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Gallai triangles in configurations of lines in the projective plane

Triangles de Gallai dans les configurations de droites du plan projective

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ABSTRACT

A question by Paul Erdős about the existence of Gallai triangles in arrangements of d real lines in the projective plane, with no more than three lines incident to each vertex, is answered in the negative for all d higher than three.

RÉSUMÉ

La note répond négativement à une question posée par Paul Erdős concernant l'existence de triangles de Gallai dans les configurations de d droites réelles du plan projective, dans la situation où $d > 3$ et où au plus trois droites concourent en chaque sommet.

A class of algebraic surfaces in the complex projective space with many ordinary double points as their only singularities was introduced in [5], in order to improve the lower bound [1] for the maximal number of singularities that an algebraic surface may have. They are based on certain bivariate polynomials that have been used also in the construction of algebraic surfaces with other kinds of singularities [6,7]. Real variants of the polynomials are associated with configurations of lines in the plane.

Let A be a configuration of d lines in the projective plane. The number of vertices of multiplicity (number of lines of A incident to a vertex) j is denoted by $t_j(A)$.

Definition 1. A triangle in a line arrangement A is called a Gallai triangle if it is formed by three lines from A such that their three intersection points have multiplicity 2.

The following problem was formulated by Paul Erdős ([3,4], p. 818, [10]).

Erdős question. *Let us suppose that an arrangement A has $t_j(A) = 0$ for j higher than 3. Then does there exist a Gallai triangle in A?*

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In [10], the authors constructed examples of arrangements of n lines, denoted by B_n, which showed that the answer is negative if n is at least four and not divisible by nine.

The folding polynomials of degree d, associated with the affine Weyl group of the root lattice A_2, are defined for $(z, w) \in \mathbf{C}^2$ as $F_d^{A_2}(z, w) := j_{1,d}(z, w) + j_{2,d}(z, w)$, where $j_{2,d}(z, w) = j_{1,d}(w, z)$ and $j_{1,d}(z, w)$ satisfies the recursion relation

$$j_{1,d}(z, w) = zj_{1,d-1}(z, w) - wj_{1,d-2}(z, w) + j_{1,d-3}(z, w) \tag{1}$$

with $j_{1,1}(z, w) = z$, $j_{1,2}(z, w) = z^2 - 2w$, $j_{1,3}(z, w) = z^3 - 3zw + 3$.

In [8] real variants of the surfaces studied in [5] were constructed. They correspond to $\tau = 0$ in the one-parameter family of degree d polynomials with real coefficients

$$J_{d,\tau}(x, y) := \mathrm{e}^{\mathrm{i}(\tau+\frac{2\pi}{3})}\widetilde{j}_{1,d}(x, y) + \mathrm{e}^{-\mathrm{i}(\tau+\frac{2\pi}{3})}\widetilde{j}_{2,d}(x, y) + 2\cos 3\tau \tag{2}$$

where $(x, y) \in \mathbf{R}^2$, $\tau \in \mathbf{R}$, and $\widetilde{j}_{1,d}(x, y) = j_{1,d}(x + \mathrm{i}y, x - \mathrm{i}y)$, $\widetilde{j}_{2,d}(x, y) = \widetilde{j}^*_{1,d}(x, y)$.

The critical points of $J_{d,\tau}(x, y)$ are determined by the critical points of $H_{d,\tau}(u, v) := 2\cos(2\pi du - \frac{2\pi}{3} - \tau) + 2\cos(2\pi dv - \frac{2\pi}{3} - \tau) + 2\cos(2\pi d(u + v) + \frac{2\pi}{3} + \tau)$, $u, v \in \mathbf{R}$ [9]. A basis of simple roots $\{\alpha_1, \alpha_2\}$ for the root lattice A_2 in the (u, v) plane is $\alpha_1 = (2, 0)$, $\alpha_2 = (-1, \sqrt{3})$. In [8] we showed that for $\tau = 0$ the positions of the critical points of $H_{d,\tau}(u, v) + 2\cos 3\tau$ with critical value $\zeta = 0$ are situated in the downscaled root lattice with basis $\{\frac{\alpha_1}{6d}, \frac{\alpha_1+\alpha_2}{6d}\}$. They determine a set of d pseudolines whose images under the generalised cosine $\widetilde{h}(u, v) := (\cos(2\pi(u + v)) + \cos(2\pi u) + \cos(2\pi v), \sin(2\pi(u + v)) - \sin(2\pi u) - \sin(2\pi v))$ are the lines in the (x, y) plane defining $J_{d,0}(x, y)$. The results in [8] can be extended by taking into account the positions of the corresponding critical points of $J_{d,\tau}(x, y)$ [9]. We get the lines $M_{d,\tau,\nu}(x, y) = 0$, $\nu = -\lfloor\frac{d-2}{2}\rfloor, -\lfloor\frac{d-2}{2}\rfloor + 1, \ldots, \lfloor\frac{d+1}{2}\rfloor$ with

