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Number theory

On the lower bound of the discrepancy of Halton's sequence I

Sur la limite inférieure de la discrépance de la suite de Halton I

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ABSTRACT

Let $(H_s(n))_{n\geq 1}$ be an s-dimensional Halton's sequence. Let D_N be the discrepancy of the sequence $(H_s(n))_{n=1}^N$. It is known that $ND_N = O(\ln^s N)$ as $N \to \infty$. In this paper, we prove that this estimate is exact:

$$\overline{\lim}_{N\to\infty} N \ln^{-s}(N) D_N > 0.$$

RÉSUMÉ

Soit $(H_s(n))_{n\geq 1}$ une suite de Halton á s dimensions. Soit D_N la discrépance de la suite $(H_s(n))_{n=1}^N$. Il est connu que $ND_N = O(\ln^s N)$ lorsque $N \to \infty$. Dans cet article, nous montrons que cette estimation est exacte :

$$\overline{\lim}_{N\to\infty} N \ln^{-s}(N) D_N > 0.$$

1. Introduction

Let $(\beta_n)_{n\geq 1}$ be a sequence in the unit cube $[0,1)^s$, $\mathbf{y} = (y_1, ..., y_s)$, $B(\mathbf{y}) = [0, y_1) \times \cdots \times [0, y_s) \subseteq [0,1)^s$,

$$\Delta(B(\mathbf{y}), (\beta_n)_{n=1}^N) = \sum_{1\leq n\leq N} (\mathbf{1}_{B(\mathbf{y})}(\beta_n) - y_1 \cdots y_s), \quad \text{where} \quad \mathbf{1}_{B(\mathbf{y})}(\mathbf{x}) = 1, \text{ if } \mathbf{x} \in B(\mathbf{y}), \tag{1}$$

and $\mathbf{1}_{B(\mathbf{y})}(\mathbf{x}) = 0$, if $\mathbf{x} \notin B(\mathbf{y})$. We define the star *discrepancy* of a N-point set $(\beta_n)_{n=1}^N$ as

$$D^*((\beta_n)_{n=1}^N) = \sup_{0<y_1,...,y_s\leq 1} |\frac{1}{N} \Delta(B(\mathbf{y}), (\beta_n)_{n=1}^N)|. \tag{2}$$

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Let $(\beta_n)_{n\geq 1}$ be an arbitrary sequence in $[0,1)^s$. In 1954, Roth proved that

$$\limsup_{N\to\infty} N(\ln N)^{-\frac{s}{2}} D^*((\beta_n)_{n=1}^N) > 0.$$

According to the well-known conjecture (see, e.g., [1, p. 283]), this estimate can be improved

$$\limsup_{N\to\infty} N(\ln N)^{-s} D^*((\beta_n)_{n=1}^N) > 0. \tag{3}$$

In 1972, W. Schmidt proved this conjecture for $s=1$. For $s=2$, Faure and Chaix [4] proved (3) for a class of (t,s)-sequences. For a review of research on this conjecture, see, for example, [2].

Definition. An s-dimensional sequence $(\beta_n)_{n\geq 1}$ is of `low discrepancy` (abbreviated l.d.s.) if $D^*((\beta_n)_{n=1}^N) = O(N^{-1}(\ln N)^s)$ for $N\to\infty$.

Let $p\geq 2$ be an integer,

$$n=\sum_{i\geq 0} e_{p,i}(n)p^i, \text{ with } e_{p,i}(n)\in\{0,1,\dots,p-1\}, \qquad \text{and} \qquad \phi_p(n)=\sum_{i\geq 0} e_{p,i}(n)p^{-i-1}. \tag{4}$$

Van der Corput (see [3, Ref. 1891]) proved that $(\phi_p(n))_{n\geq 0}$ is a 1-dimensional l.d.s. Let

$$H_s(n)=(\phi_{p_1}(n),\dots,\phi_{p_s}(n)), \quad n=0,1,2,\dots, \tag{5}$$

where $p_1,\dots,p_s\geq 2$ are pairwise coprime integers. Halton (see [3, Ref. 729]) proved that $(H_s(n))_{n\geq 0}$ is an s-dimensional l.d.s. For other examples of l.d.s. see e.g. [1,3]. In §2 we will prove the following theorem.

