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Spectral analysis near the low ground energy of magnetic Pauli operators ☆

Analyse spectrale près du bas niveau d'énergie pour des opérateurs de Pauli magnétiques

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ABSTRACT

We are interested in 3-D magnetic Pauli operators perturbed by a 2×2 Hermitian matrix-valued potential $V = V(x)$, $x \in \mathbb{R}^3$. We extend to the Pauli case the Breit–Wigner-type approximation and trace formula results obtained for the 3-D Schrödinger operator near the Landau levels. Hence, we give a link between the resonances and the spectral shift function near the low ground energy of the operators.

RÉSUMÉ

On s'intéresse à des opérateurs magnétiques 3-D de Pauli perturbés par un potentiel matriciel 2×2 hermitien $V = V(x)$, $x \in \mathbb{R}^3$. Nous étendons au cas Pauli des résultats d'approximation de type Breit–Wigner et de formule trace obtenus pour l'opérateur de Schrödinger 3-D près des niveaux de Landau. Ainsi, nous établissons un lien entre les résonances et la fonction de décalage spectrale près du bas niveau d'énergie des opérateurs.

Version française abrégée

On considère des opérateurs de Pauli H_V définis comme suit. Notant $x = (x_1, x_2, x_3)$ les variables habituelles de $\mathbb{R}^3$, soit $\mathbf{B} = (0, 0, b)$ un champ magnétique de direction constante tel que $b = b(x_1, x_2)$ soit *un champ magnétique admissible*, c'est-à-dire qu'il existe une constante $b_0 > 0$ satisfaisant $b(x_1, x_2) = b_0 + \tilde{b}(x_1, x_2)$, où $\tilde{b}$ est une fonction telle que l'équation de Poisson $\Delta\tilde{\varphi} = \tilde{b}$ admette une solution $\tilde{\varphi} \in C^2(\mathbb{R}^2)$ vérifiant $\sup_{(x_1,x_2)\in\mathbb{R}^2} |D^\alpha \tilde{\varphi}(x_1, x_2)| < \infty$, $\alpha \in \mathbb{N}^2$, $|\alpha| \le 2$. Considérons

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$\mathbf{A}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ un potentiel magnétique associé (i.e. $\mathbf{B} = \mathrm{rot}\,\mathbf{A}$) tel que $\mathbf{A} = \big(A_1(x_1,x_2), A_2(x_1,x_2), 0\big)$. Pour $V = \{V_{\ell k}(x)\}_{\ell,k=1}^{2}$ une matrice 2×2 hermitienne, l'opérateur de Pauli H_V agissant sur $L^2(\mathbb{R}^3,\mathbb{C}^2)$ est défini par

$$H_V := \begin{pmatrix} (-\mathrm{i}\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-\mathrm{i}\nabla - \mathbf{A})^2 + b \end{pmatrix} + V.$$

Pour $V = 0$, il est connu que le spectre de H_0 est $[0,+\infty)$. Dans cette note, V est supposée vérifier

$$0 \not\equiv V \in C^0(\mathbb{R}^3), \quad |V_{\ell k}(x)| \lesssim \langle (x_1,x_2)\rangle^{-m_\perp} e^{-\delta\langle x_3\rangle}, \quad 1 \le \ell, k \le 2, \tag{$\mathbf{H}$}$$

où $m_\perp > 2$, $\delta > 0$ sont des constantes fixées, et $\langle y\rangle := \sqrt{1+|y|^2}$ pour $y \in \mathbb{R}^d$. Sous l'hypothèse (**H**), pour z assez petit, $z \mapsto \mathrm{e}^{-\delta\langle x_3\rangle/2}\big(H_V - z\big)^{-1}\mathrm{e}^{-\delta\langle x_3\rangle/2}$ admet un prolongement méromorphe sur une surface de Riemann localement à deux feuillets $\mathcal{M}$ de $\mathbb{C}^*\setminus[\zeta,\infty)$, où $\zeta > 0$ est une constante explicite. Les *résonances* de H_V près de 0 sont définies comme étant les pôles de cette extension.

