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Numerical analysis

NURBS or not NURBS?

NURBS ou pas ?

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ABSTRACT

The view adopted here is that the largest class of splines which are useful for design is composed of all spaces of geometrically continuous piecewise quasi-Chebyshevian splines that contain constants and possess blossoms. We recently described an iterative construction of all spaces of this class. This note announces the possibility of building associated rational spline spaces (and therefore, associated NURBS) while remaining in the same class. This is obtained when applying in an appropriate way one step of this construction.

RÉSUMÉ

On peut assez naturellement considérer que la plus grande classe d'espaces de splines utiles pour le design est celle des espaces de splines à sections dans différents espaces « quasi-Chebyshev généralisés » reliées entre elles par des matrices de connexion, espaces qui, de plus, contiennent les constantes et possèdent des floraisons. Nous avons récemment donné une construction itérative étonnamment simple de cette classe très difficile. Une application adéquate d'une étape de cette itération peut être interprétée comme la construction de splines rationnelles (donc de NURBS) associées, tout se passant à l'intérieur de la même classe.

1. Geometrically continuous piecewise Chebyshevian splines for design

Throughout this note, we work with a fixed interval $[a, b]$, $a < b$, and a fixed positive number n. An $(n+1)$-dimensional space $\mathbb{E} \subset C^n([a, b])$ is an *Extended Chebyshev-space (for short EC-space) on* $[a, b]$ if any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a, b]$, counting multiplicities up to $(n+1)$, or equivalently if any Hermite interpolation problem in $(n+1)$ data on $[a, b]$ has a unique solution in $\mathbb{E}$. Because we are working on a closed bounded interval, this important class of spaces coincides with the class of all spaces of the form $EC(w_0, \dots, w_n)$, defined as the set of all functions $F \in C^n([a, b])$ for which $L_n F$ is constant on $[a, b]$, where L_n is the differential operator built from the *system* $(w_0, \dots, w_n)$ *of weight functions on* $[a, b]$ (which means that each w_i is C^{n-i} and positive on $[a, b]$) via the classical procedure [19]

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$$L_0F := \frac{F}{w_0}, \quad L_iF := \frac{1}{w_i}DL_{i-1}F, \quad 1 \leqslant i \leqslant n. \tag{1}$$

The EC-spaces which are *good for design* on $[a,b]$ are those in which we can take $w_0 = \mathbb{1}$, where $\mathbb{1}(x) = 1$ for all $x \in [a,b]$.

We now consider a fixed sequence of interior knots $a < t_1 < \cdots < t_q < b$ and a fixed associated sequence of multiplicities m_k, with $0 \leqslant m_k \leqslant n$ for $1 \leqslant k \leqslant q$. With $t_0 := a$, $t_{q+1} = b$ and $m_0 := m_{q+1} = n+1$, and with $x^{[k]}$ standing for x repeated k times, this provides us with the knot-vector

$$\mathbb{K} := \left({t_0}^{[m_0]}, {t_1}^{[m_1]}, \dots, {t_q}^{[m_q]}, {t_{q+1}}^{[m_{q+1}]}\right) = (\xi_{-n}, \dots, \xi_{m+n+1}), \quad \text{where } m := \sum_{k=1}^{q} m_k.$$

We denote by $\mathcal{C}(\mathbb{K})$ the class of all $(n+1+m)$-dimensional spaces of *piecewise Chebyshevian splines* (for short PEC-splines) based on $\mathbb{K}$. To build a space $\mathbb{S}$ in the class $\mathcal{C}(\mathbb{K})$, we need the following ingredients:

– a sequence of section spaces $\mathbb{E}_k$, $0 \leqslant k \leqslant q$: for each k, $\mathbb{E}_k \subset C^n([t_k, t_{k+1}])$ contains constants and the space $D\mathbb{E}_k$ is an n-dimensional EC-space on $[t_k, t_{k+1}]$ (*i.e.*, $\mathbb{E}_k$ is good for design on $[t_k, t_{k+1}]$);
– a sequence of connection matrices M_k, $1 \leqslant k \leqslant q$: for each k, M_k is a lower triangular square matrix of order $(n - m_k)$ with positive diagonal entries.

