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Partial differential equations

Leray's problem for the Navier–Stokes equations revisited

Le problème de Leray pour les équations de Navier–Stokes réexaminé

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A B S T R A C T

In a seminal work in 1934, J. Leray constructed solutions $\boldsymbol{u}(\cdot,t) \in L^\infty([0,\infty), L^2_\sigma(\mathbb{R}^3)) \cap C^0_w([0,\infty), L^2(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3))$ of the Navier–Stokes equations for arbitrary initial data $\boldsymbol{u}(\cdot,0) \in L^2_\sigma(\mathbb{R}^3)$ and left it open whether $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ would necessarily tend to zero as $t \to \infty$. This question was answered positively fifty years later by T. Kato, using a different approach. Here, we reexamine Leray's problem and solve this and other important related questions using Leray's original ideas and some standard tools (Fourier transform, Duhamel's principle, heat kernel estimates) already in use in his time.

R É S U M É

En 1934, J. Leray a construit des solutions faibles $\boldsymbol{u}(\cdot,t) \in L^\infty([0,\infty), L^2_\sigma(\mathbb{R}^3)) \cap C^0_w([0,\infty), L^2(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3))$ pour les équations de Navier–Stokes avec des données initiales $\boldsymbol{u}(\cdot,0) \in L^2_\sigma(\mathbb{R}^3)$ arbitraires, où il a laissé non résolue la question de savoir si $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ tendrait toujours vers zéro quand $t \to \infty$, à laquelle a été répondu positivement en 1984 par T. Kato, au moyen d'une autre approche. Ici, on reconsidère le problème de Leray et quelques-unes de ses extensions, qui sont résolus en n'employant que des idées développées par Leray en 1934 et des techniques classiques, très utilisées déjà à cette époque.

1. Introduction

In this work, we derive some old and new results concerning the important solutions introduced by Leray in [6] for the incompressible Navier–Stokes equations in three-dimensional space,

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \Delta\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.1a}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^3), \tag{1.1b}$$

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where $L^2_\sigma(\mathbb{R}^3)$ denotes the space of functions $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_1, \mathrm{u}_3) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \mathbf{u} = 0$ in distributional sense. As shown in [6], Leray's solutions are globally defined and weakly continuous in $L^2(\mathbb{R}^3)$, satisfying (1.1a) as distributions and (1.1b) in the sense that $\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)} \to 0$ as $t \searrow 0$. Moreover, $\boldsymbol{u}(\cdot,t) \in L^\infty([0,\infty), L^2_\sigma(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3))$, with $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ monotonically decreasing in $t \in [0,\infty) \setminus E$ (for some bounded set $E \subset (0,\infty)$ of zero Lebesgue measure) and such that the energy inequality

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|D\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \le \|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)} \tag{1.2}$$

holds for all $t \ge 0$. (Other properties are also obtained in [6]. See Section 2 for a quick review of Leray's construction.) Furthermore, Leray showed that, for some $t_* \gg 1$, one actually has $\boldsymbol{u} \in C^\infty(\mathbb{R}^3 \times [t_*,\infty))$, and, for each $m \ge 1$,

$$\boldsymbol{u}(\cdot,t) \in C([t_*,\infty), H^m(\mathbb{R}^3)) \tag{1.3}$$

(see [6], p. 246),[1] where, as usual, $H^m(\mathbb{R}^3)$ denotes the Sobolev space of functions (in this case, with values in $\mathbb{R}^3$) belonging to $L^2(\mathbb{R}^3)$ and such that all their distributional derivatives of order m are also in $L^2(\mathbb{R}^3)$. Regarding the asymptotic ($t \to \infty$) limit of the eventually decreasing functional $W(t) := \|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}$, Leray observed that (cf. [6], p. 248):

J'ignore si $W(t)$ tend nécessairement vers 0 quand t augmente indéfiniment.

