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Note on conditional quantiles for functional ergodic data

Note sur les quantiles conditionnels pour variables fonctionnelles ergodiques

Fatima Benziadi [a], Ali Laksaci [b], Fethallah Tebboune [c]

[a] *Université Moulay Taher de Saida, Algeria*
[b] *Laboratoire Statistique et Processus stochastiques, Université Djillali-Liabès, BP 89, S. B. A. 22000, Algeria*
[c] *Université Djillali-Liabes, Sidi Bel Abbès, Algeria*

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ABSTRACT

In this Note, we study the recursive kernel estimator of the conditional quantile of a scalar response variable Y given a random variable (rv) X taking values in a semi-metric space. We establish the almost complete consistency of this estimate when the observations are sampled from a functional ergodic process.

RÉSUMÉ

Dans cette Note, nous étudions l'estimateur à noyau récursif des quantiles conditionnels d'une variable réponse réelle Y sachant une variable aléatoire fonctionnelle X. Nous établissons la convergence presque complète de cet estimateur estimation lorsque les observations ont une corrélation ergodique.

Version française abrégée

Soit $(X_i, Y_i)_{i=1,\dots,n}$ une suite de variables aléatoires, strictement stationnaire, à valeurs dans $\mathcal{F} \times \mathbb{R}$, où $\mathcal{F}$ est un espace semi-métrique. On note d la semi-métrique sur $\mathcal{F}$. On suppose que la version régulière de la probabilité conditionnelle de Y sachant X existe et admet une densité bornée par rapport à la mesure de Lebesgue sur $\mathbb{R}$. Pour tout $x \in \mathcal{F}$, on désigne par F^x la fonction de répartition conditionnelle de Y sachant $X = x$ (resp. par f^x la densité conditionnelle de Y sachant $X = x$).

Par la suite, on fixe un point x dans $\mathcal{F}$ et on note N_x un voisinage de ce point. Pour $\alpha \in]0, 1[$, le quantile conditionnel d'ordre α notée $t_\alpha(x)$ est

$$t_\alpha(x) = \inf\{y \in \mathbb{R} : \mathbb{P}(Y_1 \le y | X_1 = x) \ge \alpha\}. \quad (1)$$

E-mail address: alilak@yahoo.fr (A. Laksaci).

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Dans cette Note, on se propose d'étudier l'estimation récursive du quantile conditionnel $t_\alpha(x)$ sous des conditions d'ergodicité. Plus précisément, on estime $t_\alpha(x)$ par

$$\widehat{t}_\alpha(x) = \inf\{y \in \mathbb{R} : \widehat{F}^x(y) \geq \alpha\}$$

où

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n K(a_i^{-1}d(x,X_i))\mathbb{1}_{\{Y_i \leq y\}}}{\sum_{i=1}^n K(a_i^{-1}d(x,X_i))}$$

avec $\mathbb{1}_{\{\cdot\}}$ désigne la fonction indicatrice. K est un noyau et a_n est une suite de nombres réels positifs telle que $\lim_{n\to\infty} a_n = 0$.

L'objectif principal de cette note est d'étudier la convergence presque complète de l'estimateur $\widehat{t}_\alpha$ vers t_α. Ce travail constitue une généralisation des résultats de Laksaci et al. [14] dans deux directions différentes (la méthode d'estimation et la corrélation des données). En effet, d'une part, la méthode du noyau classique peut être considérée comme un cas particulier de la présente étude. D'autre part, l'hypothèse d'ergodicité, considérée ici, est impliquée par les conditions de mélange fort considérées par Laksaci et al. [14]. Le résultat obtenu est donné par le théorème suivant (voir la version en anglais pour les notations et les hypothèses) :

Théorème 1. *Sous les conditions (H1)–(H4), on a*

$$\widehat{t}_\alpha(x) - t_\alpha(x) = O_{\text{a.co.}}\left(\frac{1}{\varphi_n(x)}\sum_{i=1}^n a_i^{\beta_1}\phi_i(x,a_i) + \sqrt{\frac{\varphi_n(x)\log n}{n^2\psi_n^2(x)}}\right).$$

1. Introduction

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be a sequence of strictly stationary dependent random variables valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space. For $x \in \mathcal{F}$ and $\alpha \in]0,1[$, the conditional quantile of order α, denoted $t_\alpha(x)$ is defined by

$$t_\alpha(x) = \inf\{y \in \mathbb{R} : \mathbb{P}(Y_1 \leq y | X_1 = x) \geq \alpha\}. \tag{2}$$

In this Note, we deal with the nonparametric estimation of this model by using a recursive kernel method when the observations are strictly stationary ergodic data. Specifically, our estimate is defined by

$$\widehat{t}_\alpha(x) = \inf\{y \in \mathbb{R} : \widehat{F}^x(y) \geq \alpha\}$$

where

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n K(a_i^{-1}d(x,X_i))\mathbb{1}_{\{Y_i \leq y\}}}{\sum_{i=1}^n K(a_i^{-1}d(x,X_i))}$$

with $\mathbb{1}_{\{\cdot\}}$ being the indicator function K is a kernel and a_n is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$.

