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On $\mathbb{A}^1$-fundamental groups of isotropic reductive groups

Sur le groupe fondamental au sens de la A1-homotopie des groupes réductifs isotropes

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A B S T R A C T

For an isotropic reductive group G satisfying a suitable rank condition over an infinite field k, we show that the sections of the $\mathbb{A}^1$-fundamental group sheaf of G over an extension field L/k can be identified with the second group homology of $G(L)$. For a split group G, we provide explicit loops representing all elements in the $\mathbb{A}^1$-fundamental group. Using $\mathbb{A}^1$-homotopy theory, we deduce a Steinberg relation for these explicit loops.

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R É S U M É

Pour un groupe réductif isotrope G défini sur un corps infini k, satisfaisant une condition de rang approprié, nous montrons que l'ensemble des sections du $\mathbb{A}^1$-faisceau de groupe fondamental de G sur une extension des corps L/k s'identifient avec la deuxième homologie des groupes de $G(L)$. Pour un groupe déployé G, nous définissons des lacets explicites représentant tous les elements du groupe $\mathbb{A}^1$-fondamental. En utilisant la théorie de la $\mathbb{A}^1$-homotopie, on déduit une rélation de Steinberg pour ces lacets explicites.

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1. Introduction

The goal of the present note is to describe the $\mathbb{A}^1$-fundamental group sheaves for isotropic reductive groups, improving the computations of [13, Proposition 5.2]. Moreover, for split groups, we obtain more precise information on the $\mathbb{A}^1$-fundamental groups by providing explicit loops representing elements in the $\mathbb{A}^1$-fundamental groups. The precise statement of our result is the following, cf. Lemma 2.2 and Proposition 3.2.

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Theorem 1. *Let k be an infinite field and let G be an isotropic reductive group over k, assuming that all components of the relative root system of G have at least rank 2. Then there is an isomorphism*

$$\mathrm{H}_2(G(k),\mathbb{Z}) \cong \pi_1(G(k[\Delta^\bullet])) \cong \pi_1^{\mathbb{A}^1}(G)(k).$$

In the case of split G, the isomorphism can be described by an explicit map

$$K_2^{\mathrm{M(W)}}(k) \xrightarrow{\sim} \pi_1(G(k[\Delta^\bullet])).$$

There, $K_2^{\mathrm{M(W)}}$ means K_2^{MW} or K_2^{M} depending on whether G is symplectic or not.

To prove the result, we use the homotopy invariance of the group homology, cf. [14], and a definition of Steinberg's groups based on the work by Petrov and Stavrova to identify $\mathrm{H}_2(G(k),\mathbb{Z})$ with $\pi_1(G(k[\Delta^\bullet]))$. The results of [1] and [2] on affine excision and descent for isotropic groups relate the latter to $\mathbb{A}^1$-homotopy theory. A slightly different approach is described in Remark 2. The Steinberg relation for explicit loops in $\mathrm{H}_2(G(k),\mathbb{Z})$ follows from the results of Hu and Kriz.

Using Morel's theory of strictly $\mathbb{A}^1$-invariant sheaves [8], we also get the following:

Corollary 1.1. *Let k be an infinite perfect field and let G be as above. Then the assignment $L/k \mapsto \mathrm{H}_2(G(L),\mathbb{Z})$ extends to a strictly $\mathbb{A}^1$-invariant sheaf of Abelian groups.*

Another implication of the above theorem is that Rehmann's computation of $\mathrm{H}_2(\mathrm{SL}_n(D),\mathbb{Z})$, cf. [10], can be seen as a description of $\pi_1^{\mathbb{A}^1}(\mathrm{SL}_n(D))$, for $n \geq 3$. The corollary implies the existence of well-behaved residue maps on $\mathrm{H}_2(\mathrm{SL}_n(D),\mathbb{Z})$, which seem to be new.

2. Preliminaries

In this article, we always assume k to be an infinite field. We consider reductive groups G over k, and we assume that they are *isotropic*, as in [9], so that all irreducible components of the relative root system of such G are of rank at least 2. This implies that the results of [9] and [2] are applicable.

For a commutative unital k-algebra R, the (abstract) group of R-points of the group scheme G is denoted by $G(R)$. The elementary subgroup $\mathrm{E}(R) \subset G(R)$ is defined, as in [9, §1], as being the subgroup of $G(R)$ generated by R-points of unipotent radicals of opposite parabolics P^+, P^- of G. By [9, Theorem 1], $\mathrm{E}(R)$ is normal in $G(R)$, and by [6, Theorem 1], the group $\mathrm{E}(R)$ is perfect. Moreover, by [11, Theorem 1.3], $K_1^G(R) := G(R)/\mathrm{E}(R)$ is invariant under polynomial extensions.