[J. Leray, 1934]

While the *uniqueness* of the solutions [6] remains a fundamental open question to this day, it has been shown by Kato [4] and Masuda [8] in 1984 (and later by other authors as well, see, e.g., [3,11]) that *all* Leray's solutions, whether uniquely defined by their initial data or not, do possess this property, that is, one always has

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{1.4}$$

(For more decay results and a broader discussion that goes beyond Leray's solutions [6], see, e.g., [2–5,8,9,11] and references therein.) It appears that Leray was not very worried about this question regarding his solutions, for — as it will become clear in Sections 2 and 3 — he had already laid out all that was actually needed to show (1.4) affirmatively. In addition, it would have not been hard after Leray's work [6] to obtain, for his solutions,

$$\lim_{t\to\infty} t^{3/4}\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = 0 \tag{1.5}$$

and more involving properties as (1.7), (1.8), (1.9b) below. Note that (1.4), (1.5) are easy to get (see e.g. [1], Theorem 3.3) for solutions $\boldsymbol{v}(\cdot,t) \in L^\infty([t_0,\infty), L^2(\mathbb{R}^3))$ of the associated linear *heat flow* problems

$$\boldsymbol{v}_t = \Delta \boldsymbol{v}, \qquad t > t_0, \tag{1.6a}$$

$$\boldsymbol{v}(\cdot,t_0) = \boldsymbol{u}(\cdot,t_0), \tag{1.6b}$$

given $t_0 \ge 0$ (arbitrary). The solution to (1.6) is, using modern notations: $\boldsymbol{v}(\cdot,t) = \mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)$, where $\mathrm{e}^{\Delta\tau}$, $\tau \ge 0$, is the heat semigroup. This suggests a close relationship between the solutions to (1.1) and (1.6), and one has indeed

$$\lim_{t\to\infty} t^{1/4}\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0, \tag{1.7}$$

as shown by Wiegner (see [11], Theorem (*c*), p. 305) in arbitrary space dimension and for a broader solution class using a very involved argument based on Fourier splitting techniques [9]. In low dimension ($n = 2, 3$), estimates like (1.7) can be derived in a *much* simpler way in the case of Leray's solutions [6] using Leray's original ideas, which can be similarly adapted to give us the important supnorm result

$$\lim_{t\to\infty} t\,\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = 0. \tag{1.8}$$

In this note, we show how to use Leray's approach [6] to get such easy derivations of (1.4), (1.5), (1.7), (1.8) for his solutions. Although we restrict our attention to dimension $n = 3$, it will be clear that our analysis can also be used in the case $n = 2$. The results obtained for $n = 2, 3$ can be summarized as follows. One has, for each $2 \le q \le \infty$ (and $n = 2, 3$):

$$\lim_{t\to\infty} t^{\frac{n}{4} - \frac{n}{2q}}\|\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0, \tag{1.9a}$$

$$\lim_{t\to\infty} t^{\frac{n-1}{2} - \frac{n}{2q}}\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0, \tag{1.9b}$$

uniformly for $2 \le q \le \infty$, where $\boldsymbol{v}(\cdot,t) = \mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)$, $t_0 \ge 0$ arbitrary, see (1.6), assuming only that $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n)$.

[1] See also [7,10] and Theorem 2.3 in Section 2 below. From [5], p. 235, it follows that one always has $t_* < 0.212 \cdot \|\boldsymbol{u}_0\|^4_{L^2(\mathbb{R}^3)}$.

Remark. The algebraic rates in (1.5), (1.7), (1.8), (1.9) are consistent with previous findings, e.g., [2,5,9,11]. For example, if $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n)$ is so that $\|e^{\Delta t}\boldsymbol{u}_0\|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$, $0 < \alpha < 1/2$, Wiegner obtained $t^{d(\alpha)}\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = O(1)$, $d(\alpha) = n/4 + 2\alpha - 1/2$ ([11], p. 305). Whether $d(\alpha)$ or the rate values in (1.7), (1.8), (1.9b) are *optimal* remains unresolved.

Notation. As indicated above, boldface letters will be used for vector quantities, as in $\boldsymbol{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$. Also, $D_j = \partial/\partial x_j$, $|\cdot|_2$ is the Euclidean norm, and

$$\|\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^3)} = \Big\{ \sum_{i=1}^{3} \int_{\mathbb{R}^3} |u_i(x,t)|^q \, dx \Big\}^{1/q}, \tag{1.10a}$$

$$\|D\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^3)} = \Big\{ \sum_{i,\,j=1}^{3} \int_{\mathbb{R}^3} |D_j u_i(x,t)|^q \, dx \Big\}^{1/q}, \tag{1.10b}$$

$$\|D^2\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^3)} = \Big\{ \sum_{i,\,j,\,\ell=1}^{3} \int_{\mathbb{R}^3} |D_j D_\ell\, u_i(x,t)|^q \, dx \Big\}^{1/q} \tag{1.10c}$$

if $1 \le q < \infty$, and $\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = \max\big\{\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} : 1 \le i \le 3\big\}$ if $q = \infty$. It will be also convenient to define

$$\|\boldsymbol{u}(\cdot,t)\|_\infty = \text{ess sup}\,\big\{ |\boldsymbol{u}(x,t)|_2 : x \in \mathbb{R}^3 \big\}. \tag{1.10d}$$