Il est bien connu que, puisque la différence $(H_V - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1}$ est de trace classe, il existe une unique $\xi = \xi(\cdot; H_V, H_0) \in L^1\big(\mathbb{R}; (1+E^2)^{-1}\mathrm{d}E\big)$, avec la condition de normalisation $\xi(E; H_V, H_0) = 0$ pour tout $E \in \big(-\infty, \inf\sigma(H_V)\big)$. La fonction $\xi(\cdot; H_V, H_0)$ est appelée *la fonction de décalage spectrale* associée à la paire d'opérateurs (H_V, H_0).

Dans la suite, on fixe la constante $N_{\delta,\zeta} := \min\big(\frac{\delta}{2}, \sqrt{\zeta}\big)$. Soient $\mathscr{W}_\pm \Subset \Omega_\pm$ des ouverts relativement compacts de $\pm]0, N_{\delta,\zeta}^2[\mathrm{e}^{\pm\mathrm{i}]-2\theta_0, 2\varepsilon_0[}$ tels que $0 < \min(\theta_0, \varepsilon_0)$ et $\max(\theta_0,\varepsilon_0) < \frac{\pi}{2}$. Soit $r > 0$ un petit paramètre, et supposons que $\mathscr{W}_\pm$ et $\Omega_\pm$ soient simplement connexes ne dépendant pas de r. On suppose aussi que les intersections de $\pm]0, N_{\delta,\zeta}^2[$ avec $\mathscr{W}_\pm$, $\Omega_\pm$ sont des intervalles. On pose $I_\pm := \mathscr{W}_\pm \cap \pm]0, N_{\delta,\zeta}^2[$. L'ensemble des résonances de H_V est noté $\mathrm{Res}\,(H_V)$. Nos résultats sont les suivants.

Théorème 0.1 *(Approximation de Breit–Wigner). Supposons l'hypothèse* (**H**) *vérifiée. Soient $\mathscr{W}_\pm \Subset \Omega_\pm$ des ouverts relativement compacts comme ci-dessus. Fixons $0 < s_1 < \sqrt{\mathrm{dist}\big(\Omega_\pm, 0\big)}$. Il existe une valeur $r_0 > 0$ et des fonctions $g_\pm$ holomorphes dans $\Omega_\pm$ vérifiant, pour tout $E \in rI_\pm$ et $r < r_0$,*

$$\xi'(E) = \frac{1}{r\pi}\mathrm{Im}\, g_\pm'\left(\frac{E}{r}, r\right) + \sum_{\substack{w\in \mathrm{Res}(H_V)\cap r\Omega_\pm \\ \mathrm{Im}(w)\neq 0}} \frac{\mathrm{Im}(w)}{\pi|E-w|^2} - \sum_{w\in\mathrm{Res}(H_V)\cap rI_\pm} \delta(E-w), \tag{0.1}$$

où $g_\pm(z,r) = \mathcal{O}\big(|\ln r| r^{-1/m_\perp}\big)$, uniformément par rapport à $0 < r < r_0$ et $z \in \Omega_\pm$.

Théorème 0.2 *(Formule trace). Considérons des domaines $\mathscr{W}_\pm \Subset \Omega_\pm$ comme dans le Théorème 0.1. Supposons que $f_\pm$ soient holomorphes dans un voisinage de $\Omega_\pm$, et soient $\psi_\pm \in C_0^\infty\big(\Omega_\pm \cap \mathbb{R}\big)$ telles que $\psi_\pm(\lambda) = 1$ près de $\Omega_\pm \cap \mathbb{R}$. Sous les hypothèses du Théorème 0.1, on a la formule*

$$\mathrm{Tr}\left[(\psi_\pm f_\pm)\left(\frac{H_V}{r}\right) - (\psi_\pm f_\pm)\left(\frac{H_0}{r}\right)\right] = \sum_{w\in\mathrm{Res}(H_V)\cap r\mathscr{W}_\pm} f_\pm\left(\frac{w}{r}\right) + E_{f_\pm,\psi_\pm}(r), \tag{0.2}$$

avec $|E_{f_\pm,\psi_\pm}(r)| \le M(\psi_\pm)\sup\big\{|f_\pm(z)| : z \in \Omega_\pm \setminus \mathscr{W}_\pm : \mathrm{Im}(z) \le 0\big\} \times N(r)$, où $N(r) = \mathcal{O}\big(|\ln r| r^{-1/m_\perp}\big)$.

Remarque 0.1. Dans le cas « $-$ », les résonances de H_V près de 0 dans Ω_- sont des valeurs propres négatives. Puisque dans ce cas ξ est une fonction de comptage, les Théorèmes 0.1 et 0.2 sont triviaux avec g_- et E_{f_-,ψ_-} nulles.