The associated PEC-spline space $\mathbb{S}$ (containing constants) is the set of all continuous functions $S : I \to \mathbb{R}$ such that

1) for $k = 0, \dots, q$, the restriction of S to $[t_k, t_{k+1}]$ belongs to $\mathbb{E}_k$;
2) for $k = 1, \dots, q$, the following connection condition is fulfilled:

$$\left(S'(t_k^+), \dots, S^{(n-m_k)}(t_k^+)\right)^T = M_k \cdot \left(S'(t_k^-), \dots, S^{(n-m_k)}(t_k^-)\right)^T. \tag{2}$$

The expression “PEC-splines” is used to stress that the pieces are taken from different EC-spaces. Due to the presence of connection matrices, PEC-splines are implicitly allowed to be *geometrically continuous*. By contrast, we use the expression *Chebyshevian spline space* in the simpler case where there exists a system $(w_1, \dots, w_n)$ of weight functions on $[a,b]$ such that $EC(\mathbb{1}, w_1, \dots, w_n) \subset \mathbb{S}$, *i.e.*, when all section-spaces are obtained as restrictions of a single EC-space good for design on the whole of $[a,b]$, and when the splines are *parametrically continuous* (*i.e.*, all M_k are identity matrices). The ordinary polynomial spline space of degree n (based on $\mathbb{K}$) is obtained when $w_1 = \cdots = w_n = \mathbb{1}$.

Not all spaces of the class $\mathcal{C}(\mathbb{K})$ are of interest. This is why we consider the subclass $\mathcal{C}_0(\mathbb{K})$ of all $\mathbb{S} \in \mathcal{C}(\mathbb{K})$ which are *good for design* in the sense that they possess *blossoms*. Readers more precisely interested in blossoms are referred to [16] and other references therein. We limit ourselves to mentioning that, when $\mathbb{S} \in \mathcal{C}_0(\mathbb{K})$, each PEC-spline $S \in \mathbb{S}$ *blossoms* into a symmetric function s of n variables (its blossom), which, by nature, is defined on a restricted subset of $[a,b]^n$ containing the diagonal of $[a,b]^n$, on which s gives S. In this very difficult PEC-context, a major difficulty consists in proving that blossoms are *pseudoaffine* in each variable. This is the precise property which permits the development of all the classical CAGD algorithms (evaluation, knot insertion, subdivision…). This leads to the following statements [13], which highly justify our terminology *good for design*.

Theorem 1.1. *Any PEC-spline space $\mathbb{S} \in \mathcal{C}_0(\mathbb{K})$ possesses a B-spline basis which is its optimal normalised totally positive basis. Conversely, given $\mathbb{S} \in \mathcal{C}(\mathbb{K})$, if $\mathbb{S}$ and any spline space derived from $\mathbb{S}$ by knot insertion possess B-spline bases, then $\mathbb{S} \in \mathcal{C}_0(\mathbb{K})$.*

As is classical, a B-spline basis in $\mathbb{S}$ is a sequence $N_\ell \in \mathbb{S}$, $-n \leqslant \ell \leqslant m$, which is *normalised* (*i.e.*, $\sum_{\ell=-n}^{m} N_\ell = \mathbb{1}$), each N_ℓ being positive on the interior of its support $[\xi_\ell, \xi_{\ell+n+1}]$, with some additional condition on its zeroes at the endpoints of its support. The *total positivity* of such bases ensures shape-preserving control (see [7]), and optimality should simply be understood as “the best possible” from this viewpoint, see [11] and references therein.