Definition 2.1. Let G be an isotropic reductive group over a commutative ring R. We define the *Steinberg group* $\mathrm{St}^G(R)$ to be the abstract group generated by elements $\widetilde{X_A}(u)$, $u \in V_A(R)$ subject to the commutator formulas from [9, Lemma 9, 10]. We define the group $K_2^G(R) := \ker\left(\mathrm{St}^G(R) \to \mathrm{E}^G(R)\right)$.

Remark 1. It is known that $K_2^G(k[\Delta^n]) \hookrightarrow \mathrm{St}^G(k[\Delta^n]) \twoheadrightarrow E^G(k[\Delta^n])$ is a universal central extension for G split of type $A_l, l \geq 3$ (van der Kallen), $C_l, l \geq 3$ (Lavrenov) and E_l (Sinchuk). It is not even a central extension for split rank-2 groups.

Using the standard cosimplicial object given by polynomial rings, one can associate a simplicial group with the reductive group G and a unital commutative k-algebra A, cf. [5]. This is denoted by $G(A[\Delta^\bullet])$ or (more commonly in the $\mathbb{A}^1$-homotopy literature) by $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(A)$. The $\mathbb{A}^1$-homotopy groups of an isotropic reductive group can be computed from the singular resolution, cf. [2, Corollary 4.3.3].

Lemma 2.2. *Let k be an infinite field and let G be an isotropic reductive group over k.*

Then $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ has affine Nisnevich excision in the sense of [1, Definition 3.2.1] and there are isomorphisms

$$\pi_i(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(A)) \xrightarrow{\sim} \pi_i^{\mathbb{A}^1}(G)(A)$$

for any essentially smooth k-algebra A and any $i \geq 0$.

Remark 2. Alternatively, one can prove the affine Nisnevich excision exactly as in [13, Theorem 4.10], using homotopy invariance for unstable K_1^G of isotropic groups from [11, Theorem 1.3]. The above result then follows from the general representability result [1, Theorem 3.3.5]. This was the approach taken in an earlier version of the present paper (arXiv:1207.2364v1), before the appearance of [1,2].

3. The second homology as a fundamental group

We now show how homotopy invariance for homology of linear groups can be used to identify the fundamental group of the singular resolution $G(k([\Delta^\bullet]))$ with the second group homology. We define for an isotropic reductive group G simplicial groups $G(k[\Delta^\bullet])$, $\mathrm{E}^G(k[\Delta^\bullet])$ and $\mathrm{St}^G(k[\Delta^\bullet])$ associated with the group, its elementary subgroup and its Steinberg group. The first thing to note is that homotopy invariance of K_1^G implies an isomorphism $\pi_1(G(k[\Delta^\bullet])) \cong \pi_1(\mathrm{E}(k[\Delta^\bullet]))$, which allows us to work with $\mathrm{E}(k[\Delta^\bullet])$ henceforth. We define further simplicial objects: denote by $K_2^G(k[\Delta^\bullet])$ the singular resolution of the functor

$$A \mapsto K_2^G(A) := \ker\left(\mathrm{St}^G(A) \to \mathrm{E}^G(A)\right),$$

by $\mathrm{UE}^G(k[\Delta^\bullet])$ the singular resolution of the functor $A \mapsto \mathrm{UE}^G(A)$, which assigns to each algebra A the universal central extension $\mathrm{UE}^G(A)$ of the perfect group $E^G(A)$, and by $\mathrm{H}_2^G(k[\Delta^\bullet])$ the singular resolution of the functor

$$A \mapsto \mathrm{H}_2^G(A) := \mathrm{H}_2(G(A), \mathbb{Z}) = \ker\left(\mathrm{UE}^G(A) \to \mathrm{E}^G(A)\right).$$

We chose slightly unusual notation in H_2^G to distinguish the above object from $\mathrm{H}_2(G(k[\Delta^\bullet]), \mathbb{Z})$, which has a different meaning.

With these notations, we have the following.

