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Statistics

Nonparametric trigonometric orthogonal regression estimation

Estimation non paramétrique de la fonction de régression par des séries trigonométriques

Nora Saadi, Smail Adjabi

LAMOS Laboratory, University of Bejaia, Algeria

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ABSTRACT

Eubank, Hart, and Speckman (1990) [2] have investigated the nonparametric trigonometric regression estimator. They assumed that the observation x_i points satisfy $\int_a^{x_i} \psi(s)\,ds = \frac{(1+i)}{n}$, $i = 1, \ldots, n$, where $\psi \in L^1[a,b]$ is a density satisfying certain smoothness conditions, and in a work by E. Rafajłowicz (1987) [3], the observation points coincide with knots of numerical quadratures. The aim of the present work is to introduce a new estimator of the regression function based on trigonometric series, for fixed point designs different from the ones considered so far, under milder restrictions on the observation points. This seems to be important since it may be numerically difficult to determine exactly the points x_i satisfying the recent condition or the knots of appropriate numerical quadratures, especially when their number is large.

RÉSUMÉ

Eubank, Hart et Speckman (1990) [2] ont étudié l'estimation non paramétrique de la fonction de régression par des séries trigonométriques. Ils ont supposé que les observations x_i satisfont la condition $\int_a^{x_i} \psi(s)\,ds = \frac{(1+i)}{n}$, $i = 1, \ldots, n$, où $\psi \in L^1[a,b]$ est une densité vérifiant certaines conditions de régularité. Dans un travail de Rafajłowicz (1987) [3], les observations coincident avec les nœuds des fonctions numériques quadratiques. Ce travail a pour objectif d'introduire un nouvel estimateur de la fonction de régression basé sur un système trigonométrique. On supposera que les observations sont prises en des points équidistants, car il est difficile de déterminer numériquement avec précision les points x_i satisfaisant aux précédentes conditions, spécialement quand le nombre d'observations est grand.

E-mail addresses: saadi.nora@gmail.com (N. Saadi), adjabi@hotmail.com (S. Adjabi).

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1. Introduction

The field of nonparametric statistics has broadened its appeal in recent years with an array of new tools for statistical analysis. These new tools offer sophisticated alternatives to traditional parametric models for exploring large amounts of univariate or multivariate data without making specific distributional assumption. As one of these tools, nonparametric regression function estimation has become a prominent statistical research topic. In many areas of mathematical analysis, the smoothness of a function is more readily determined by the behaviour of its Fourier series. Our thesis in this paper is that the latter approach is natural and convenient when analyzing the properties of orthogonal trigonometric regression function estimators. Consider the partition of the interval $A = [a, b] \subset \mathbb{R}$ into n subintervals $A_1 = [a_0, a_1]$, $A_i = (a_{i-1}, a_i]$, $i = 1, ..., n$, where $a = a_0 < a_1 < ... < a_n$. Suppose that the observations $y_1, ..., y_n$ follow the model $y_i = R(x_i) + \eta_i$, where $R(x) : [a, b] \longrightarrow \mathbb{R}$ is an unknown function satisfying certain smoothness conditions specified below, $x_i \in A_i$, $i = 1, ..., n$, and η_i, $i = 1, ..., n$, are independent identically distributed random variables with mean zero and finite variance $\sigma^2 > 0$. Let the functions e_k, $k = 0, ...,$ form an orthogonal system in the space $L^2[a, b]$ i.e.:

$$\int_a^b e_k(x)e_j(x)\,\mathrm{d}x = \delta_{kj}, 0 \le k, j < \infty. \tag{1}$$

Where δ_{kj} is the Kroncker delta. We assume that the regression function R is an element of this space and consequently it has a representation

$$R(x) = \sum_{k=0}^{\infty} c_k e_k(x), k = 0, \dots, x \in [a, b]. \tag{2}$$

As an estimator of R, we take

$$\hat{R}_{d_n}(x) = \sum_{k=0}^{d_n} \hat{c}_k e_k(x), \tag{3}$$

where

$$\hat{c}_k = \sum_{i=1}^{n} y_i \int_{a_{i-1}}^{a_i} e_k(x)\,\mathrm{d}x, \tag{4}$$

and d_n it is a sequence of positive number chosen so that $d_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