1. Introduction and results

In this note, we consider some magnetic Pauli operators H_V defined as follows. Denoting $x = (x_1,x_2,x_3)$ the usual variables of $\mathbb{R}^3$, let $\mathbf{B} = (0,0,b)$ be a nice scalar magnetic field with constant direction such that $b = b(x_1,x_2)$ is *an admissible magnetic field*. That is, there exists a constant $b_0 > 0$ satisfying $b(x_1,x_2) = b_0 + \tilde{b}(x_1,x_2)$, where $\tilde{b}$ is a function such that the Poisson equation $\Delta\tilde{\varphi} = \tilde{b}$ admits a solution $\tilde{\varphi} \in C^2(\mathbb{R}^2)$ verifying $\sup_{(x_1,x_2)\in\mathbb{R}^2} |D^\alpha\tilde{\varphi}(x_1,x_2)| < \infty$, $\alpha \in \mathbb{N}^2$, $|\alpha| \le 2$. Consider $\mathbf{A}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ an associated magnetic potential (i.e. $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$) such that $\mathbf{A} = \big(A_1(x_1,x_2), A_2(x_1,x_2), 0\big)$. Then, for a 2×2 Hermitian matrix $V = \{V_{\ell k}(x)\}_{\ell,k=1}^{2}$, the magnetic Pauli operator H_V acting on $L^2(\mathbb{R}^3) := L^2(\mathbb{R}^3,\mathbb{C}^2)$ is defined by

$$H_V := \begin{pmatrix} (-\mathrm{i}\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-\mathrm{i}\nabla - \mathbf{A})^2 + b \end{pmatrix} + V.$$

For $V = 0$, it is known that the spectrum of H_0 is $[0,+\infty)$ (see, e.g., [8]). Throughout our exposition, we assume that V satisfies

$$0 \not\equiv V \in C^0(\mathbb{R}^3), \quad |V_{\ell k}(x)| \lesssim \langle (x_1,x_2)\rangle \mathrm{e}^{-m_\perp}\, \mathrm{e}^{-\delta\langle x_3\rangle}, \quad 1 \le \ell, k \le 2, \tag{$\mathbf{H}$}$$

for some constants $m_\perp > 2$, $\delta > 0$, and $\langle y\rangle := \sqrt{1+|y|^2}$ for $y \in \mathbb{R}^d$.

Under hypothesis (**H**), for z small enough,

$$z \mapsto e^{-\delta\langle x_3\rangle/2}\big(H_0 - z\big)^{-1} e^{-\delta\langle x_3\rangle/2} \tag{1.1}$$

has a holomorphic extension on a locally 2-sheeted covering $\mathcal{M}$ of $\mathbb{C}^* \setminus [\zeta, \infty)$ (see [9, Proposition 3.1]), for some explicit constant $\zeta > 0$. Actually, in the constant magnetic field case $b = b_0$, we have $\zeta = 2b_0$ (the first Landau level of $H_+ := (-i\nabla - \mathbf{A})^2 + b$). Then, by using the resolvent equation and the analytic Fredholm theorem, from (1.1) we deduce that, for z small enough,

$$z \mapsto e^{-\delta\langle x_3\rangle/2}\big(H_V - z\big)^{-1} e^{-\delta\langle x_3\rangle/2}$$

has a meromorphic extension on $\mathcal{M}$ (see [9, Proposition 3.2]). The resonances near 0 of H_V are defined as the poles of this meromorphic extension.

It is well known that since the resolvent difference $(H_V - i)^{-1} - (H_0 - i)^{-1}$ is of trace-class, there exists a unique $\xi = \xi(\cdot; H_V, H_0) \in L^1\big(\mathbb{R}; (1+E^2)^{-1}dE\big)$ such that the Lifshits–Krein trace formula

$$\mathrm{Tr}\big(f(H_V) - f(H_0)\big) = \int_{\mathbb{R}} \xi(E; H_V, H_0) f'(E)\, dE \tag{1.2}$$

holds for any $f \in C_0^\infty(\mathbb{R})$, and the normalization condition $\xi(E; H_V, H_0) = 0$ is fulfilled for any $E \in \big(-\infty, \inf\sigma(H_V)\big)$ (see the original works [7,6] or [12, Chapter 8]). The function $\xi(\cdot; H_V, H_0)$ is called *the spectral shift function* for the operator pair (H_V, H_0). By the *Birman–Krein formula*, for almost every $E > 0$, it coincides with *the scattering phase* for the operator pair (H_V, H_0) (see the original work [1] or the monograph [12]). Further, for almost every $E < 0$, we have $-\xi(E; H_V, H_0) = \#\{\mathrm{Eig}(H_V) \in (-\infty, E)\}$, $\mathrm{Eig}(H_V)$ denoting the set of eigenvalues of H_V.