Lemma 3.1. *There are fibre sequences of simplicial sets:*

$$\mathrm{H}_2^G(k[\Delta^\bullet]) \to \mathrm{UE}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet]), \text{ and}$$
$$K_2^G(k[\Delta^\bullet]) \to \mathrm{St}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet]).$$

Proof. It follows from Moore's lemma, e.g., [3, Lemma I.3.4], that the morphisms $\mathrm{UE}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet])$ and $\mathrm{St}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet])$ are fibrations of fibrant simplicial sets. The fibres are by definition $\mathrm{H}_2^G(k[\Delta^\bullet])$ and $K_2^G(k[\Delta^\bullet])$, respectively. □

Proposition 3.2. *Let k be an infinite field, and let G be an isotropic reductive group over k. Then the boundary morphism $\Omega\,\mathrm{E}^G(k[\Delta^\bullet]) \to \mathrm{H}_2^G(k[\Delta^\bullet])$ associated with the fibration $\mathrm{UE}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet])$ induces an isomorphism:*

$$\pi_1(\mathrm{E}^G(k[\Delta^\bullet]), 1) \xrightarrow{\sim} \mathrm{H}_2(G(k), \mathbb{Z}).$$

If the Steinberg group does not have non-trivial central extensions, i.e. for all n

$$\mathrm{St}^G(k[\Delta^n])/\left[K_2^G(k[\Delta^n]), \mathrm{St}^G(k[\Delta^n])\right] \to \mathrm{E}^G(k[\Delta^n])$$

is the universal central extension, then the boundary morphism $\Omega\,\mathrm{E}^G(k[\Delta^\bullet]) \to K_2^G(k[\Delta^\bullet])$ associated with the fibration $\mathrm{St}^G(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet])$ induces an isomorphism

$$\pi_1(\mathrm{E}^G(k[\Delta^\bullet]), 1) \xrightarrow{\sim} K_2^G(k).$$

Proof. By [14, Theorem 1.1], all the usual maps (inclusion of constants, evaluation at 0) induce the isomorphisms $\mathrm{H}_2(G(k), \mathbb{Z}) \cong \mathrm{H}_2(G(k[T]), \mathbb{Z})$. Therefore, we have

$$\pi_0(\mathrm{H}_2^G(k[\Delta^\bullet])) = \mathrm{H}_2(G(k), \mathbb{Z}), \text{ and } \pi_1(\mathrm{H}_2^G(k[\Delta^\bullet])) = 0.$$

Moreover, $\mathrm{E}^G(k)$ and $\mathrm{St}^G(k)$ are generated by $X_A(u)$, $u \in V_A$. These elements are all homotopic to the identity by the homotopy $X_A(uT)$. Therefore,

$$\pi_0(\mathrm{E}^G(k[\Delta^\bullet])) \cong \pi_0(\mathrm{St}^G(k[\Delta^\bullet])) = 0.$$

The long exact sequence associated with the first fibre sequence from Lemma 3.1 yields via the above computations a short exact sequence

$$0 \to \pi_1(\mathrm{UE}^G(k[\Delta^\bullet])) \to \pi_1(\mathrm{E}^G(k[\Delta^\bullet])) \to \pi_0(\mathrm{H}_2^G(k[\Delta^\bullet])) \to 0.$$

Now let $\widetilde{\mathrm{E}^G}(k[\Delta^\bullet]) \to \mathrm{E}^G(k[\Delta^\bullet])$ be the universal covering of the simplicial group $\mathrm{E}^G(k[\Delta^\bullet])$. This has the structure of a simplicial group, and by uniqueness of liftings is degree-wise a central extension by $\pi_1(\mathrm{E}^G(k[\Delta^\bullet]))$. Therefore, the above injective map factors as $\pi_1(\mathrm{UE}^G(k[\Delta^\bullet])) \to \pi_1(\widetilde{\mathrm{E}^G}(k[\Delta^\bullet])) \to \pi_1(\mathrm{E}^G(k[\Delta^\bullet]))$, which together with $\pi_1(\widetilde{E^G}(k[\Delta^\bullet])) = 0$ implies the required isomorphism.

The second claim concerning K_2 follows by the same argument, replacing UE^G by

$$\mathrm{St}^G(k[\Delta^n])/[K_2^G(k[\Delta^n]), \mathrm{St}^G(k[\Delta^n])]. \qquad \square$$

Remark 3. It should be noted that the isomorphism in Proposition 3.2 has been established in the case of Chevalley groups over algebraically closed fields in [5, Theorem 2.1]. Jardine's proof uses the spectral sequence for the homology of $G(k[\Delta^\bullet])$ to establish this isomorphism. This is not too far away from our proof above; however, there are better methods available now to establish the necessary $\mathbb{A}^1$-invariance of H_2.