To cite a few specific works, Rutkowski [5] investigated sufficient conditions for almost sure convergence of the above estimator, constructed using the trigonometric functions and Legendre polynomials. Results concerning the convergence rates of the integrated mean-square error of the estimator constructed using the Fourier series were obtained in [2]. However, it is assumed that the observation points satisfy the condition $\int_a^{x_i} \psi(s)\,\mathrm{d}s = \frac{(1+i)}{n}$, $i = 1, ..., n$, where $\psi \in L^1[a, b]$ is a density satisfying certain smoothness conditions, and in [3] the observation points coincide with knots of numerical quadratures. The aim of the present work is to introduce a new estimator of the regression function based on trigonometric basis introduced by Saadi and Adjabi [6]. We obtain asymptotic results, in particular convergence rates for IMSE and the pointwise mean-square error of the estimator for fixed point designs different from the ones considered so far, under milder restrictions on the observation points. This seems to be important, since it may be numerically difficult to determine exactly the points x_i satisfying the recent condition or the knots of appropriate numerical quadratures, especially when their number is large.

2. Construction of the estimator

Consider the partition of the interval $A = [-\pi, \pi]$ into n subintervals $A_1 = [a_0, a_1]$, $A_i = (a_{i-1}, a_i]$, $i = 1, ..., n$, where $-\pi = a_0 < a_1 < ... < a_n = \pi$. Suppose that the observations $y_1, ..., y_n$ follow the model $y_i = R(x_i) + \eta_i$, where $R(x) : [-\pi, \pi] \longrightarrow \mathbb{R}$ is an unknown function satisfying certain smoothness conditions specified below, $x_i \in A_i$, $i = 1, ..., n$, and η_i, $i = 1, ..., n$, are independent identically distributed random variables with mean zero and finite variance $\sigma^2 > 0$. Let the functions $\{e_k, k = 0, ...\}$ form an orthogonal system in the space $L^2[-\pi, \pi]$ defined by Saadi and Adjabi [6]:

$$e_k(x) = \frac{1}{\sqrt{2\pi}}(\cos(kx) + \sin(kx))1_{[-\pi,\pi]}(x),\ k = 0, \dots. \tag{5}$$

We assume that $R(x) \in L^2[-\pi, \pi]$ and consequently it has a representation

$$R(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} c_k(\cos(kx) + \sin(kx)),\ \text{where } c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\cos(kx) + \sin(kx))R(x)\,\mathrm{d}x. \tag{6}$$

An estimator of $R(x)$ is given by:

$$\hat{R}_{d_n}(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{d_n} \hat{c}_k (\cos(kx) + \sin(kx))(x), \tag{7}$$

where

$$\hat{c}_k = \sum_{i=1}^{n} Y_i \int_{a_{i-1}}^{a_i} (\cos(kx)(x) + \sin(kx))\, \mathrm{d}x, \tag{8}$$

and d_n it is a sequence of a positive number chosen so that $d_n \longrightarrow \infty$ as $n \longrightarrow \infty$. As an estimator of R, we take

$$\hat{R}_{d_n}(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{d_n} \hat{c}_k (\cos(kx) + \sin(kx)) \tag{9}$$

$$= \frac{1}{2\pi} \sum_{i=1}^{n} \sum_{k=0}^{d_n} Y_i [(\cos(ka_i) + \sin(ka_i))(\cos(kx) + \sin(kx)) \tag{10}$$
$$+ (\cos(-ka_{i-1}) + \sin(-ka_{i-1}))(\cos(kx) + \sin(kx))].$$