$$M_{d,\tau,\nu}(x, y) := y - (x - \cos(\frac{2\pi}{d}(\frac{6\nu - 1}{6} - \frac{\tau}{\pi})))\tan(\frac{\pi}{d}(\frac{6\nu - 1}{6} - \frac{\tau}{\pi})) + \sin(\frac{2\pi}{d}(\frac{6\nu - 1}{6} - \frac{\tau}{\pi})) \tag{3}$$

The polynomials given in eq. (2) are formed by the union of the lines in eq. (3), up to a normalising factor [9].

The lines $L_{d,k,\nu}(x, y) := M_{d,(2k+1)\frac{\pi}{6},\nu}(x, y)$, $k = 0, 1, \ldots, 5$ have the parametric equations

$$z = \mathrm{e}^{-\mathrm{i}2\pi u} + t\mathrm{e}^{\mathrm{i}\pi u}, t \in \mathbf{R} \tag{4}$$

with $u = \frac{3\nu-k-1}{3d}$. For the values of $\tau = (2k + 1)\frac{\pi}{6}$ corresponding to $k = 0, 1, \ldots, 5$, the polynomials $J_{d,\tau}(x, y)$ have all the minima and maxima with the same absolute value (see Theorem 2.2 in [9]) and correspond to simplicial arrangements in the affine plane (all the bounded cells are triangles, as in Fig. 1). However for our purposes it is enough to study the cases $k = 0, 2$. We consider the configurations of $d > 3$ lines

$$A_{d,k} := \{L_{d,k,\nu} = 0\}_{\nu\in S} \tag{5}$$

where $S := \{-m + 1, -m + 2, \ldots, m + 1\}$ if $d = 2m + 1$ and $S := \{-m + 1, -m + 2, \ldots, m\}$ if $d = 2m$.

Lemma 1.

1. Each line in $A_{d,k}$ intersects each other line.

2. $L_{d,k,\nu_1} \cap L_{d,k,\nu_2} \cap L_{d,k,\nu_3} \neq \emptyset$ iff $\nu_1 + \nu_2 + \nu_3 \equiv k + 1$ (mod d), $\forall L_{d,k,\nu_1}, L_{d,k,\nu_2}, L_{d,k,\nu_3} \in A_{d,k}$.

3. There is no vertex of multiplicity higher than 3 in $A_{d,k}$.

Proof. 1. The incidence condition in order to have $L_{d,k,\nu_1} \cap L_{d,k,\nu_2} \neq \emptyset$ is $\frac{\nu_1-\nu_2}{d} \notin \mathbf{Z}$, which is always true because $max|\nu_1 - \nu_2| = d - 1, \forall \nu_1, \nu_2 \in S$.

2. The lines $L_{d,k,\nu_1}, L_{d,k,\nu_2}, L_{d,k,\nu_3} \in A_{d,k}$ with $u_l = \frac{3\nu_l-k-1}{3d}$ are concurrent iff $\cos\pi(2u_2 + u_1) = \cos\pi(2u_3 + u_1)$, namely when $\pi(2u_2 + u_1) = -\pi(2u_3 + u_1) + 2n\pi, n \in \mathbf{Z}$ or, equivalently, $\nu_1 + \nu_2 + \nu_3 \equiv k + 1$ (mod d).

3. We assume that there are already three lines concurrent to a vertex, therefore $u_1 + u_2 + u_3 = n_1 \in \mathbf{Z}$. If the multiplicity of the vertex is 4 then there exists u_4 such that $u_1 + u_2 + u_4 = n_2 \in \mathbf{Z}$ and $u_1 + u_3 + u_4 = n_3 \in \mathbf{Z}$, hence $u_2 - u_3 = n_2 - n_3 \in \mathbf{Z}$, which is not possible. □

Lemma 2. *For $\nu_0, \nu_1 \in S$, $\nu_0 \neq \nu_1$, we have $2\nu_0 + \nu_1 \equiv k + 1$ (mod d) iff $L_{d,k,\nu_0} \cap L_{d,k,\nu_1}$ is a vertex of multiplicity 2.*

Proof. We assume $2\nu_0 + \nu_1 \equiv k + 1$ (mod d) and we choose $\nu_j \in S \setminus \{\nu_0, \nu_1\}$. Therefore $\nu_0 + \nu_1 + \nu_j \equiv k + 1 + \nu_j - \nu_0 \not\equiv k + 1$ (mod d) because if $\nu_j > \nu_0$ then $1 \leq \nu_j - \nu_0 \leq d - 1$ and if $\nu_j < \nu_0$ then $1 - d \leq \nu_j - \nu_0 \leq -1$. This proves the first part of the lemma.