Theorem. *Let $p_0=p_1p_2\cdots p_s$, $s\geq 2$ and $m_0=[2p_0\log_2 p_0]+2$. Then*

$$\sup_{1\leq N\leq 2^{mm_0}} ND^*((H_s(n))_{n=1}^N)\geq m^s(8p_0)^{-1} \quad \textit{for} \quad m\geq p_0.$$

Remark. This result supports the conjecture (3). In [5], we received a similar result for sequences obtained from algebraic lattices. In [6], we proved a similar result for some (t,s)-sequences (see also [7]).

2. Proof of the Theorem

Let $x_i=\sum_{j\geq 1} x_{i,j}p_i^{-j}$, with $x_{i,j}\in\{0,1,\dots,p_i-1\}$, $[x_i]_r=\sum_{1\leq j\leq r} x_{i,j}p_i^{-j}$, $i=1,\dots,s$, $r=1,2,\dots$. By (4), we have $\phi_{p_i}(n)\in[[x_i]_r,[x_i]_r+p_i^{-r})$ if and only if $n\equiv \dot{x}_{i,r} \pmod{p_i^r}$, where $\dot{x}_{i,r}=\sum_{1\leq j\leq r} x_{i,j}p_i^{j-1}$. Let $\mathbf{r}=(r_1,\dots,r_s)$, $P_{\mathbf{r}}=p_1^{r_1}\cdots p_s^{r_s}$ and $M_{i,\mathbf{r}}\equiv (P_{\mathbf{r}}p_i^{-r_i})^{-1} \pmod{p_i^{r_i}}$. Using the Chinese Remainder Theorem, we get

$$\phi_{p_i}(n)\in[[x_i]_{r_i},[x_i]_{r_i}+p_i^{-r_i}), \text{ for } i=1,\dots,s \iff n\equiv \ddot{x}_{\mathbf{r}} \pmod{P_{\mathbf{r}}} \text{ with } \ddot{x}_{\mathbf{r}}=\sum_{i=1}^s M_{i,\mathbf{r}}P_{\mathbf{r}}p_i^{-r_i}\dot{x}_{i,r_i}. \tag{6}$$

It is easy to verify that if $r_i'\geq r_i$, for all $i=1,\dots,s$, then

$$\ddot{x}_{\mathbf{r}'}\equiv \ddot{x}_{\mathbf{r}} \pmod{P_{\mathbf{r}}}. \tag{7}$$

We consider the case $x_{i,r_i}\neq 0$ for all $i=1,\dots,s$. We obtain from (6) that

$$\phi_{p_i}(n)\in[[x_i]_{r_i}-p_i^{-r_i},[x_i]_{r_i}), \quad \text{for} \quad i=1,\dots,s \iff n\equiv \ddot{x}_{\mathbf{r}}-\sum_{i=1}^s M_{i,\mathbf{r}}P_{\mathbf{r}}p_i^{-1} \pmod{P_{\mathbf{r}}}. \tag{8}$$

Let $p_0=p_1p_2\cdots p_s$, $\breve{p}_i=p_0/p_i$, $\tau_i=\min\{1\leq k<\breve{p}_i | p_i^k\equiv 1 \pmod{\breve{p}_i}\}$, $i=1,\dots,s$. Let $\mathbf{y}=(y_1,\dots,y_s)$ with $y_i=\sum_{1\leq j\leq m} p_i^{-j\tau_i}$, $[y_i]_{\tau_ik_i}=\sum_{1\leq j\leq k_i} p_i^{-j\tau_i}$, and let $\dot{y}_{i,\tau_ik_i}=\sum_{1\leq j\leq k_i} p_i^{j\tau_i-1}$, $k_i\geq 1$, $i=1,\dots,s$, $\mathbf{k}=(k_1,\dots,k_s)$, $B(\mathbf{y})=[0,y_1)\times\cdots\times[0,y_s)\subset[0,1)^s$, $B_{\mathbf{k}}=\prod_{1\leq i\leq s}[[y_i]_{\tau_ik_i}-p_i^{-k_i\tau_i},[y_i]_{\tau_ik_i})$, $\boldsymbol{\tau}=(\tau_1,\dots,\tau_s)$, $u\cdot v=(u_1v_1,\dots,u_sv_s)$. We have