Using the previous calculus, the estimator $\hat{R}_{d_n}(x)$ of $R(x)$ is given by

$$\hat{R}_{d_n}(x) = \frac{1}{4\pi} \sum_{i=1}^{n} Y_i \Big[\frac{\sin[\frac{(2d_n+1)(a_i - x)}{2}]}{\sin[\frac{a_i - x}{2}]} + \frac{\sin[\frac{(2d_n+1)(\frac{\pi}{2} - (a_i + x))}{2}]}{\sin[\frac{\frac{\pi}{2} - (a_i + x)}{2}]}$$
$$+ \Big[\frac{\sin[\frac{(2d_n+1)(-a_{i-1} - x)}{2}]}{\sin[\frac{-a_i - x}{2}]} + \frac{\sin[\frac{(2d_n+1)(\frac{\pi}{2} + (a_{i-1} - x))}{2}]}{\sin[\frac{\frac{\pi}{2} + (a_{i-1} - x)}{2}]} \Big]\Big]. \tag{11}$$

3. Proprieties of the estimator

To investigate the asymptotic properties of our estimator, we need the following theorem giving bounds for the variance and squared bias of the Fourier coefficient estimators defined in (8).

Theorem 1. *If $R(x) \in L_2[-\pi, \pi]$, then the Fourier coefficient estimators $\{\hat{c}_k, k = 0, 1, \ldots\}$, given by (8), satisfy*

$$[\hat{c}_k - \mathbb{E}(\hat{c}_k)]^2 \leq 2\pi\phi^2(R, \Delta_n), \ \mathbb{V}ar(\hat{c}_k) = \sigma_\eta^2 \Delta_n,$$

where $\phi(R, \delta) = \sup_{|x-s| \leq \delta} | R(x) - R(s) |$, $\delta > 0$, and $\Delta_n = \max_{1 \leq i \leq n}(a_i - a_{i-1})$.

Proof. Since $[-\pi, \pi] = \bigcup_{i=1}^{n} A_i$ and the functions $\{e_k\}_{k=0,1,\ldots}$, form an orthonormal system in $L_2[-\pi, \pi]$.

$$\mathbb{E}(\hat{c}_k) = \mathbb{E}[\frac{1}{2\pi} \sum_{i=1}^{n} Y_i \int_{a_{i-1}}^{a_i} (\cos(kx) + \sin(kx))\, \mathrm{d}x] = \frac{1}{2\pi} \sum_{i=1}^{n} R(X_i) \int_{a_{i-1}}^{a_i} (\cos(kx) + \sin(kx))\, \mathrm{d}x. \tag{12}$$

Then we deduce that

$$\mathbb{E}[\hat{c}_k - \mathbb{E}(\hat{c}_k)]^2 = \frac{1}{2\pi} \mathbb{E}[\sum_{i=1}^{n} \eta_i \int_{a_{i-1}}^{a_i} (\cos(kx) + \sin(kx))\, \mathrm{d}x]^2 = \frac{\sigma_\eta^2}{2\pi} \sum_{i=1}^{n} [\int_{a_{i-1}}^{a_i} (\cos(kx) + \sin(kx))\, \mathrm{d}x]^2$$
$$\leq \frac{\sigma_\eta^2}{2\pi} \sum_{i=1}^{n} (a_i - a_{i-1}) \int_{a_{i-1}}^{a_i} [(\cos(kx) + \sin(kx))\, \mathrm{d}x]^2 \leq \frac{\sigma_\eta^2 \Delta_n}{2\pi} \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} [(\cos(kx) + \sin(kx))]^2\, \mathrm{d}x$$
$$\leq \frac{\sigma_\eta^2 \Delta_n}{2\pi} \int_{[-\pi,\pi]} [(\cos(kx) + \sin(kx))]^2\, \mathrm{d}x = \sigma_\eta^2 \Delta_n.$$