For the second part we assume $2\nu_0 + \nu_1 \not\equiv k + 1$ (mod d) (and therefore also $2\nu_1 + \nu_0 \not\equiv k + 1$ (mod d)). We have two possibilities (mod d)

(a) $2\nu_0 + \nu_1 < k + 1$:

In this case $\exists l = 1, 2, \ldots, k + m$ with $d = 2m$ or $d = 2m + 1$ such that $2\nu_0 + \nu_1 \equiv k + 1 - l$ (mod d) and we have $\nu_0 + \nu_1 + \nu_l \equiv k + 1$ (mod d), with $\nu_l = \nu_0 + l \in S \setminus \{\nu_0, \nu_1\}$ (mod d).

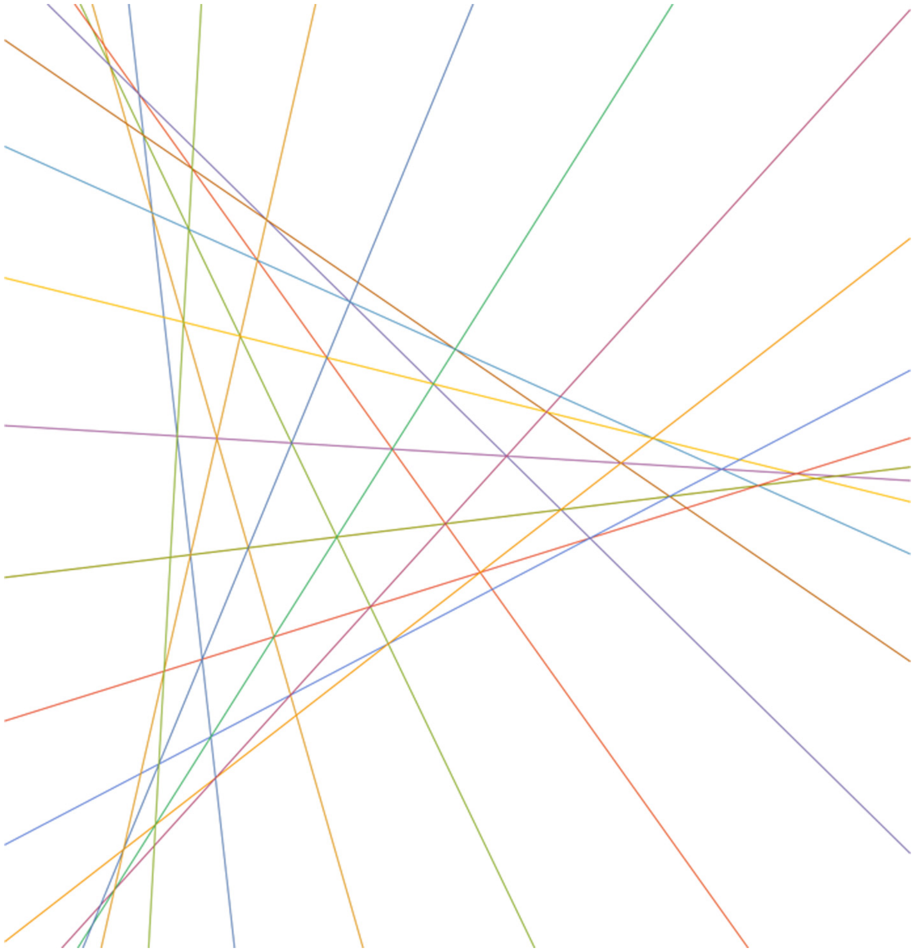

Fig. 1. The configuration of lines $A_{18,0}$.

(b) $k+1 < 2\nu_0 + \nu_1$:

Now $2\nu_0 + \nu_1 \equiv k+1+l \pmod{d}$, $l = 1, 2, \ldots, t$ with $t = m-k$ if $d = 2m+1$ or $t = m-k-1$ if $d = 2m$ and $\nu_0 + \nu_1 + \nu_l \equiv k+1 \pmod{d}$, with $\nu_l = \nu_0 - l \in S \setminus \{\nu_0, \nu_1\} \pmod{d}$.

In both cases we would get a vertex of multiplicity 3. □

In the sequel we will use the following well-known results (see for instance [2], p. 122 about linear congruences).

Proposition 1. *If* $\gcd(a, d) = 1$, *then the linear congruence* $ax \equiv b \pmod{d}$ *has exactly one solution modulo* d.