4. Explicit description of loops and relations

Fix a root system Φ. For a commutative unital ring R, denote $G(\Phi, R)$ the split Chevalley group, $\mathrm{E}(\Phi, R)$ its elementary subgroup and $\mathrm{St}(\Phi, R)$ its Steinberg group. We now describe explicit loops in $\pi_1(G(\Phi, k[\Delta^\bullet]))$, which is a direct translation of the Steinberg symbols for H_2. This also gives rise to an explicit isomorphism $\mathrm{H}_2(G(\Phi, k), \mathbb{Z}) \xrightarrow{\sim} \pi_1(G(\Phi, k[\Delta^\bullet]), 1)$

Definition 4.1. For every $\alpha \in \Phi$, we denote by $x_\alpha(u)$ the corresponding root group elements and then define morphisms

$$\begin{aligned}
X^\alpha &: \mathbb{G}_\mathrm{a}(R) \to \mathrm{E}(\Phi, R[T]), & R \ni u &\mapsto X^\alpha_T(u) := x_\alpha(Tu),\\
W^\alpha &: \mathbb{G}_\mathrm{m}(R) \to \mathrm{E}(\Phi, R[T]), & R^\times \ni u &\mapsto W^\alpha_T(u) := X^\alpha_T(u) X^{-\alpha}_T(-u^{-1}) X^\alpha_T(u),\\
H^\alpha &: \mathbb{G}_\mathrm{m}(R) \to \mathrm{E}(\Phi, R[T]), & R^\times \ni u &\mapsto H^\alpha_T(u) := W^\alpha_T(u) W^\alpha_T(1)^{-1},
\end{aligned}$$

$$\begin{aligned}
&C^\alpha : \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \to \mathrm{E}(\Phi, R[T]),\\
&\qquad R^\times \times R^\times \ni (a, b) \mapsto C^\alpha_T(a, b) := H^\alpha_T(a) H^\alpha_T(b) H^\alpha_T(ab)^{-1} \in \mathrm{E}(\Phi, R[T]).
\end{aligned}$$

We will use the same letters with an additional tilde to denote the corresponding lifts to $\mathrm{St}(\Phi, R[\Delta^\bullet])$.

Example 1. We give an example of the "symbol loops" in the group SL_2. With the obvious choice $x_\alpha(u) = e_{12}(u)$, we have

$$\begin{aligned}
C^\alpha_T(u, v) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + T(T^2 - 1)\frac{(1-u)(1-v)}{u^2 v} D^\alpha_T(u, v), \text{ where}\\
D^\alpha_T(u, v) &= \begin{pmatrix} u(1-u)T(T^2-1)(T^2-2) & -vu^2((T^2-1)^2(1-u)+u)(T^2-2) \\ (1-u)(T^2-1)^2 - 1 & -uv(1-u)T(T^2-1)(T^2-2) \end{pmatrix} \qquad \square
\end{aligned}$$

Remark 4. Philosophically, what is happening here is the following: choosing a maximal torus S in G, the associated root system and root subgroups x_α allows us to write down a contraction of the (elementary part of the) torus, i.e. a homotopy $H\colon S \times \mathbb{A}^1 \to G$, where $H(-, 0)$ factors through the identity $1 \in G$ and $H(-, 1)$ is the inclusion of S as maximal torus of G. This is nothing but a more elaborate version of the lemma of Whitehead. After fixing such a contraction, there is a preferred choice of path $H(u)$ for any $u \in S$. Given two units in the torus, one can concatenate the paths $H(u)$, $uH(v)$ and $H(uv)^{-1}$ to obtain a loop. This is basically what happens in Definition 4.1.

The translation between elements (and symbols) in the Steinberg group and loops (and symbol loops) in the singular resolution $G(\Phi, k[\Delta^\bullet])$ is given as in covering space theory:

(i) an element of the Steinberg group is given by a product $\tilde{y} = \prod_i \widetilde{x_{\alpha_i}}(u_i)$. Setting $y_T = \prod_i x_\alpha(Tu_i)$ produces a path in $\mathrm{E}(\Phi, R[T])$. If $\tilde{y}$ is in the kernel of the projection $\mathrm{St}(\Phi, R) \to \mathrm{E}(\Phi, R)$, the path y_T is in fact a loop;

(ii) a path $y_T \in \mathrm{E}(\Phi, k[T])$ with $y_T(0) = 1$ can be factored as a product of elementary matrices $\prod_i x_{\alpha_i}(f_i(T))$, which in turn can be lifted to $\mathrm{St}(\Phi, k[T])$. Evaluating at $T = 1$ yields an element $\prod_i \widetilde{x_{\alpha_i}}(f_i(1)) \in \mathrm{St}(\Phi, R)$. If the path y_T was in fact a loop, then the resulting element $\prod_i \widetilde{x_{\alpha_i}}(f_i(1)) \in \mathrm{St}(\Phi, R)$ lies in fact in the kernel of the projection $\mathrm{St}(\Phi, R) \to \mathrm{E}(\Phi, R)$.