In order to state our results, some additional notations are needed. Fix the constant

$$N_{\delta,\zeta} := \min\left(\frac{\delta}{2}, \sqrt{\zeta}\right). \tag{1.3}$$

Let $\mathscr{W}_\pm \Subset \Omega_\pm$ be open relatively compact subsets of $\pm]0, N_{\delta,\zeta}^2[e^{\pm i]-2\theta_0, 2\varepsilon_0[}$ such that $0 < \min(\theta_0, \varepsilon_0)$ and $\max(\theta_0, \varepsilon_0) < \frac{\pi}{2}$. Let $r > 0$ be a small parameter and assume that $\mathscr{W}_\pm$ and $\Omega_\pm$ are simply connected sets independent of r. We also assume that the intersections between $\pm]0, N_{\delta,\zeta}^2[$ and $\mathscr{W}_\pm$, $\Omega_\pm$ are intervals. Hence, we set $I_\pm := \mathscr{W}_\pm \cap \pm]0, N_{\delta,\zeta}^2[$; $\mathrm{Res}\,(H_V)$ denotes the resonances set of H_V.

Theorem 1.1 *(Breit–Wigner approximation). Assume that assumption* (**H**) *holds. Let $\mathscr{W}_\pm \Subset \Omega_\pm$ be open relatively compact subsets of $\pm]0, N_{\delta,\zeta}^2[e^{\pm i]-2\theta_0, 2\varepsilon_0[}$ as above. Choose moreover $0 < s_1 < \sqrt{\mathrm{dist}(\Omega_\pm, 0)}$. There exists $r_0 > 0$ and holomorphic functions $g_\pm$ in $\Omega_\pm$ satisfying for any $E \in rI_\pm$ and $r < r_0$,*

$$\xi'(E) = \frac{1}{r\pi}\mathrm{Im}\, g'_\pm\left(\frac{E}{r}, r\right) + \sum_{\substack{w \in \mathrm{Res}(H_V)\cap r\Omega_\pm \\ \mathrm{Im}(w)\neq 0}} \frac{\mathrm{Im}(w)}{\pi|E - w|^2} - \sum_{w \in \mathrm{Res}(H_V)\cap rI_\pm} \delta(E - w), \tag{1.4}$$

where $g_\pm(z, r) = \mathcal{O}\left(|\ln r| r^{-1/m_\perp}\right)$, uniformly with respect to $0 < r < r_0$ and $z \in \Omega_\pm$.

As consequence of Theorem 1.1, we have the following

Theorem 1.2 *(Trace formula). Let the domains $\mathscr{W}_\pm \Subset \Omega_\pm$ be as in Theorem 1.1. Assume that $f_\pm$ are holomorphic in a neighbourhood of $\Omega_\pm$, and let $\psi_\pm \in C_0^\infty\big(\Omega_\pm \cap \mathbb{R}\big)$ satisfy $\psi_\pm(\lambda) = 1$ near $\Omega_\pm \cap \mathbb{R}$. Under the assumptions of Theorem 1.1, we have*

$$\mathrm{Tr}\left[(\psi_\pm f_\pm)\left(\frac{H_V}{r}\right) - (\psi_\pm f_\pm)\left(\frac{H_0}{r}\right)\right] = \sum_{w \in \mathrm{Res}(H_V)\cap r\mathscr{W}_\pm} f_\pm\left(\frac{w}{r}\right) + E_{f_\pm,\psi_\pm}(r), \tag{1.5}$$

with $|E_{f_\pm,\psi_\pm}(r)| \le M(\psi_\pm)\sup\big\{|f_\pm(z)| : z \in \Omega_\pm \setminus \mathscr{W}_\pm : \mathrm{Im}(z) \le 0\big\} \times N(r)$, where $N(r) = \mathcal{O}\left(|\ln r| r^{-1/m_\perp}\right)$.