For the bias term, we easily obtain

$$
\begin{aligned}
\mid \hat{c}_k - \mathbb{E}(\hat{c}_k) \mid^2 &= \frac{1}{2\pi} \mid \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} R(x)(\cos(kx)+\sin(kx))\,\mathrm{d}x - \sum_{i=1}^{n} R(X_i) \int_{a_{i-1}}^{a_i} (\cos(kx)+\sin(kx))\,\mathrm{d}x \mid^2 \\
&\le \frac{1}{2\pi} [\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \mid R(x) - R(X_i) \mid\mid \cos(kx)+\sin(kx) \mid]^2 \,\mathrm{d}x \\
&\le \frac{1}{2\pi} [\sum_{i=1}^{n} (\int_{a_{i-1}}^{a_i} (R(x)-R(X_i))^2\,\mathrm{d}x)^{\frac{1}{2}} (\int_{a_{i-1}}^{a_i} (\cos(kx)+\sin(kx))^2)\,\mathrm{d}x)^{\frac{1}{2}}]^2 \\
&\le \frac{1}{2\pi} [\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} (R(x)-R(X_i))^2\,\mathrm{d}x \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} (\cos(kx)+\sin(kx))^2 \\
&\le \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} (R(x)-R(X_i))^2\,\mathrm{d}x \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} (\frac{1}{\sqrt{2\pi}}(\cos(kx)+\sin(kx)))^2\,\mathrm{d}x \\
&\le \phi^2(R,\Delta_n) \sum_{i=1}^{n} (a_i - a_{i-1}) \\
&\le 2\pi\phi^2(R,\Delta_n). \qquad \square
\end{aligned}
$$

3.1. Integrated mean-square error

In this section we examine the integrated mean-square error of the estimator and obtain convergence rates for the estimators constructed using trigonometric polynomials.

Theorem 2. *If $R(x) \in L_2[-\pi,\pi]$ satisfies the Lipschitz condition with exponent $0 < \alpha \le 1$ in $[-\pi,\pi]$, $\Delta_n = O(n^{-1})$, $d_n \longrightarrow \infty$, $\frac{d_n}{n^\alpha} \longrightarrow 0$, $\alpha > 0$ as $n \longrightarrow \infty$ and $\sum_{k=d_n+1}^{\infty} c_k^2 \longrightarrow 0$ as $n \longrightarrow \infty$, then*

$$\lim_{n\longrightarrow\infty} IMSE(\hat{R}_{d_n}(x)) = 0. \tag{13}$$

Proof. It is easy to show the following formula for the integrated mean-square error (IMSE) of our estimator: $IMSE(\hat{R}_{d_n}(x)) = \sum_{k=0}^{d_n} \mathbb{E}(c_k - \hat{c}_k)^2 + \sum_{k=d_n+1}^{\infty} c_k^2$. Theorem 1 yields

$$IMSE(\hat{R}_{d_n}(x)) = \sum_{k=0}^{d_n} [(\mathbb{E}\hat{c}_k - c_k)^2 + \mathbb{E}(\mathbb{E}\hat{c}_k - c_k)^2] + \sum_{k=d_{n+1}}^{\infty} c_k^2 \tag{14}$$

$$\le (d_n+1)[\sigma_\eta^2 \Delta_n + 2\pi\phi^2(R,\Delta_n)] + \sum_{k=d_n+1}^{\infty} c_k^2. \tag{15}$$

By assumption, there exist constants $\beta > 0$ such that $\Delta_n \le \frac{\beta}{n}$ and R satisfies the Lipschitz condition in $[-\pi,\pi]$ i.e. $\exists\zeta > 0$, $\alpha > 0$, $\phi(R,\Delta_n) \le \zeta\Delta_n^\alpha$. Consequently, (15) shows that

$$\lim_{n\longrightarrow\infty} IMSE(\hat{R}_{d_n}(x)) = 0. \qquad \square$$

Theorem 3. Rate of the convergence of mean integrated square error. *If $R(x) \in L_2[-\pi,\pi]$ satisfies the Lipschitz condition with exponent $0 < \alpha \le 1$ in $[-\pi,\pi]$, $\sum_{k=d_n+1}^{\infty} c_k^2 = O(d_n^{-2\alpha})$ and $\Delta_n = O(n^{-1})$, then*

$$IMSE(\hat{R}_{d_n}(x)) = O(n^{\frac{-4\alpha^2}{2\alpha+1}}), 0 < \alpha \le \frac{1}{2}, d_n \sim n^{\frac{2\alpha}{2\alpha+1}}, \tag{16}$$

$$IMSE(\hat{R}_{d_n}(x)) = O(n^{\frac{-4\alpha^2}{2\alpha+1}}), \frac{1}{2} < \alpha \le 1, d_n \sim n^{\frac{1}{2\alpha+1}}. \tag{17}$$