2. Mathematical preliminaries

In this section, we recall Leray's construction [6] and some basic properties of the resulting solutions, which we refer to as *Leray's solutions*. An important new result (Theorem 2.2 below) is also established here, taking advantage of Leray's ideas. For the construction of his solutions, Leray used an ingenious regularization procedure, which, for convenience, we briefly review: taking (any) $G \in C_0^\infty(\mathbb{R}^n)$ nonnegative with $\int_{\mathbb{R}^3} G(x)\,dx = 1$ and setting $\bar{\boldsymbol{u}}_{0,\delta}(\cdot) \in C^\infty(\mathbb{R}^3)$ by convolving $\boldsymbol{u}_0(\cdot)$ with $G_\delta(x) = \delta^{-n}G(x/\delta)$, $\delta > 0$, one defines $\boldsymbol{u}_\delta$, $p_\delta \in C^\infty(\mathbb{R}^3 \times [0,\infty[)$ as the (unique, globally defined) classical L^2 solutions to the regularized equations

$$\frac{\partial}{\partial t}\boldsymbol{u}_\delta + \bar{\boldsymbol{u}}_\delta(\cdot,t)\cdot\nabla\boldsymbol{u}_\delta + \nabla p_\delta \;=\; \Delta\boldsymbol{u}_\delta, \qquad \nabla\cdot\boldsymbol{u}_\delta(\cdot,t) = 0, \tag{2.1a}$$

$$\boldsymbol{u}_\delta(\cdot,0) \;=\; \bar{\boldsymbol{u}}_{0,\,\delta} := G_\delta * \boldsymbol{u}_0 \in \bigcap_{m=1}^{\infty} H^m(\mathbb{R}^3), \tag{2.1b}$$

where $\bar{\boldsymbol{u}}_\delta(\cdot,t) := G_\delta * \boldsymbol{u}_\delta(\cdot,t)$. It was then shown by Leray that, for some sequence $\delta' \to 0$, one has the weak convergence

$$\boldsymbol{u}_{\delta'}(\cdot,t) \rightharpoonup \boldsymbol{u}(\cdot,t) \quad \text{as } \delta' \to 0, \qquad \forall\, t \ge 0, \tag{2.2}$$

that is, $\boldsymbol{u}_{\delta'}(\cdot,t) \to \boldsymbol{u}(\cdot,t)$ weakly in $L^2(\mathbb{R}^3)$, for every $t \ge 0$ (see [6], p. 237). This gives $\boldsymbol{u}(\cdot,t) \in L^\infty([0,\infty), L^2_\sigma(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3)) \cap C^0_w([0,\infty), L^2(\mathbb{R}^3))$, with $\boldsymbol{u}(\cdot,t)$ continuous in L^2 at $t = 0$ and solving the Navier–Stokes equations (1.1) in the distributional sense. Moreover, the energy inequality (1.2) is satisfied for all $t \ge 0$, so that, in particular,

$$\int_0^\infty \|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\,dt \;\le\; \frac{1}{2}\,\|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)}. \tag{2.3}$$

A similar estimate for the regularized solutions $\boldsymbol{u}_\delta(\cdot,t)$ is also valid, since we have, from (2.1) above, that

$$\|\boldsymbol{u}_\delta(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|D\boldsymbol{u}_\delta(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\,ds \le \|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)} \tag{2.4}$$

for all $t > 0$ (and $\delta > 0$ arbitrary). Another important property shown in [6] is that $\boldsymbol{u} \in C^\infty([t_*,\infty[)$ for some $t_* \gg 1$, with $D^m\boldsymbol{u}(\cdot,t) \in L^\infty_{\text{loc}}([t_*,\infty), L^2(\mathbb{R}^3))$ for each $m \ge 1$, cf. (1.3). This fact, together with Theorems 2.2 and 2.3, will greatly simplify our main analysis in Section 3. Other results needed there have to do with the Helmholtz projection of $-\boldsymbol{u}(\cdot,t)\cdot\nabla\boldsymbol{u}(\cdot,t)$ into $L^2_\sigma(\mathbb{R}^3)$, that is, the divergence-free field $\boldsymbol{Q}(\cdot,t) \in L^2_\sigma(\mathbb{R}^3)$ given by

$$\boldsymbol{Q}(\cdot,t) := -\boldsymbol{u}(\cdot,t)\cdot\nabla\boldsymbol{u}(\cdot,t) - \nabla p(\cdot,t), \qquad \text{a.e. } t > 0. \tag{2.5}$$

