta \rfloor$, the largest integer no bigger than n^β.

Observe for $k \le n$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_k} Z_{n-k}(u, B - S_u). \tag{14}$$

We obtain the following key decomposition:

$$\begin{aligned}\frac{1}{m^n} Z_n((-\infty, t_n]) &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \Big[W_{n-k_n}(u, t_n - S_u) - \mathbb{E}\Big(W_{n-k_n}(u, t_n - S_u)\Big| S_u\Big)\Big] \\ &\quad + \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}\Big(W_{n-k_n}(u, t_n - S_u)\Big| S_u\Big) =: \mathbb{A}_n + \mathbb{B}_n.\end{aligned} \tag{15}$$

Theorems 1.2 and 1.3 follow from Lemmas 2.1 and 2.2 respectively.

Lemma 2.1. *Under the conditions of Theorem 1.2, for each fixed $t \in \mathbb{R}$, the following hold a.s. as $n \to \infty$:*

$$(i)\ \sqrt{n}\mathbb{A}_n \to 0,$$

$$(ii)\ \mathbb{B}_n = \Phi(t)W_{k_n} + \frac{1}{\sqrt{n}}\left[\frac{\sigma^{(3)}}{6\sigma^3}(1-t^2)\phi(t)W_{k_n} - \frac{1}{\sigma}\phi(t)N_{1,k_n}\right] + o(\frac{1}{\sqrt{n}}),$$

$$(iii)\ W_{k_n} - W = o(\frac{1}{\sqrt{n}}).$$

Lemma 2.2. *Under the conditions of Theorem 1.3, for each fixed $t \in \mathbb{R}$, the following hold a.s. as $n \to \infty$:*

$$(i)\ n\mathbb{A}_n \to 0,$$

$$(ii)\ \mathbb{B}_n = \Phi(t)W_{k_n} + \frac{1}{\sqrt{n}}\left[\frac{\sigma^{(3)}}{6\sigma^3}(1-t^2)\phi(t)W_{k_n} - \frac{1}{\sigma}\phi(t)N_{1,k_n}\right] + \frac{1}{n}\Bigg[-\frac{1}{2\sigma^2}t\phi(t)N_{2,k_n}$$
$$- \frac{\sigma^{(3)}}{6\sigma^4}H_3(t)\phi(t)N_{1,k_n} - \Big(\frac{(\sigma^{(3)})^2}{72\sigma^6}H_5(t) + \frac{(\sigma^{(4)}-3\sigma^4)}{24\sigma^4}H_3(t)\Big)\phi(t)W_{k_n}\Bigg] + o(\frac{1}{n}),$$

$$(iii)\ W_{k_n} - W = o(\frac{1}{n}), \quad N_{1,k_n} - V_1 = o(\frac{1}{\sqrt{n}}).$$

The main terms in the expansion of $\mathbb{B}_n$ come from the use of the Edgeworth expansions in the central limit theorem (see, e.g., [12, P. 159 Theorem 1]). In the proofs of the lemmas, we need to carry out a careful analysis by combining the tools including the Borel–Cantelli Lemma, truncating arguments, moment inequalities for sums of independent random variables, and so on. The reader may refer to [5,6] for details.

Acknowledgements

The authors thank the anonymous referee for valuable comments and suggestions. The kind hospitality of LMBA, Université de Bretagne-Sud, where part of this work was carried out, is well acknowledged. The work has been partially supported by the National Natural Science Foundation of China (Grants No. 11101039, No. 11271045, No. 11571052, No. 11401590), by the Fundamental Research Funds for the Central Universities (2013YB52), and by the Natural Science Foundation of Guangdong Province (China), Grant No. 2015A030313628.

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