Proof. By assumption, there exist constants $\beta > 0$, $\xi > 0$ such that $\Delta_n \le \frac{\beta}{n}$, $\sum_{k=d_n+1}^{\infty} c_k^2 \le \frac{\xi}{d_n^{2\alpha}}$ and R satisfies the Lipschitz condition in $[-\pi,\pi]$ i.e. $\exists\zeta > 0$, $\alpha > 0$, $\phi(R,\Delta_n) \le \zeta\Delta_n^\alpha$. According to (15) we deduce that

$$IMSE(\hat{R}_{d_n}(x)) \leq (d_n+1)[\frac{\beta\sigma_\eta^2}{n} + \frac{2\pi\zeta^2}{n^{2\alpha}}] + \frac{\xi}{d_n^{2\alpha}},$$

which can be rewritten as

$$IMSE(\hat{R}_{d_n}(x)) \leq d_n[\frac{F_1}{n} + \frac{F_2}{n^{2\alpha}}] + \frac{F_3}{d_n^{2\alpha}}.$$

Where F_1, F_2, F_3 are constants > 0. Now, for $0 < \alpha \leq \frac{1}{2}$, $d_n \sim n^{\frac{2\alpha}{2\alpha+1}}$ we deduce (16).

The proof of the (17) is analogous to that of (16) except that now we use inequality (15) with $\frac{1}{2} < \alpha \leq 1$, $d_n \sim n^{\frac{1}{2\alpha+1}}$. □

4. Mean-square error MSE

In this section we examine the pointwise mean-square error of our estimator and obtain convergence rates for the estimators constructed using trigonometric series. We have

$$MSE(\hat{R}_{d_n}(x)) = \mathbb{E}[\hat{R}_{d_n}(x) - R(x)]^2 \tag{18}$$

$$= \mathbb{E}[\sum_{k=0}^{d_n} \hat{c}_k e_k(x) - \sum_{k=0}^{\infty} \hat{c}_k e_k(x)]^2 = \sum_{k=0}^{d_n} \mathbb{E}[(\hat{c}_k - c_k)e_k(x) - \sum_{k=d_{n+1}}^{\infty} c_k e_k(x)]^2 \tag{19}$$

$$= \mathbb{E}[\sum_{k=0}^{d_n} (\hat{c}_k - c_k)e_k(x)]^2 + [\sum_{k=d_{n+1}}^{\infty} c_k e_k(x)]^2 + 2\mathbb{E}[\sum_{k=0}^{d_n} (\hat{c}_k - c_k)e_k(x)] \sum_{k=d_{n+1}}^{\infty} c_k e_k(x). \tag{20}$$

Theorem 4. *If $R(x)$ satisfies the Lipschitz condition with exponent $0 < \alpha \leq 1$ in $[-\pi, \pi]$, $\Delta_n = O(n^{-1})$, $\alpha > 0$ as and $(\sum_{k=d_n+1}^{\infty} | a_k k^\beta |^q)^{\frac{2}{q}} < \infty$, $\beta > 0$ $q > 0$, then*

$$\mathbb{E}[\hat{R}_{d_n}(x) - R(x)]^2 \leq d_n^2(\frac{G_1}{n} + \frac{G_2}{n^{2\alpha}}) + \frac{d_n^{1-\alpha} G_3}{n^\alpha} + \frac{c^2}{d_n^{2\alpha}}, \tag{21}$$

almost everywhere in $[-\pi, \pi]$, where $G_1, G_2, G_3 > 0$ are constants.

In this section, we can prove the main results concerning the convergence rates of the pointwise mean-square error of our estimator.