Of similar interest is the quantity $\boldsymbol{Q}_\delta(\cdot,t) := -\bar{\boldsymbol{u}}_\delta(\cdot,t)\cdot\nabla\boldsymbol{u}_\delta(\cdot,t) - \nabla p_\delta(\cdot,t)$, which will be important in Theorem 2.2 below.

Theorem 2.1. *For almost every $s > 0$ (and every $s \geq t_*$, with t_* given in (1.3) above), one has*

$$\| e^{\Delta(t-s)} \boldsymbol{Q}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \leq K (t-s)^{-3/4} \| \boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \tag{2.6a}$$

and

$$\| e^{\Delta(t-s)} \boldsymbol{Q}(\cdot, s) \|_{\infty} \leq K (t-s)^{-3/4} \| \boldsymbol{u}(\cdot, s) \|_{\infty} \| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \tag{2.6b}$$

for all $t > s$, where $K = (8\pi)^{-3/4}$.

Proof. This is shown in [5], p. 236, using Fourier transforms. The following more direct argument was suggested by the reviewer, as follows. Let $\mathbb{P}$ be the Helmholtz projection. Since, by definition, $\mathbb{P}$ is an orthogonal projection in the Hilbert space of L^2-vector fields, we have $\|\mathbb{P}\boldsymbol{f}\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ for any L^2-vector field $\boldsymbol{f}$. Hence we also have $\|e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_{L^2} = \|e^{\Delta(t-s)}\mathbb{P}[-\boldsymbol{u}(\cdot,s)\cdot\nabla\boldsymbol{u}(\cdot,s)]\|_{L^2} = \|\mathbb{P}[e^{\Delta(t-s)}(-\boldsymbol{u}(\cdot,s)\cdot\nabla\boldsymbol{u}(\cdot,s))]\|_{L^2} \leq \|e^{\Delta(t-s)}(-\boldsymbol{u}(\cdot,s)\cdot\nabla\boldsymbol{u}(\cdot,s))\|_{L^2} \leq \|\Gamma(t-s)\|_{L^2}\|\boldsymbol{u}(\cdot,s)\cdot\nabla\boldsymbol{u}(\cdot,s)\|_{L^1}$, where Γ denotes the heat kernel, so that $\|e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_{L^2} \leq \|\Gamma(t-s)\|_{L^2}\|\boldsymbol{u}(\cdot,s)\|_{L^2}\|\nabla\boldsymbol{u}(\cdot,s)\|_{L^2}$. This is (2.6a). Similarly, $\|e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_{L^\infty} \leq \|\Gamma(t-s)\|_{L^2}\|\boldsymbol{Q}(\cdot,s)\|_{L^2} \leq \|\Gamma(t-s)\|_{L^2}\|\boldsymbol{u}(\cdot,s)\cdot\nabla\boldsymbol{u}(\cdot,s)\|_{L^2} \leq \|\Gamma(t-s)\|_{L^2}\|\boldsymbol{u}(\cdot,s)\|_{L^\infty} \times \|\nabla\boldsymbol{u}(\cdot,s)\|_{L^2}$, which is (2.6b). □

In a completely similar way, considering the solutions to the regularized Navier–Stokes equations (2.1), one obtains

$$\| e^{\Delta(t-s)} \boldsymbol{Q}_{\delta}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \leq K (t-s)^{-3/4} \| \boldsymbol{u}_{\delta}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}_{\delta}(\cdot, s) \|_{L^2(\mathbb{R}^3)} \tag{2.8}$$

for all $t > s > 0$ (and similarly for $\|e^{\Delta(t-s)}\boldsymbol{Q}_{\delta}(\cdot,s)\|_{\infty}$), where $K = (8\pi)^{-3/4}$, $\boldsymbol{Q}_{\delta}(\cdot,s) = -\bar{\boldsymbol{u}}_{\delta}(\cdot,s)\cdot\nabla\boldsymbol{u}_{\delta}(\cdot,s) - \nabla p_{\delta}(\cdot,s)$.