It is then possible to derive elementary relations between the above loops in just the same way as the relations for Steinberg symbols in [7]. The contraction of the torus $H^\alpha_T(u)$ is chosen such that $H^\alpha_T(1)$ is the constant loop. From this, it follows immediately that $C^\alpha_T(x, 1) = C^\alpha_T(1, x) = 1$ for all $x, y \in k^\times$. The symbol loops $C^\alpha_T(x, y)$ in $G(\Phi, k[T])$ are not central on the nose, but are central up to homotopy because the fundamental group of a simplicial group is Abelian, and conjugation by paths acts trivially on the fundamental group. Then the conjugation formulas in [7, Lemma 5.2] can be translated into statements of homotopies between corresponding products of paths $W^\alpha_T(u)$ resp. $H^\alpha_T(u)$. In particular, the (weak) bilinearity of symbol loops in the fundamental group can be proved exactly as in [7]. For details, cf. [12]. It is not clear how to prove the Steinberg relation simply by computing with loops and homotopies inside $\mathrm{E}(k[\Delta^\bullet])$. We derive a general Steinberg relation from $\mathbb{A}^1$-homotopy theory in the next section.

5. The Steinberg relation from $\mathbb{A}^1$-homotopy theory

In the case of split groups, the Steinberg relation in $\mathrm{H}_2(G(k), \mathbb{Z})$ can be deduced from $\mathbb{A}^1$-homotopy as follows. We denote by Σ and Ω the simplicial suspension and loop space functors, respectively.

Proposition 5.1. *Let $C\colon \mathbb{G}_m \wedge \mathbb{G}_m \to \Omega G_\bullet$ be any morphism with $G_\bullet$ a simplicial group satisfying the affine Nisnevich excision. Let $s\colon \mathbb{A}^1 \setminus \{0,1\} \to \mathbb{G}_m \wedge \mathbb{G}_m$ be the Steinberg morphism $a \mapsto (a, 1-a)$. Then the composition of C with the Steinberg morphism $C \circ s\colon \mathbb{A}^1 \setminus \{0,1\} \to \Omega G_\bullet$ has trivial homotopy class in the simplicial and $\mathbb{A}^1$-local homotopy category.*

Proof. We have the natural adjunction $[\Sigma X, Y] \cong [X, \Omega Y]$ both in the simplicial and $\mathbb{A}^1$-local homotopy category. Choose a fibrant resolution $r\colon G_\bullet \to \mathrm{Ex}^\infty_{\mathbb{A}^1}(G_\bullet)$. Under the adjunction, the morphism $r \circ C \circ s$ corresponds to the composition

$$\Sigma\mathbb{A}^1 \setminus \{0,1\} \xrightarrow{\Sigma s} \Sigma\mathbb{G}_m \wedge \mathbb{G}_m \xrightarrow{C^{\mathrm{ad}}} \mathrm{Ex}^\infty_{\mathbb{A}^1}(G_\bullet).$$

By [4, Prop. 1], this composition factors through the $\mathbb{A}^1$-contractible space $\Sigma\mathbb{A}^1$ and is therefore trivial. More specifically, we have the following equality in $[\Sigma\mathbb{A}^1 \setminus \{0,1\}, \mathrm{Ex}^\infty_{\mathbb{A}^1}(G_\bullet)]_{\mathbb{A}^1}$:

$$r \circ C^{\mathrm{ad}} \circ \Sigma s = r \circ C^{\mathrm{ad}} \circ \Sigma\tilde{s} \circ \Sigma\iota = r \circ C^{\mathrm{ad}} \circ 0 = 0.$$

This implies the $\mathbb{A}^1$-local statement. The simplicial statement follows from [1, Theorem 3.3.5], which gives a bijection

$$[\mathbb{A}^1 \setminus \{0,1\}, G_\bullet]_s \cong [\mathbb{A}^1 \setminus \{0,1\}, \mathrm{Ex}^\infty_{\mathbb{A}^1}(G_\bullet)]_{\mathbb{A}^1}. \qquad \square$$