Theorem 5. *If R satisfies the Lipschitz condition with exponent $0 < \alpha \leq 1$, $\Delta_n = O(n^{-1})$, then*

(1) *for $0 < \alpha \leq \frac{1}{2}$ and $d_n \sim n^{\frac{\alpha}{\alpha+1}}$, we have*

$$\mathbb{E}(\hat{R}_{d_n}(x) - R(x))^2 = O(n^{\frac{-2\alpha^2}{\alpha+1}}) \text{ almost everywhere in } [-\pi, \pi];$$

(2) *for $\frac{1}{2} < \alpha \leq 1$ and $d_n \sim n^{\frac{1}{2(\alpha+1)}}$, we have*

$$\mathbb{E}(\hat{R}_{d_n}(x) - R(x))^2 = O(n^{\frac{-\alpha}{(\alpha+1)}}) \text{ almost everywhere in } [-\pi, \pi].$$

Proof. According to Theorem 4 and for $0 < \alpha \leq \frac{1}{2}$, $d_n \sim n^{\frac{\alpha}{\alpha+1}}$, we have

$$\mathbb{E}(\hat{R}_{d_n}(x) - R(x))^2 \leq H_1 n^{\frac{-2\alpha^2}{\alpha+1}} + H_2 n^{\frac{-\alpha(\alpha+1)+\alpha(\alpha-1)}{\alpha+1}} + H_3 n^{\frac{-2\alpha^2}{\alpha+1}},$$

almost everywhere in $[-\pi, \pi]$, where $H_1, H_2, H_3 > 0$. Consequently, we deduce that

$$\mathbb{E}(\hat{R}_{d_n}(x) - R(x))^2 \leq (H_1 + H_2 + H_3) n^{\frac{-2\alpha^2}{\alpha+1}},$$

almost everywhere in $[-\pi, \pi]$, which proves (1).

The proof of (2) is analogous to that of (1) if we use the Theorem 4, except that $\frac{1}{2} < \alpha \leq 1$, $d_n \sim n^{\frac{1}{2(\alpha+1)}}$. □

5. Concluding remarks

In the present work, it has been shown that the estimators considered can attain, at least for regression functions satisfying the Lipschitz condition with exponent $1/2 < \alpha \leq 1$, the same optimal IMSE convergence rate as the least squares polynomial estimators considered in [4] and orthogonal series estimators based on Legendre polynomials and numerical quadratures [3]. Our trigonometric series estimator attains the same IMSE convergence rate as the estimator for the equidistant point design considered in [1], where only the case of regression functions satisfying the Lipschitz condition with exponent $\alpha = 1$ was examined. Hence, the asymptotic properties of our estimator seem to be competitive in comparison to series regression estimators based on the orthogonal polynomials considered so far. Earlier Rafajłowicz [4] obtained sufficient conditions for uniform consistency of that estimator in the sense of pointwise mean-square error for other observation point designs, but they are satisfied only by functions of higher smoothness. In the present work, we go one step further, showing that there exist trigonometric series estimators uniformly consistent in the sense of pointwise mean-square error, and obtained the convergence rate for that error, under the same smoothness conditions on the regression function as in [7] and less restrictive assumptions on the observation points. The present work also shows that there exist series-type regression function estimators with asymptotic properties comparable to or even better than those of the least-squares polynomial estimators investigated in [4] and [7], which were examined only for special fixed point designs, when the observation points are equidistant.