Theorem 2.2. *Let $\boldsymbol{u}(\cdot,t)$, $t > 0$, be any particular Leray's solution to (1.1). Given any pair of initial values $\tilde{t}_0 > t_0 \geq 0$, one has*

$$\| \boldsymbol{v}(\cdot,t) - \tilde{\boldsymbol{v}}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \leq \frac{K}{\sqrt{2}} \| \boldsymbol{u}_0 \|^2_{L^2(\mathbb{R}^3)} (\tilde{t}_0 - t_0)^{1/2} (t - \tilde{t}_0)^{-3/4} \tag{2.9}$$

and

$$\| \boldsymbol{v}(\cdot,t) - \tilde{\boldsymbol{v}}(\cdot,t) \|_{L^\infty(\mathbb{R}^3)} \leq \frac{\Gamma}{\sqrt{2}} \| \boldsymbol{u}_0 \|^2_{L^2(\mathbb{R}^3)} (\tilde{t}_0 - t_0)^{1/2} (t - \tilde{t}_0)^{-3/2} \tag{2.10}$$

for all $t > \tilde{t}_0$, where $\boldsymbol{v}(\cdot,t) = e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)$, $\tilde{\boldsymbol{v}}(\cdot,t) = e^{\Delta(t-\tilde{t}_0)}\boldsymbol{u}(\cdot,\tilde{t}_0)$ are the corresponding heat flows associated with t_0, $\tilde{t}_0$, respectively, and $K = (8\pi)^{-3/4}$, $\Gamma = (4\pi)^{-3/2}$.

Proof. The following argument combines Leray's construction [6] with the widely used strategy of handling nonlinear terms as a Duhamel-type correction. We start by writing $\boldsymbol{v}(\cdot,t) = e^{\Delta(t-t_0)}[\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_{\delta}(\cdot,t_0)] + e^{\Delta(t-t_0)}\boldsymbol{u}_{\delta}(\cdot,t_0)$, $t > t_0$, with $\boldsymbol{u}_{\delta}(\cdot,t)$ given in (2.1), $\delta > 0$. Because

$$\boldsymbol{u}_{\delta}(\cdot,t_0) = e^{\Delta t_0}\bar{\boldsymbol{u}}_{0,\delta} + \int_0^{t_0} e^{\Delta(t_0-s)}\boldsymbol{Q}_{\delta}(\cdot,s)\,ds,$$

where $\bar{\boldsymbol{u}}_{0,\delta} = G_{\delta} * \boldsymbol{u}_0$ and $\boldsymbol{Q}_{\delta}(\cdot,s) = -\bar{\boldsymbol{u}}_{\delta}(\cdot,s)\cdot\nabla\boldsymbol{u}_{\delta}(\cdot,s) - \nabla p_{\delta}(\cdot,s)$, we get

$$\boldsymbol{v}(\cdot,t) = e^{\Delta(t-t_0)}[\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_{\delta}(\cdot,t_0)] + e^{\Delta t}\bar{\boldsymbol{u}}_{0,\delta} + \int_0^{t_0} e^{\Delta(t-s)}\boldsymbol{Q}_{\delta}(\cdot,s)\,ds,$$

for $t > t_0$. A similar expression holds for $\tilde{\boldsymbol{v}}(\cdot,t)$ as well, giving

$$\tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) = e^{\Delta(t-\tilde{t}_0)}[\boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_{\delta}(\cdot,\tilde{t}_0)] - e^{\Delta(t-t_0)}[\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_{\delta}(\cdot,t_0)] + \int_{t_0}^{\tilde{t}_0} e^{\Delta(t-s)}\boldsymbol{Q}_{\delta}(\cdot,s)\,ds.$$