$$B(\mathbf{y})=\bigcup_{1\leq k_1,\dots,k_s\leq m} B_{\mathbf{k}}, \quad \text{and} \quad \mathbf{1}_{B(\mathbf{y})}(\mathbf{z})-y_1\cdots y_s=\sum_{1\leq k_1,\dots,k_s\leq m}(\mathbf{1}_{B_{\mathbf{k}}}(\mathbf{z})-P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}). \tag{9}$$

Let $\hat{y}_{\boldsymbol{\tau}\cdot\mathbf{k}}=\sum_{i=1}^s M_{i,\boldsymbol{\tau}\cdot\mathbf{k}},P_{\boldsymbol{\tau}\cdot\mathbf{k}}p_i^{-\tau_ik_i}\dot{y}_{i,\tau_i(k_i-1)}$ and

$$A_{\mathbf{k}} \equiv -\sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\mathbf{k}} P_{\boldsymbol{\tau}\cdot\mathbf{k}} p_i^{-1} \pmod{P_{\boldsymbol{\tau}\cdot\mathbf{k}}}, \quad \text{with} \quad A_{\mathbf{k}} \in [0, P_{\boldsymbol{\tau}\cdot\mathbf{k}}). \tag{10}$$

From (6) we get $\ddot{y}_{\boldsymbol{\tau}\cdot\mathbf{k}} = \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\mathbf{k}}, P_{\boldsymbol{\tau}\cdot\mathbf{k}} p_i^{-\tau_i k_i} \dot{y}_{i,\tau_i k_i} \equiv \hat{y}_{\boldsymbol{\tau}\cdot\mathbf{k}} - A_{\mathbf{k}} \pmod{P_{\boldsymbol{\tau}\cdot\mathbf{k}}}$. By (5) and (8), we obtain

$$H_s(n) \in B_{\mathbf{k}} \iff \phi_{p_i}(n) \in [\dot{y}_{i,\tau_i k_i} - p_i^{-\tau_i k_i}, \dot{y}_{i,\tau_i k_i}), \quad \text{for} \quad i = 1, \dots, s \iff n \equiv \hat{y}_{\boldsymbol{\tau}\cdot\mathbf{k}} \pmod{P_{\boldsymbol{\tau}\cdot\mathbf{k}}}. \tag{11}$$

Let

$$\tilde{y}_m :\equiv \hat{y}_{\boldsymbol{\tau}(m+1)} \pmod{P_{\boldsymbol{\tau}(m+1)}}, \text{ with } \tilde{y}_m \in [0, P_{\boldsymbol{\tau}(m+1)}), \text{ where } \boldsymbol{\tau}(m+1) = (\tau_1(m+1), \dots, \tau_s(m+1)). \tag{12}$$

Using (7), we get $\hat{y}_{\boldsymbol{\tau}\cdot\mathbf{k}} - A_{\mathbf{k}} \equiv \ddot{y}_{\boldsymbol{\tau}\cdot\mathbf{k}} \equiv \ddot{y}_{\boldsymbol{\tau}(m+1)} \equiv \hat{y}_{\boldsymbol{\tau}(m+1)} - A_{\boldsymbol{\tau}(m+1)} \equiv \tilde{y}_m \pmod{P_{\boldsymbol{\tau}\cdot\mathbf{k}}}$, with $k_1, \dots, k_s \in [1, m]$. Applying (11), we have

$$H_s(n) \in B_{\mathbf{k}} \iff n \equiv \tilde{y}_m + A_{\mathbf{k}} \pmod{P_{\boldsymbol{\tau}\cdot\mathbf{k}}}.$$