By using the Euclid's algorithm it can be shown that if $\gcd(a, d) = m$ then there are two integers s and t such that $m = as + dt$ (Bézout's identity). Therefore when $m = 1$, the solution to $ax \equiv b \pmod{d}$ is $x = bs$.

Proposition 2. *If* $\gcd(a, d) = m$, *then the congruence* $ax \equiv b \pmod{d}$ *has solution iff* $m|b$. *In that case there are exactly* m *solutions modulo* d *which can be written as* $x_1, x_1 + d_1, \ldots, x_1 + (m-1)d_1$, *where* $d = md_1$ *and* x_1 *is the solution to the congruence* $a_1 x \equiv b_1 \pmod{d_1}$, $a = ma_1$, $b = mb_1$.

Now we can get the main result of this article:

Theorem 1. *The configurations of lines* $A_{d,k}$ *in the following cases have no Gallai triangles:*

1. $d = 3q+1$, $q = 1, 2, 3, \ldots$; $k = 0$
2. $d = 3q+2$, $q = 1, 2, 3, \ldots$; $k = 0$
3. $d = 9n$, $n = 1, 2, 3, \ldots$; $k = 0$
4. $d = 3q$, $q \neq 3n$, $q, n = 1, 2, 3, \ldots$; $k = 2$

Proof. 1. According to Lemma 2 there is a Gallai triangle for $k=0$ when we can find $\nu_0 \not\equiv \nu_1 \not\equiv \nu_2 \pmod{d}$ such that $2\nu_0+\nu_1 \equiv 2\nu_1+\nu_2 \equiv 2\nu_2+\nu_0 \equiv 1 \pmod{d}$ and as a consequence $9\nu_0 \equiv 3 \pmod{d}$, $36\nu_1 \equiv 12 \pmod{d}$, $18\nu_2 \equiv 6 \pmod{d}$.

The equation $9\nu_0 \equiv 3 \pmod{3q+1}$ has a unique solution because $\gcd(9, 3q+1)=1$. The solution is $\nu_0 \equiv -q \pmod{3q+1}$, but $\nexists \nu_1 \not\equiv \nu_0 \pmod{3q+1}$ satisfying $2\nu_0+\nu_1 \equiv 1 \pmod{3q+1}$ therefore there is no Gallai triangle.

2. Now $\gcd(9, 3q+2)=1$ and the unique solution to $9\nu_0 \equiv 3 \pmod{3q+2}$ is $\nu_0 \equiv q+1 \pmod{3q+2}$. Again there is no solution for $\nu_1 \not\equiv \nu_0$ of $2\nu_0+\nu_1 \equiv 1 \pmod{3q+2}$.

3. $\gcd(9, 9n)=9$ but 9 does not divide 3 therefore $9\nu_0 \equiv 3 \pmod{9n}$ has no solution according to Proposition 2. Alternatively if $d=9n$, then we must have $3(\nu_0 - n \cdot l)=1$ for some $l \in \mathbf{Z}$, which is not possible. The configuration of lines $A_{18,0}$ can be seen in Fig. 1.

4. A consequence of $2\nu_0+\nu_1 \equiv 2\nu_1+\nu_2 \equiv 2\nu_2+\nu_0 \equiv 3 \pmod{d}$ is $9\nu_0 \equiv 9 \pmod{d}$, $36\nu_1 \equiv 36 \pmod{d}$, $18\nu_2 \equiv 18 \pmod{d}$. For $k=2$ we analyse $9\nu_0 \equiv 9 \pmod{d}$. When $d=3q, q \neq 3n$, we have $\gcd(9, d)=3$ and $3|d$. There are exactly three solutions: $\nu_0 \in \{1-q, 1, 1+q\}$, which are also solutions (not the only ones) of $36\nu_1 \equiv 36 \pmod{d}$, $18\nu_2 \equiv 18 \pmod{d}$. But they can not be the vertices of a Gallai triangle because if $\{\nu_0, \nu_1, \nu_2\} = \{1-q, 1, 1+q\}$ then $\nu_0+\nu_1+\nu_2 \equiv 3 \pmod{d}$ (see Lemma 1). On the other hand $\nexists \nu_1' \not\equiv \nu_0$ such that $2\nu_0+\nu_1' \equiv 3 \pmod{d}$ with $\nu_0 \in \{1-q, 1, 1+q\}$. □

There are other examples with different values of k without Gallai triangles, but the results of Theorem 1 are sufficient to answer the Erdős question: for each integer $d \geq 4$ there are configurations of d lines in the plane with no more than three lines incident to each vertex and having no Gallai triangles.

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