We now consider a system $(w_0, \dots, w_n)$ of *piecewise weight functions on* $[a,b]$, with the meaning that each w_i is C^{n-i} and positive separately on each $[t_k^+, t_{k+1}^-]$. With such a system we can associate linear piecewise differential operators $L_0, \dots, L_n$ via the procedure already recalled in (1). We denote by $ECP(w_0, \dots, w_n)$ the set of all piecewise functions on $[a,b]$ such that L_nF is constant on $[t_k^+, t_{k+1}^-]$ for $k = 0, \dots, q$, with the additional requirement that

$$L_iF(t_k^+) = L_iF(t_k^-) \text{ for } i = 0, \dots, n, \text{ and for } k = 1, \dots, q.$$

This space is $(n+1)$-dimensional and it is an *Extended Chebyshev Piecewise space on* $[a,b]$, in the sense that we can count the total number of zeroes of each of its non-zero elements, including multiplicities up to $(n+1)$, and this number is at most n. We conclude this section with a constructive characterisation of the subclass $\mathcal{C}_0(\mathbb{K})$ (see [16,17]), of which the most difficult is the “only if” part. It implies in particular that splines in $\mathcal{C}_0(\mathbb{K})$ are examples of the so-called *CCT-splines* (see Chapter 9 of [19] and [1]).

Theorem 1.2. *Let $\mathbb{S} \in \mathcal{C}(\mathbb{K})$ be given. Then, $\mathbb{S} \in \mathcal{C}_0(\mathbb{K})$ if and only if there exists a system $(w_1, \dots, w_n)$ of piecewise weight functions on $[a,b]$ such that $ECP(\mathbb{1}, w_1, \dots, w_n) \subset \mathbb{S}$.*

2. Geometrically continuous piecewise Chebyshevian NURBS

That a space $\mathbb{S}$ in the class $\mathcal{C}(\mathbb{K})$ belongs to the subclass $\mathcal{C}_0(\mathbb{K})$ can as well be characterised by the existence of B-spline-like bases in the space $D\mathbb{S}$ obtained from $\mathbb{S}$ by (possibly left or right) differentiation and also in all spline spaces obtained from it via knot-insertion [16]. Note that, a priori, splines in $D\mathbb{S}$ are not functions but piecewise functions. B-spline-like bases satisfy properties similar to those of a B-spline basis, not including normalisation.

In the rest of this section, **we consider a fixed PEC-spline space $\mathbb{S}$ in $\mathcal{C}_0(\mathbb{K})$**, and we denote by N_k, $k = -n, \ldots, m$ its B-spline basis. One key-point to establish the "only if" part of Theorem 1.2 is the following result (see [16,17]).

Theorem 2.1. *For a spline $\Sigma = \sum_{-n}^{m} \sigma_k N_k \in \mathbb{S}$ the following properties are equivalent:*

(i) *the poles $\sigma_{-n}, \ldots, \sigma_m$ of Σ form a strictly increasing sequence;*
(ii) *$W := D\Sigma$ has positive coordinates in any B-spline-like basis of $D\mathbb{S}$;*
(iii) *the piecewise function W is positive on each $[t_k^+, t_{k+1}^-]$ and, if we define the piecewise differential operator L by $LF = DF/W$, then the spline space $L\mathbb{S}$ lies in the class $\mathcal{C}_0(\mathbb{K})$.*

This theorem has been exploited in [17] to show the existence of infinitely many Schoenberg-type operators in $\mathbb{S}$, permitting simultaneous approximation of a function and its first derivative. Here, we exploit it in a different way, after observing that its equivalence (ii) $\Leftrightarrow$ (iii) is actually an equivalence within the space $D\mathbb{S}$. We can thus restate it in $\mathbb{S}$ rather than in $D\mathbb{S}$, which yields:

Theorem 2.2. *Given a spline $\Omega = \sum_{-n}^{m} \omega_k N_k \in \mathbb{S}$, the following properties are equivalent:*

(i) *the poles $\omega_{-n}, \ldots, \omega_m$ of Ω are all positive;*
(ii) *the spline Ω is positive on $[a, b]$ and the space obtained after division of all elements of $\mathbb{S}$ by Ω belongs to the subclass $\mathcal{C}_0(\mathbb{K})$.*

Definition 2.1. For each spline $\Omega \in \mathbb{S}$ with positive poles, the space obtained after division of all elements of $\mathbb{S}$ by Ω is called the rational spline space based on $\mathbb{S}$ and Ω. We denote it by $\mathcal{R}(\mathbb{S}; \Omega)$.