The result and Lemma 2.2 imply that for split G, all the loops $C^\alpha(u, 1-u)$, $u \in k^\times$, described in Section 3 are contractible in the singular resolution $G(k[\Delta^\bullet])$: the symbol $C^\alpha(x,y)$ can be interpreted as a morphism of simplicial groups $\mathbb{G}_m \times \mathbb{G}_m \to \Omega\,\mathrm{Sing}^{\mathbb{A}^1}_\bullet G$. But since $C^\alpha(1,y) = C^\alpha(x,1) = 1$ is equal to the identity, this morphism factors through a morphism of simplicial presheaves $\mathbb{G}_m \wedge \mathbb{G}_m \to \Omega\,\mathrm{Sing}^{\mathbb{A}^1}_\bullet G$. The above corollary then yields the Steinberg relation. Even better, since $\mathrm{Sing}^{\mathbb{A}^1}_\bullet G$ has affine excision, there is a single algebraic morphism $\mathbb{A}^1 \setminus \{0,1\} \times \mathbb{A}^1 \to G$ realizing all the Steinberg loops $C^\alpha(u, 1-u)$, $u \in k^\times \setminus \{1\}$ at once; and there is a single algebraic homotopy $(\mathbb{A}^1 \setminus \{0,1\} \times \mathbb{A}^1) \times \mathbb{A}^1 \to G$ providing all the contractions of the Steinberg loops at once. This is one instance where a computation in group homology can be deduced from $\mathbb{A}^1$-homotopy theory.

We want to point out the following generalization of the Steinberg relation for non-split groups. Let D be a central simple algebra over k. There is an associated reduced norm which can be interpreted as a regular morphism $\mathrm{Nrd}_D\colon \mathbb{A}^{\dim D} \to \mathbb{A}^1$. In $\mathbb{A}^{\dim D}$ we have two open subschemes, the linear algebraic group $\mathrm{GL}_1(D)$ defined by $\mathrm{Nrd}_D(u) \neq 0$, and another open subscheme $\mathcal{U}_D$ defined by $\mathrm{Nrd}_D(u) \neq 0$ and $\mathrm{Nrd}_D(1-u) \neq 0$. There is an obvious analogue of the Steinberg morphism:

$$s_D : \mathcal{U}_D \to \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \to \mathrm{GL}_1(D) \wedge \mathrm{GL}_1(D)\colon u \mapsto (u, 1-u).$$

Proposition 5.2. *Let $s_D : \mathcal{U}_D \to \mathrm{GL}_1(D) \wedge \mathrm{GL}_1(D)$ be the Steinberg morphism defined above. Then there exists a space $\mathcal{X}_D$ and a commutative diagram*

$$\begin{array}{ccc} \mathcal{U}_D & \xrightarrow{\quad s_D \quad} & \mathrm{GL}_1(D) \wedge \mathrm{GL}_1(D) \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\psi_D} \\ \mathbb{A}^{\dim D} & \xrightarrow{\quad \tilde{s}_D \quad} & \mathcal{X}_D \end{array}$$

with the suspension $\Sigma\psi_D$ of ψ_D being an $\mathbb{A}^1$-local weak equivalence.

Proof. The argument is the same as in [4, Prop. 1], replacing $\mathbb{A}^1$ by $\mathbb{A}^{\dim D}$, $\mathbb{G}_m$ by $\mathrm{GL}_1(D)$, and $\mathbb{A}^1 \setminus \{0,1\}$ by $\mathcal{U}_D$. The varieties V and W have to be replaced by $\mathcal{V}_D = [y - 1 = x \cdot_D z, y \neq 0]$ and $\mathcal{W}_D = [x - 1 = y \cdot_D z, x \neq 0]$. The space $\mathcal{X}_D$ is then the pushout $\mathcal{V}_D \cup_{\mathrm{GL}_1(D) \times \mathrm{GL}_1(D)} \mathcal{W}_D$. $\square$

This provides an $\mathbb{A}^1$-homotopy proof of the Steinberg relation in $\mathrm{H}_2(\mathrm{SL}_n(D), \mathbb{Z})$, $n \geq 3$. All Steinberg relations are given by a single algebraic map $\mathcal{U}_D \times \mathbb{A}^1 \to \mathrm{SL}_n(D)$, and they are all contracted by a single (inexplicit) algebraic homotopy $(\mathcal{U}_D \times \mathbb{A}^1) \times \mathbb{A}^1 \to \mathrm{SL}_n(D)$.

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