Therefore, given any $\mathbb{K} \subset \mathbb{R}^3$ compact, we get, for each $t > \tilde{t}_0$, $\delta > 0$:

$$\begin{aligned}
\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{K})} &\leq J_{\delta}(t) + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)}\boldsymbol{Q}_{\delta}(\cdot,s) \|_{L^2(\mathbb{K})}\,ds \\
&\leq J_{\delta}(t) + K \int_{t_0}^{\tilde{t}_0} (t-s)^{-3/4} \| \boldsymbol{u}_{\delta}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}_{\delta}(\cdot,s) \|_{L^2(\mathbb{R}^3)}\,ds
\end{aligned}$$

$$\leq J_\delta(t) \,+\, \frac{K}{\sqrt{2}}\,(\tilde{t}_0 - t_0)^{\frac{1}{2}}\,\|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)}\,(t-\tilde{t}_0)^{-\frac{3}{4}}$$

by (2.4) and (2.8), where $K = (8\pi)^{-3/4}$ and $J_\delta(t) = \|e^{\Delta(t-\tilde{t}_0)}[\boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_\delta(\cdot,\tilde{t}_0)]\|_{L^2(\mathbb{K})} + \|e^{\Delta(t-t_0)}[\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_\delta(\cdot,t_0)]\|_{L^2(\mathbb{K})}$. Taking $\delta = \delta' \to 0$ according to (2.2), we get $J_\delta(t) \to 0$, since, by Lebesgue's Dominated Convergence Theorem and (2.2), we have, for any, $\sigma\,\tau > 0$: $\|e^{\Delta\tau}[\boldsymbol{u}(\cdot,\sigma) - \boldsymbol{u}_{\delta'}(\cdot,\sigma)]\|_{L^2(\mathbb{K})} \to 0$ as $\delta' \to 0$. This shows (2.9). The proof of (2.10) is entirely similar. □

Theorem 2.3. *Let $\boldsymbol{u}(\cdot,t)$, $t > 0$, be any particular Leray's solution to (1.1). Then $\lim_{t\to\infty} t^{1/2}\|D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0$.*

Proof. The following argument is taken from [5], p. 236. From Leray's theory, $\|D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ is monotonically decreasing everywhere on $[t_0,\infty)$ if $t_0 \geq t_*$ (see (1.3)) and $\|\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} < 1$. Therefore, by (1.2), there must exist $t_{**} \geq t_*$ such that $\|D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ is a smooth decreasing function of t on $[t_{**},\infty)$. Recalling (2.3), this immediately gives the result. □

3. Leray's problem in $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$

In this section, we derive (1.4), (1.5), (1.7) and (1.8) above. Let $t_* \gg 1$ be given in (1.3). Taking $t_0 \geq t_*$ (arbitrary), we then have the representation

$$\boldsymbol{u}(\cdot,t) \,=\, e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) \,+ \int_{t_0}^{t} e^{\Delta(t-s)}\,\boldsymbol{Q}\,(\cdot,s)\,ds, \qquad t \geq t_0 \tag{3.1}$$

(in modern notation), by Duhamel's principle, where $\boldsymbol{Q}\,(\cdot,s)$ is defined in (2.5).

Theorem 3.1 (Leray's classical L^2 problem). *One has*

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0 \tag{3.2}$$

and, for any $t_0 \geq 0$:

$$\lim_{t\to\infty} t^{1/4}\|\boldsymbol{u}(\cdot,t) - e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.3}$$

Proof. It is sufficient to show (3.3); moreover, by (2.9), Theorem 2.2, we need only consider $t_0 \geq t_*$, so that (3.1) is valid. Now, given $\epsilon > 0$ arbitrary, let $t_\epsilon > t_0$ be taken large enough that, by Theorem 2.3, we have

$$t^{1/2}\,\|D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon \qquad \forall t \geq t_\epsilon. \tag{3.4}$$

We then get

$$\begin{aligned}
&(t-t_\epsilon)^{1/4}\|\boldsymbol{u}(\cdot,t) - e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\\
&\quad\leq (t-t_\epsilon)^{1/4}\int_{t_0}^{t}\|e^{\Delta(t-s)}\,\boldsymbol{Q}\,(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,ds \qquad [\text{by (3.1)}]\\
&\quad\leq I(t,t_\epsilon) + K(t-t_\epsilon)^{1/4}\int_{t_\epsilon}^{t}(t-s)^{-3/4}\|\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,ds \qquad [\text{by (2.6a)}]\\
&\quad\leq I(t,t_\epsilon) + K\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}\epsilon\,(t-t_\epsilon)^{1/4}\int_{t_\epsilon}^{t}(t-s)^{-3/4}\,s^{-1/2}\,ds \qquad [\text{by (1.2), (3.4)}]\\
&\quad\leq I(t,t_\epsilon) + 0.636\,\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}\epsilon
\end{aligned}$$

for all $t > t_\epsilon$, where $K = (8\pi)^{-3/4}$ and

$$I(t,t_\epsilon) = K(t-t_\epsilon)^{1/4}\int_{t_0}^{t_\epsilon}(t-s)^{-3/4}\|\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s$$

$$\le K\,\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}(t-t_\epsilon)^{-1/2}\int_{t_0}^{t_\epsilon}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s.$$