We now assume that (i) of Theorem 2.2 holds. The B-spline basis in $\mathcal{R}(\mathbb{S}; \Omega)$ is the sequence

$$\frac{\omega_k N_k}{\Omega}, \quad k = -n, \ldots, m. \tag{3}$$

The rational spline space $\mathcal{R}(\mathbb{S}; \Omega)$ can thus be described as the set of all continuous functions of the form

$$\frac{\sum_{-n}^{m} \alpha_k \omega_k N_k}{\sum_{-n}^{m} \omega_k N_k}, \quad \alpha_{-n}, \ldots, \alpha_m \in \mathbb{R}. \tag{4}$$

By analogy with the classical rational splines, we say the functions in (3) are *geometrically continuous piecewise Chebyshevian NURBS*. One can check that, for each $\Omega \in \mathbb{S}$ satisfying (i) of Theorem 2.2

$$\mathbb{S} = \mathcal{R}\left(\mathcal{R}(\mathbb{S}; \Omega); \frac{1}{\Omega}\right). \tag{5}$$

3. Geometrically continuous piecewise quasi-Chebyshevian NURBS

An $(n+1)$-dimensional space $\mathbb{E} \subset C^{n-1}([a, b])$ is said to be a *Quasi-Extended Chebyshev-space (for short QEC-space) on* $[a, b]$ if any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a, b]$, counting multiplicities up to n, or, equivalently, if any Hermite interpolation problem in $(n+1)$ data involving at least two distinct points in $[a, b]$ has a unique solution in $\mathbb{E}$. If $\mathbb{C}$ is a two-dimensional *Chebyshev space (C-space) on* $[a, b]$ (which, in dimension two, is the same as being a QEC-space on $[a, b]$) and if L_{n-1} is associated with a system $(w_0, \ldots, w_{n-1})$ of weight functions on $[a, b]$, then the set $QEC(w_0, \ldots, w_{n-1}; \mathbb{C})$ composed of all functions $F \in C^{n-1}([a, b])$ for which $L_{n-1}F \in \mathbb{C}$ is an $(n+1)$-dimensional QEC-space on $[a, b]$ [8]. Actually, all $(n+1)$-dimensional QEC-space on $[a, b]$ are of this form [14]. For the design, we have to take $w_0 = \mathbb{1}$. Note that the QEC-context implies many more difficulties (see [12]). As an example, for any real numbers $p, q > n-1$, the space $\mathbb{E}_{p,q}$ spanned on $[0, 1]$ by the $(n+1)$ functions $1, x, \ldots, x^{n-2}, x^p, (1-x)^q$ is a QEC-space on $[0, 1]$, while it is an EC-space on $[0, 1]$ if and only if $p = q = n$.

To define the class $\mathcal{QC}(\mathbb{K})$ of all $(n+1+m)$-dimensional spaces of (geometrically continuous) piecewise quasi-Chebyshevian (PQEC) splines, we weaken the requirements on the section-spaces: if each $\mathbb{E}_k$ is still assumed to contain constants, we only require that $\mathbb{E}_k \subset C^{n-1}([t_k, t_{k+1}])$ and that $D\mathbb{E}_k$ be an n-dimensional QEC-space on $[t_k, t_{k+1}]$. Apart from this change, the PQEC-spline space $\mathbb{S}$ is then defined exactly as previously when all multiplicities are positive. Without going into details, let us mention that if $m_k = 0$ for some $k \in \{1, \dots, q\}$, to a connection of the type (2) between the $(n-1)$ first left and right derivatives at t_k, we have to add a convenient relation between left and right Bézier points. To have a flavour of what such a relation should be, the reader can refer to [15] and [18]. As previously, we have to introduce the subclass $\mathcal{QC}_0(\mathbb{K})$ composed of all $\mathbb{S} \in \mathcal{QC}(\mathbb{K})$, which are good for design in the sense that they possess blossoms. Of course this is a larger class than $\mathcal{C}_0(\mathbb{K})$. In this larger framework, we can then state the exact analogue of Theorem 1.1, simply replacing "B-spline basis" by "Quasi-B-spline basis". The term "quasi" refers to the fact that the count of zeroes at the endpoints of the supports must take into account that the section spaces are not EC-spaces, but QEC-spaces.