By (11), we get

$$\sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}}^{\tilde{y}_m+(N_1+1)P_{\boldsymbol{\tau}\cdot\mathbf{k}}-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}) = 0, \quad \text{and} \quad \sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}+N_2-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1})$$
$$= \sum_{n\in[\tilde{y}_m,\tilde{y}_m+N_2)} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}) = \sum_{\substack{n\in[\tilde{y}_m,\tilde{y}_m+N_2) \\ n=\tilde{y}_m+A_{\mathbf{k}}}} 1 - N_2 P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1} = \mathbf{1}_{[0,N_2)}(A_{\mathbf{k}}) - N_2 P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}, \tag{13}$$

with $N_1 \geq 0$ and $N_2 \in [0, P_{\boldsymbol{\tau}\cdot\mathbf{k}})$, $N_1, N_2 \in \mathbb{Z}$. From (1) and (9), we get

$$\Delta(B(\mathbf{y}), (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}) = \sum_{y_{0,m}\leq n< y_{0,m}+N} (\mathbf{1}_{B(\mathbf{y})}(H_s(n)) - y_1 \cdots y_s)$$
$$= \sum_{1\leq k_1,\dots,k_s\leq m} \rho(\mathbf{k}, N), \quad \text{with} \quad \rho(\mathbf{k}, N) = \sum_{y_{0,m}\leq n< y_{0,m}+N} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}). \tag{14}$$

Let

$$\alpha_m := \frac{1}{P_{\boldsymbol{\tau}m}} \sum_{N=1}^{P_{\boldsymbol{\tau}m}} \Delta(B(\mathbf{y}), (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}) = \sum_{1\leq k_1,\dots,k_s\leq m} \alpha_{m,\mathbf{k}}, \text{ with } \alpha_{m,\mathbf{k}} = \frac{1}{P_{\boldsymbol{\tau}m}} \sum_{N=1}^{P_{\boldsymbol{\tau}m}} \rho(\mathbf{k}, N). \tag{15}$$

Bearing in mind (13) and (14), we derive

$$\alpha_{m,\mathbf{k}} = \frac{1}{P_{\boldsymbol{\tau}m}} \sum_{N_1=0}^{P_{\boldsymbol{\tau}m}/P_{\boldsymbol{\tau}\cdot\mathbf{k}}-1} \sum_{N_2=1}^{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \Big(\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1})$$
$$+ \sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau}\cdot\mathbf{k}}+N_2-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1}) \Big) = \frac{1}{P_{\boldsymbol{\tau}m}} \sum_{N_1=0}^{P_{\boldsymbol{\tau}m}/P_{\boldsymbol{\tau}\cdot\mathbf{k}}-1} \sum_{N_2=1}^{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \Big(\mathbf{1}_{[0,N_2)}(A_{\mathbf{k}}) - N_2 P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1} \Big)$$
$$= \frac{1}{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \sum_{N_2=1}^{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \Big(\mathbf{1}_{[0,N_2)}(A_{\mathbf{k}}) - N_2 P_{\boldsymbol{\tau}\cdot\mathbf{k}}^{-1} \Big) = \frac{P_{\boldsymbol{\tau}\cdot\mathbf{k}} - A_{\mathbf{k}}}{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} - \frac{P_{\boldsymbol{\tau}\cdot\mathbf{k}}(P_{\boldsymbol{\tau}\cdot\mathbf{k}}+1)}{2P_{\boldsymbol{\tau}\cdot\mathbf{k}}^2} = \frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} - \frac{1}{2P_{\boldsymbol{\tau}\cdot\mathbf{k}}}.$$

Using (15), we have

$$\alpha_m = \sum_{1\leq k_1,\dots,k_s\leq m} \Big(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} - \frac{1}{2P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \Big). \tag{16}$$

Taking into account that $M_{i,\boldsymbol{\tau}\cdot\mathbf{k}} \equiv (P_{\boldsymbol{\tau}\cdot\mathbf{k}} p_i^{-\tau_i k_i})^{-1} \equiv \prod_{1\leq j\leq s, j\neq i} p_j^{-\tau_j k_j} \pmod{p_i^{\tau_i k_i}}$, and that $p_j^{\tau_j} \equiv 1 \pmod{p_i}$ $(i \neq j)$, we obtain $M_{i,\boldsymbol{\tau}\cdot\mathbf{k}} \equiv 1 \pmod{p_i}$, $i = 1, \dots, s$. From (10), we get

$$[0, 1) \ni \frac{A_{\mathbf{k}}}{P_{\boldsymbol{\tau}\cdot\mathbf{k}}} \equiv -\sum_{1\leq i\leq s} M_{i,\boldsymbol{\tau}\cdot\mathbf{k}} P_{\boldsymbol{\tau}\cdot\mathbf{k}} p_i^{-1} / P_{\boldsymbol{\tau}\cdot\mathbf{k}} \equiv -\frac{1}{p_1} - \cdots - \frac{1}{p_s} \pmod{1}.$$