Remark 1.1. Notice that in the case "−", the resonances of H_V near zero in Ω_- are negative eigenvalues. Since in this case ξ is a counting function, Theorems 1.1 and 1.2 are trivial with g_- and E_{f_-,ψ_-} equal to zero.

2. Strategy of proofs

2.1. An auxiliary result

Let $\mathscr{H}$ be a separable Hilbert space. We denote $\mathbf{S}_\infty(\mathscr{H})$ (resp. $\mathbf{S}_1(\mathscr{H})$, resp. $\mathbf{S}_2(\mathscr{H})$) the set of compact (resp. trace-class, resp. Hilbert–Schmidt) operators acting in $\mathscr{H}$. For $T \in \mathbf{S}_2(\mathscr{H})$, the regularized determinant $\det_2(I-T)$ is defined by $\det_2(I-T) := \det\big((I-T)\mathrm{e}^T\big)$.

Let $p := p(b)$ be the spectral projection in $L^2(\mathbb{R}^2)$ onto the (infinite dimensional) kernel of $H_1 := (-\mathrm{i}\partial_{x_1} - A_1)^2 + (-\mathrm{i}\partial_{x_2} - A_2)^2 - b$. Set $P := p \otimes 1$, $Q := I - P$ and define in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ the orthogonal projection $\mathrm{Q} := \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$. Introduce $e_\pm$ the multiplication operators on $L^2(\mathbb{R}^3)$ by the functions $\mathrm{e}^{\pm\frac{\delta}{2}\langle\cdot\rangle}$. Let $c : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ be given by $c(u) := \langle u, \mathrm{e}^{-\frac{\delta}{2}\langle\cdot\rangle}\rangle$. Define the operator $K : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^2)$ by

$$K := \frac{1}{\sqrt{2}}(p \otimes c)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_+ |V|^{\frac{1}{2}}. \tag{2.1}$$

We denote $s(k)$ the operator acting from $\mathrm{e}^{-\frac{\delta}{2}\langle t\rangle} L^2(\mathbb{R})$ to $\mathrm{e}^{\frac{\delta}{2}\langle t\rangle} L^2(\mathbb{R})$ with the integral kernel $\frac{1-\mathrm{e}^{ik|x_3-x_3'|}}{2ik}$.

Near $z = 0$, $\mathcal{M}$ can be parametrized by $z(k) = k^2$, $k \in \mathbb{C}^*$, $|k| \ll 1$ (for more details, see [9, Section 2]). In the sequel, we set $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{C}^+_{1/2} := \{k \in \mathbb{C} : k^2 \in \mathbb{C}^+\}$. With respect to the variable k, we define the punctured disk $D(0,\epsilon)^* := \{k \in \mathbb{C} : 0 < |k| < \epsilon\}$, $\epsilon < N_{\delta,\zeta}$, where $N_{\delta,\zeta}$ is the constant defined by (1.3).

As in [9, Propositions 4.2–4.3] and [2, Propositions 3–4], we have the following.

Proposition 2.1. *Assume that V satisfies assumption (**H**). Then, for k small enough,*

(**i**) *The operator-valued function $\mathbb{C}^+_{1/2} \cap D(0,\epsilon)^* \ni k \longmapsto \mathcal{T}_V\big(z(k)\big) := J|V|^{1/2}\big(H_0 - z(k)\big)^{-1}|V|^{1/2}$, where $J := \mathrm{sign}(V)$, has a holomorphic extension to $D(0,\epsilon)^*$ with values in $\mathbf{S}_2\left(L^2(\mathbb{R}^3)\right)$. This extension is denoted $\mathcal{T}_V\big(z(k)\big)$ again. Further, $\partial_z \mathcal{T}_V\big(z(k)\big) \in \mathbf{S}_1\left(L^2(\mathbb{R}^3)\right)$ is holomorphic.*

(**ii**) *The following assertions are equivalent:*

a) *$z = z(k)$ is a resonance of H_V near zero,*

b) $\det_2\big(I + \mathcal{T}_V(z)\big) = 0$.