Appendix A

Calculation of the

$$S = \sum_{k=0}^{d_n} \frac{1}{2\pi}[\cos(k(y-x)) + \sin(k(y+x))]$$

$$\begin{aligned}
S &= [\frac{1}{2\pi} + \frac{1}{2\pi}[\cos(y-x) + \sin(y+x) + \ldots + \cos(d_n(y-x)) + \sin(d_n(y+x))]] \\
&= \frac{1}{2\pi}[1 + \cos(y-x) + \sin(y+x) + \cdots + \cos(d_n(y-x)) + \sin(d_n(y+x))] \\
&= \frac{1}{2\pi}[\frac{1}{2} + \cos(y-x) + \cos 2(y-x) + \ldots + \cos(d_n(y-x)) \\
&\quad + \frac{1}{2} + \sin(y+x) + \sin 2(y+x) + \cdots + \sin(d_n(y+x))] \\
&= \frac{1}{2\pi}[\frac{1}{2} + \cos(y-x) + \cos 2(y-x) + \cdots + \cos(d_n(y-x)) \\
&\quad + \frac{1}{2} + \cos(\frac{\pi}{2} - (y+x)) + \cos 2(\frac{\pi}{2} - (y+x)) + \ldots + \cos d_n(\frac{\pi}{2} - (y+x))] \\
&= \frac{1}{2\pi}[\frac{1}{2} + \sum_{k=1}^{d_n} \cos(k(y-x)) + \frac{1}{2} + \sum_{k=1}^{d_n} \cos(k(\frac{\pi}{2} - (y+x)))] \\
&= \frac{1}{2\pi}\left[\frac{\sin[\frac{(2d_n+1)(y-x)}{2}]}{2\sin[\frac{y-x}{2}]} + \frac{\sin[\frac{(2d_n+1)(\frac{\pi}{2}-(y+x))}{2}]}{2\sin[\frac{\frac{\pi}{2}-(y+x)}{2}]}\right] \\
&= \frac{1}{4\pi}\left[\frac{\sin[\frac{(2d_n+1)(y-x)}{2}]}{\sin[\frac{y-x}{2}]} + \frac{\sin[\frac{(2d_n+1)(\frac{\pi}{2}-(y+x))}{2}]}{\sin[\frac{\frac{\pi}{2}-(y+x)}{2}]}\right].
\end{aligned}$$

Proof of Theorem 4. We have

$$\begin{aligned}
MSE(\hat{R}_{d_n}(x)) &= \mathbb{E}[\hat{R}_{d_n}(x) - R(x)]^2 \\
&= \mathbb{E}[\sum_{k=0}^{d_n}(\hat{c}_k - c_k)e_k(x)]^2 + [\sum_{k=d_{n+1}}^{\infty} c_k e_k(x)]^2 + 2\mathbb{E}[\sum_{k=0}^{d_n}(\hat{c}_k - c_k)e_k(x)] \sum_{k=d_{n+1}}^{\infty} c_k e_k(x). \qquad (22)
\end{aligned}$$

From the Cauchy–Schwarz inequality it further follows that

$$MSE(\hat{R}_{d_n}(x)) \leq \sum_{k=0}^{d_n}[(c_{k-\mathbb{E}\hat{c}_k})^2 + \mathbb{E}(c_{k-\mathbb{E}\hat{c}_k})^2]\sum_{k=0}^{d_n} e_k^2(x) + [\sum_{k=d_{n+1}}^{\infty} c_k e_k(x)]^2 \qquad (23)$$

$$+2\mathbb{E}[\sum_{k=0}^{d_n}(c_{k-\mathbb{E}\hat{c}_k})^2]^{\frac{1}{2}}[\sum_{k=0}^{d_n}e_k^2(x)]^{\frac{1}{2}} \mid \sum_{k=d_{n+1}}^{\infty} c_k e_k(x) \mid$$

and according to Theorem 1, we finally obtain the inequality

$$MSE(\hat{R}_{d_n}(x)) \le (d_n+1)[\sigma_n^2\Delta_n + 2\pi\phi^2(R,\Delta_n)]\sum_{k=0}^{d_n} e_k^2(x) + [\sum_{k=d_{n+1}}^{\infty} c_k e_k(x)]^2 \tag{24}$$

$$+2\sqrt{2\pi}(\sqrt{d_n+1}\phi(R,\Delta_n))(\sum_{k=0}^{d_n} e_k^2(x))^{\frac{1}{2}} \mid \sum_{k=d_{n+1}}^{\infty} c_k e_k(x) \mid .$$

$$\mid \sum_{k=d_n+1}^{\infty} c_k e_k(x) \mid \le \sum_{k=d_n+1}^{\infty} \mid c_k \mid\mid e_k(x) \mid \le \sup_{x\in[-\pi,\pi]} \mid e_k(x) \mid \sum_{k=d_n+1}^{\infty} \mid c_k \mid . \tag{25}$$