Therefore, $(t-t_\epsilon)^{1/4}\|\boldsymbol{u}(\cdot,t)-\mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} < \|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}\epsilon$ for all $t > t_\epsilon$ sufficiently large. This shows (3.3). □

Theorem 3.2 (Leray's L^∞ problem). *One has*

$$\lim_{t\to\infty} t^{3/4}\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = 0 \tag{3.5}$$

and, for any $t_0 \ge 0$:

$$\lim_{t\to\infty} t\,\|\boldsymbol{u}(\cdot,t)-\mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^\infty(\mathbb{R}^3)} = 0. \tag{3.6}$$

Proof. Again, it is sufficient to show (3.6), which can be done as follows. (For an alternative derivation, see [10].) By (2.10), Theorem 2.2, we need only consider the case $t_0 \ge t_*$ [see (1.3)], to which (3.1) applies. Given $\epsilon > 0$ small, let (by Theorem 2.3) $t_\epsilon > t_0$ be so large that (3.4) holds. Setting $w(t) := (t-t_\epsilon)^{3/2}\|\boldsymbol{u}(\cdot,t)-\mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_\infty$, $\mu(t) := (t+t_\epsilon)/2$, $\boldsymbol{v}(\cdot,t) := \mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)$, we have $w(t) \le I(t) + J(t)$ for $t > t_\epsilon$, where (using elementary heat kernel estimates):

$$I(t) = (t-t_\epsilon)^{3/2}\int_{t_0}^{\mu(t)}\|\mathrm{e}^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_\infty\,\mathrm{d}s \;\le\; (4\pi)^{-3/4}(t-t_\epsilon)^{3/2}\int_{t_0}^{\mu(t)}(t-s)^{-3/4}\,\|\mathrm{e}^{\frac{1}{2}\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s$$

$$\le (4\pi)^{-3/2}(t-t_\epsilon)^{3/2}\int_{t_0}^{\mu(t)}(t-s)^{-3/2}\,\|\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \quad \text{[by (2.6a)]}$$

$$\le (4\pi)^{-3/2}\|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^3)}(t_\epsilon-t_0)^{1/2} + (2\pi)^{-3/2}\epsilon\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}\int_{t_\epsilon}^{\mu(t)}s^{-1/2}\,\mathrm{d}s \quad \text{[by (1.2), (3.4)]}$$

and

$$J(t) = (t-t_\epsilon)^{3/2}\int_{\mu(t)}^{t}\|\mathrm{e}^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\|_\infty\,\mathrm{d}s$$

$$\le (8\pi)^{-3/4}(t-t_\epsilon)^{3/2}\int_{\mu(t)}^{t}(t-s)^{-3/4}\,\|\boldsymbol{u}(\cdot,s)\|_\infty\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \quad \text{[by (2.6b)]}$$

$$\le (8\pi)^{-3/4}\epsilon\,(t-t_\epsilon)^{3/2}\int_{\mu(t)}^{t}(t-s)^{-3/4}\,s^{-1/2}\,\|\boldsymbol{u}(\cdot,s)-\boldsymbol{v}(\cdot,s)\|_\infty\,\mathrm{d}s$$

$$+\,(8\pi)^{-3/4}\epsilon\,(t-t_\epsilon)^{3/2}\int_{\mu(t)}^{t}(t-s)^{-3/4}\,s^{-1/2}\,\|\boldsymbol{v}(\cdot,s)\|_\infty\,\mathrm{d}s$$

$$\le (2\pi)^{-3/4}\epsilon\int_{\mu(t)}^{t}(t-s)^{-3/4}\,(s-t_\epsilon)^{-1/2}\,w(s)\,\mathrm{d}s + (2\pi)^{-3/2}\epsilon\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}(t-t_\epsilon)^{1/2}$$

for all $t > t_\epsilon$, so that $(t-t_\epsilon)\|\boldsymbol{u}(\cdot,t)-\mathrm{e}^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_\infty < \|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^3)}\epsilon$ for all $t > t_\epsilon$ sufficiently large. This shows (3.6). □

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