(**iii**) *The following decomposition holds:*

$$\mathcal{T}_V\big(z(k)\big) = \frac{\mathrm{i}J}{k}\mathscr{B} + \mathscr{A}(k), \quad \mathscr{B} := K^*K, \tag{2.2}$$

where the operator $\mathscr{A}(k) \in \mathbf{S}_2\big(L^2(\mathbb{R}^3)\big)$ is given by

$$\mathscr{A}(k) := J|V|^{\frac{1}{2}}\,\mathrm{e}_+\, p \otimes s(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_+|V|^{\frac{1}{2}} + J|V|^{\frac{1}{2}}\big(H_0 - z(k)\big)^{-1}\mathrm{Q}|V|^{\frac{1}{2}}, \tag{2.3}$$

and is holomorphic on the open disk $D(0,\epsilon) := \{k \in \mathbb{C} : 0 \le |k| < \epsilon\}$.

2.2. Sketch of proof of Theorem 1.1

As in [2], the proof is based on complex analysis results due to Sjöstrand. The difference with [2] is that we take into account the privileged role near 0 of the half of the Pauli operator H_0, namely the operator $H_- := (-\mathrm{i}\nabla - \mathbf{A})^2 - b$. Unlike the operator $H_+ := (-\mathrm{i}\nabla - \mathbf{A})^2 + b$ whose spectrum belongs to $[\zeta, +\infty)$, its spectrum coincides with $[0, +\infty)$. We introduce $\mathbf{W}$ the multiplication operator on $L^2(\mathbb{R}^2)$ by the function

$$\mathbf{W}(x_\perp) := \int_{\mathbb{R}} |V|_{11}(x_\perp, x_3)\,\mathrm{d}x_3, \quad x_\perp := (x_1, x_2) \in \mathbb{R}^2,$$

where $|V|_{\ell k}$, $1 \le \ell, k \le 2$, are the coefficients of the matrix $|V|$. Notice that we have $KK^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{p\mathbf{W}p}{2}$ (see [9, Subsection 4.2] for more details), where K is the operator defined by (2.1). Under the hypothesis (**H**), [8, Lemma 2.3] implies that the positive self-adjoint Toeplitz operator $p\mathbf{W}p$ is of trace-class. For our purpose, it is more convenient to introduce the 2-regularized spectral shift function

$$\xi_2(\lambda) = \frac{1}{\pi}\lim_{\varepsilon \longrightarrow 0^+} \mathrm{Arg}\,\det_2\big(I + V(H_0 - \lambda - \mathrm{i}\varepsilon)^{-1}\big), \tag{2.4}$$

whose derivative is given by the distribution

$$\xi_2' : f \longmapsto -\mathrm{Tr}\left(f(H_V) - f(H_0) - \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(H_0 + \varepsilon V)_{|\varepsilon=0}\right), \quad f \in C_0^\infty(\mathbb{R}),$$

(see, e.g., [5,3]). With the help of the Helffer–Sjöstrand formula (see, e.g., [4]) and the Green formula, it can be proved, as in [2, Lemma 8], that

$$\xi' = \xi_2' + \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left(\partial_z \mathcal{T}_V(\cdot)\right) \tag{2.5}$$

on $]-N^2_{\delta,\zeta}, N^2_{\delta,\zeta}[\setminus\{0\}$. By **(iii)** of Proposition 2.1, $k \longmapsto \mathscr{A}(k)$ is holomorphic near zero. Then, for $0 < s < |k| \le s_0$ small enough, there exists $\mathscr{A}_0$, a finite-rank operator independent of k, and $\tilde{\mathscr{A}}(k)$ an analytic operator near zero with $\|\tilde{\mathscr{A}}(k)\| < \frac{1}{4}$, such that $\mathscr{A}(k) = \mathscr{A}_0 + \tilde{\mathscr{A}}(k)$.

From **(ii)** of Proposition 2.1, to analyse the resonances of H_V near 0, we are reduced to the investigation of the zeros of $\det_2\left(I + \mathcal{T}_V(\cdot)\right)$. Fix $0 < s_1 < \sqrt{\mathrm{dist}(\Omega_\pm, 0)}$. Then, using Sjöstrand's representation theorems (see [10,11]) on zeros of holomorphic functions, similarly to [2, Proof of Proposition 8], we have the existence of holomorphic functions $g_{0,\pm}$, g_1 in $\Omega_\pm$ such that