Given a system $(w_0, \dots, w_{n-1})$ of piecewise weight functions on $[a, b]$, the associated piecewise differential operator L_{n-1}, and a two-dimensional C-space $\mathbb{C} \subset C^0([a, b])$ on $[a, b]$, denote by $QECP(w_0, w_1, \dots, w_{n-1}; \mathbb{C})$ the $(n+1)$-dimensional space of all piecewise functions F on $[a, b]$ such $L_{n-1}F \in \mathbb{C}$ and $L_iF(t_k^+) = L_iF(t_k^+)$ for $i = 0, \dots, n-1$ and $k = 1, \dots, q$. This space is a *Quasi-Extended Chebyshev Piecewise space* in the sense that the total number of zeroes of a non-zero element is bounded above by n, multiplicities included up to n. Below, the analogue of Theorem 1.2 describes the class $\mathcal{QC}_0(\mathbb{K})$ (including the possibility of zero multiplicities) see [15].

Theorem 3.1. *Assume that $\mathbb{S} \in \mathcal{QC}(\mathbb{K})$. Then $\mathbb{S} \in \mathcal{QC}_0(\mathbb{K})$ if and only if there exists a system $(w_1, \dots, w_{n-1})$ of piecewise weight functions on $[a, b]$ and a two-dimensional C-space $\mathbb{C}$ on $[a, b]$ such $QECP(\mathbb{1}, w_1, \dots, w_{n-1}; \mathbb{C}) \subset \mathbb{S}$.*

The "only if" part relies on an analogue of Theorem 2.1 for $n \geqslant 2$. However, moving from $\mathcal{C}_0(\mathbb{K})$ to $\mathcal{QC}_0(\mathbb{K})$ introduces many additional difficulties in the proofs. This in turn leads to the exact analogue of Theorem 2.2, simply replacing $\mathcal{C}_0(\mathbb{K})$ by $\mathcal{QC}_0(\mathbb{K})$. Within $\mathcal{QC}_0(\mathbb{K})$, we can thus define *geometrically continuous piecewise quasi-Chebyshevian NURBS* as in (3), the rational spaces satisfying (4) and (5).

4. Conclusion

The class $\mathcal{QC}_0(\mathbb{K})$ can be seen as the largest class of spline spaces with ordinary differentiability assumptions on the section-spaces which can be used for Geometric Design (or Approximation or Isogeometric Analysis). The most famous examples of spaces in $\mathcal{QC}_0(\mathbb{K}) \setminus \mathcal{C}_0(\mathbb{K})$ are the so-called *variable "degree" splines*, that is, splines with, up to affine changes of variables, different $\mathbb{E}_{p,q}$ as section-spaces [3,4,9,8,10,15,2]. To make the note accessible, we deliberately avoided details on the difficulties specific to QEC-spaces [8,12,14], preferring to insist on the smaller class $\mathcal{C}_0(\mathbb{K})$. This choice also enables readers to have access to detailed versions of the recent results recalled in Section 1, since they have already appeared, while it is not yet so concerning their analogues in the even more difficult class $\mathcal{QC}_0(\mathbb{K})$, see [18].

Theorem 2.2 says in particular that **the classical rational spline spaces are examples of spaces of parametrically continuous splines in the class** $\mathcal{C}_0(\mathbb{K})$. They can thus benefit from all properties developed within $\mathcal{C}_0(\mathbb{K})$, see [17] for instance. Compared to the degree n polynomial B-splines, one of the main interests of introducing the classical NURBS (see [5,6]) was the shape effects permitted by the parameters $\omega_{-n}, \dots, \omega_m$ defining them. The class $\mathcal{QC}_0(\mathbb{K})$ provides us with such a great variety of shape parameters (coming either from the section-spaces or from the connection matrices) that it may seem useless to add new parameters to introduce NURBS in it, all the more so as this does not increase the class $\mathcal{QC}_0(\mathbb{K})$. Nevertheless, given $\mathbb{S} \in \mathcal{QC}_0(\mathbb{K})$, it will be interesting to investigate how its own shape parameters interact with the positive parameters defining all rational spline spaces based on $\mathbb{S}$. The whole question will be addressed in details in a further article.

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