Applying (16), we derive

$$\alpha_m = m^s\Big(\frac{1}{2} - \{-\beta\}\Big) - \sum_{1\le k_1,\dots,k_s\le m} \frac{1}{2P_{\boldsymbol{\tau}\cdot\mathbf{k}}}, \quad \text{with} \quad \beta = \frac{1}{p_1} + \dots + \frac{1}{p_s}, \tag{17}$$

where $\{x\}$ is the fractional part of x. Let $\beta \equiv 1/2 \pmod 1$. Hence $p_0 = p_1 p_2 \cdots p_s \equiv 0 \pmod 2$. Let $p_\nu \equiv 0 \pmod 2$ for some $\nu \in [1, s]$. Then

$$b_1 := p_0(p_\nu/2 - 1)/p_\nu \equiv p_0(\beta - 1/p_\nu) = p_0 \sum_{1\le i\le s, i\ne\nu} 1/p_i (\text{mod } 1) \text{ and } b_1 \equiv b_2 (\text{mod } p_0), \text{ with } b_2 = \sum_{i\ne\nu} p_0/p_i.$$

Let $j \in [1, s]$ and $j \ne \nu$. We see that $b_1 \equiv 0 \pmod{p_j}$ and $b_2 \not\equiv 0 \pmod{p_j}$. We get a contradiction. Hence $\beta \not\equiv 1/2 \pmod 1$. We have

$$0 \ne \Big|\frac{1}{2} - \Big\{-\beta\Big\}\Big| = \Big|\frac{1}{2} - \Big\{-\Big(\frac{1}{p_1} + \dots + \frac{1}{p_s}\Big)\Big\}\Big| = \frac{|a|}{2p_0}, \quad \text{with some integer } a.$$

Thus $|1/2 - \{-\beta\}| \ge 1/(2p_0)$. Bearing in mind that $P_{\boldsymbol{\tau}\cdot\mathbf{k}} \ge 2^{k_1+k_2+\dots+k_s}$, we obtain from (17)

$$|\alpha_m| \ge \frac{m^s}{2p_0} - \frac{1}{2} = \frac{m^s}{2p_0}(1 - \frac{p_0}{m^s}) \ge \frac{m^s}{4p_0} \quad \text{for} \quad m \ge p_0 > 4. \tag{18}$$

It is easy to see that $\tau_i \le p_0$, $i = 1, \dots, s$, and $2P_{\boldsymbol{\tau}(m+1)} = 2p_1^{\tau_1(m+1)} \cdots p_s^{\tau_s(m+1)} \le 2^{1+p_0(m+1)\log_2 p_0} \le 2^{m(1+2p_0\log_2 p_0)} \le 2^{mm_0}$ with $m_0 = [2p_0 \log_2 p_0] + 2$. Using (12), we have that $\tilde{y}_m + P_{\boldsymbol{\tau}m} < 2P_{\boldsymbol{\tau}(m+1)} \le 2^{mm_0}$. By (18), (15) and (2), we get

$$m^s(4p_0)^{-1} \le |\alpha_m| \le \sup_{1\le N\le P_{\boldsymbol{\tau}m}} ND^*((H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1})$$
$$\le \sup_{1\le L, L+N\le 2P_{\boldsymbol{\tau}(m+1)}} ND^*((H_s(n))_{n=L}^{L+N-1}) \le 2 \sup_{1\le N\le 2^{m\dot{m}_0}} ND^*((H_s(n))_{n=1}^{N}) \quad \text{for} \quad m \ge p_0.$$

Hence, the Theorem is proved.

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