$$\det_2\left(I + \mathcal{T}_V(z)\right) = \prod_{w \in \mathrm{Res}(H_V)\cap r\Omega_\pm} \left(\frac{zr - \omega}{r}\right) \mathrm{e}^{g_{0,\pm}(z,r) + g_1(z,r)} \mathrm{e}^{-\mathrm{Tr}\left(\mathcal{T}_V(z) - \mathfrak{A}(k)\right)}, \; z = z(\sqrt{r}k), \tag{2.6}$$

where $\mathfrak{A}(k) := \frac{\mathrm{i}J}{\sqrt{r}k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B}) + \tilde{\mathscr{A}}(\sqrt{r}k)$, $\frac{\mathrm{d}}{\mathrm{d}z} g_{0,\pm}(z,r) = \mathcal{O}\left(\mathrm{Tr}\,\mathbf{1}_{(s_1\sqrt{r},\infty)}(p\mathbf{W}p)|\ln r|\right)$ and $\frac{\mathrm{d}}{\mathrm{d}z} g_1(z,r) = \mathcal{O}\left(\tilde{n}_2\left(\frac{1}{2}\sqrt{r}s_1\right)\right)$ uniformly with respect to $z \in \mathscr{W}_\pm$. Here, we set $\tilde{n}_p(r) := \left\|\frac{\mathscr{B}}{r}\mathbf{1}_{[0,r]}(\mathscr{B})\right\|^p_{\mathbf{S}_p}$, $p = 1, 2$. Therefore, we deduce from (2.4) and (2.6) that for $E = z(\sqrt{r}k) = rk^2 \in r(\Omega_\pm \cap \mathbb{R})$,

$$\begin{aligned}\xi_2'(E) = {} & \frac{1}{\pi r}\mathrm{Im}\,\partial_\lambda(g_{0,\pm} + g_1)\left(\frac{E}{r}, r\right) + \sum_{\substack{w \in \mathrm{Res}(H_V)\cap r\Omega_\pm \\ \mathrm{Im}(w)\neq 0}} \frac{\mathrm{Im}(w)}{\pi|E - w|^2} - \sum_{w \in \mathrm{Res}(H_V)\cap rI_\pm} \delta(E - w) \\ & + \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left(\frac{1}{2k}\partial_k\left(\frac{\mathrm{i}J}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\right)\right) - \frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}\,\partial_z \mathcal{T}_V(E + i0),\end{aligned}$$

with $k = \sqrt{\mu}$ if $\mu > 0$, and $k = \mathrm{i}\sqrt{-\mu}$ if $\mu < 0$. We have $\mathrm{Tr}\left(\frac{1}{2k}\partial_k\left(\frac{\mathrm{i}J}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B})\right)\right) = -\frac{\mathrm{i}Js_1\sqrt{r}}{4k^3}\tilde{n}_1\left(\frac{1}{2}\sqrt{r}s_1\right)$. According to **(i)** of Proposition 2.1, $\partial_z\mathcal{T}_V(z)$ is of trace class. Then, since $\mathscr{B} \in \mathbf{S}_1\left(L^2(\mathbb{R}^3)\right)$, the operator $\partial_k\tilde{\mathscr{A}}(k) = \partial_k\mathscr{A}(k) = \partial_k\Big(\mathcal{T}_V\left(z(k)\right) - \frac{\mathrm{i}J}{k}\mathscr{B}\Big)$ is of trace class. Moreover, the definition (2.3) of $\mathscr{A}(k)$ implies that $\mathrm{Tr}\left(\frac{1}{2k}\partial_k\mathscr{A}(k)\right) = \mathrm{Tr}\left(J|V|^{\frac{1}{2}}(H_0 - \mu)^{-2}\mathrm{Q}|V|^{\frac{1}{2}}\right)$. Thus, by setting $g_\pm = g_{0,\pm} + g_1 + g_2$ with $g_2(z) = -\frac{\mathrm{i}Js_1}{2\sqrt{z}}\tilde{n}_1\left(\frac{1}{2}\sqrt{r}s_1\right)$, Theorem 1.1 follows thanks to (2.5). The estimation of the function $g_\pm$ is an immediate consequence of the analogue of [2, Corollary 1], where for $q \in \mathbb{N}$, the projection p_q associated with a Landau level $2bq$ is replaced by $p = p(b)$, and B_q by $\mathscr{B}$.

2.3. *Sketch of Proof of Theorem 1.2*

The proof goes in the same lines as that of [2, Corollary 3].

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