In addition,

$$\sup_{x\in[-\pi,\pi]} \mid e_k(x) \mid \sum_{k=d_n+1}^{\infty} \mid c_k \mid = \frac{2}{\sqrt{2\pi}} \sum_{k=d_n+1}^{\infty} \mid c_k \mid = \frac{2}{\sqrt{2\pi}} \sum_{k=d_n+1}^{\infty} \mid c_k k^\beta \mid \frac{1}{k^\beta}. \tag{26}$$

However,

$$\frac{2}{\sqrt{2\pi}} \sum_{k=d_n+1}^{\infty} \mid c_k k^\beta \mid \frac{1}{k^\beta} \le \frac{2}{\sqrt{2\pi}} (\sum_{k=d_n+1}^{\infty} \mid c_k k^\beta \mid^q)^{\frac{1}{q}} (\sum_{k=d_n+1}^{\infty} \frac{1}{k^{p\beta}})^{\frac{1}{p}} = \frac{2}{\sqrt{2\pi}} M (\sum_{k=d_n+1}^{\infty} \frac{1}{k^{p\beta}})^{\frac{1}{p}},$$

with

$$M = (\sum_{k=d_n+1}^{\infty} \mid c_k k^\beta \mid^q)^{\frac{1}{q}}, p>0, \frac{1}{p}+\frac{1}{q}=1. \tag{27}$$

We have

$$\sum_{k=d_n+1}^{\infty} \frac{1}{k^{p\beta}} \le \int_{d_n}^{\infty} \frac{1}{x^{p\beta}}\,\mathrm{d}x = \frac{1}{\beta p - 1}(\frac{1}{d_n})^{\beta p - 1}. \tag{28}$$

It follows that:

$$\mid \sum_{k=d_n+1}^{\infty} c_k e_k(x) \mid^2 \le \frac{2}{\pi} M^2 \frac{1}{(\beta p-1)^{\frac{2}{p}}} (\frac{1}{d_n})^{2\beta - \frac{2}{p}} \tag{29}$$

putting

$$c = \frac{2}{\pi} M^2 [\frac{1}{\beta p - 1}]^{\frac{2}{p}}. \tag{30}$$

Then, we deduce that

$$\mid \sum_{k=d_n+1}^{\infty} c_k e_k(x) \mid^2 \le c(\frac{1}{d_n})^{2\alpha}, \alpha = \beta - \frac{1}{p}. \tag{31}$$

By assumption, there exist constants $\varrho, \eta > 0$ such that $\Delta_n \le \frac{\varrho}{n}$ and $\phi(R,\gamma) \le \eta\gamma^\alpha$ for $\gamma > 0$, so by (24), and according to (31), we have

$$MSE(\hat{R}_{d_n}(x)) \le (d_n+1)^2(\frac{\varrho\sigma_n^2}{n} + 2\pi\frac{\eta^2\varrho^{2\alpha}}{n^{2\alpha}}) \sup_{x\in[-\pi,\pi]} e_k^2(x) + \frac{c^2}{d_n^{2\alpha}} + 2\sqrt{2\pi}(d_n+1) \sup_{x\in[-\pi,\pi]} \mid e_k(x) \mid \frac{\eta\varrho^\alpha}{d_n^\alpha}\frac{c}{d_n^\alpha},$$

which can be rewritten as

$$\mathbb{E}[\hat{R}_{d_n}(x) - R(x)]^2 \le d_n^2(\frac{G_1}{n} + \frac{G_2}{n^{2\alpha}}) + \frac{d_n^{1-\alpha}G_3}{n^\alpha} + \frac{c^2}{d_n^{2\alpha}}, \tag{32}$$

almost everywhere in $[-\pi,\pi]$, where $G_1, G_2, G_3 > 0$ are constants. □

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