Let us note that the recursive estimation method permits to update the estimate for each additional information. This feature is very useful in this area of functional data analysis. Because the observations are available by real time monitoring, this implies a real-time update of the information.

It is well documented that the conditional quantile estimation is useful in prediction setting. Indeed, it provides an alternative approach to classical regression estimation. The first results in this topic date back to Stone [18] and have been widely studied, when the explanatory variable lies in a finite-dimensional space (see, for instance, Samanta [17] for previous results and Dabo-Niang and Thiam [5] for more recent advances). This model was introduced in nonparametric functional statistics by Ferraty and Vieu [8]. They established the almost complete convergence of the kernel estimator in the independent, identically distributed (i.i.d.) case. The asymptotic normality of this estimator was studied by Ezzahrioui and Ould-Saïd [7]. Laksaci et al. [13,14] proposed an alternative estimate based on the L^1-method. Recently, Dabo-Niang and Laksaci [4] stated the convergence in L^p-norm under less restrictive hypotheses. Among the wide literature on functional data analysis, we only refer to the overviews for parametric models given by Bosq [3], Ramsay and Silverman [15], and to the monograph of Ferraty and Vieu [8] for the prediction problem in functional nonparametric statistics. It should also be noted that the literature on nonfunctional nonparametric modeling of ergodic data is quite extensive; we refer the reader to Delecroix and Rosa [6] or to Laib and Ould-Said [10], and the references therein.

Our main purpose in this paper is to study the almost complete convergence[1] of a functional kernel estimate of the conditional quantile $t_\alpha(x)$ under the ergodicity condition. More precisely, we establish this asymptotic property for the

[1] Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of real r.v.'s; we say that z_n converges almost completely (a.co.) to zero if, and only if, $\forall\epsilon > 0$, $\sum_{n=1}^\infty P(|z_n| > \epsilon) < \infty$. Moreover, we say that the rate of almost complete convergence of z_n to zero is of order u_n (with $u_n \to 0$) and we write $z_n = O_{\text{a.co.}}(u_n)$ if, and only if, $\exists\epsilon > 0$, $\sum_{n=1}^\infty P(|z_n| > \epsilon u_n) < \infty$. This kind of convergence implies both almost sure convergence and convergence in probability.

L^1-recursive kernel estimate of $t_\alpha(x)$. We point out that the results of this work generalize those of Laksaci et al. [14] in two different directions (estimation method and data correlation). Indeed, firstly, the classical kernel method can be viewed as a particular case of the present study. Secondly, the ergodicity hypothesis, considered here, is implied by the mixing conditions studied in this cited work. It should be noted that, although the statistical modeling for functional ergodic data has a great importance in various domains such as statistical physics, thermodynamics, or signal processing, it has not yet been fully explored. As far as we know, only the papers by Laib and Louani [11,12] and Gheriballah et al. [9] have paid attention to study the kernel estimation of the regression function. On the other hand, as noticed by Roussas and Tran [16], the recursive estimate is more relevant than the classical kernel method in time series analysis. So, we can say that the adaptation of the recursive estimate in ergodic functional time series is very motivating. It is worth noting that this work is the first contribution that considers a recursive estimate in ergodic statistic.

The paper is organized as follows: we present our general framework in Section 2. The asymptotic properties of the recursive L^1-estimate are given in Section 3. We study some particular cases in Section 4.

2. The functional ergodic data framework

The general framework of our study is nonparametric modeling in functional ergodic data. Such dependence structure is an alternative to the strong mixing condition usually assumed in functional time series analysis. The ergodicity condition is easier to handle than the mixing one, which needs to calculate the supremum over two sigma algebras. Moreover, it is well known that ergodicity is less restrictive than the mixing hypothesis. Recall that, in multivariate statistics, the ergodicity assumption is defined with respect to an ergodic transformation. In our functional context, we adopt the definition introduced by Laib and Louani [11]. In addition, as usually for nonparametric modeling in functional statistics, the contribution of the functional component to our asymptotic study is controlled by the concentration property of the probability measure of the functional variable. Specifically, in addition to the classical concentration hypothesis, we have to take into account the dependency setting, modeled by the ergodic condition. Thus, our functional ergodic data is carried out by the following considerations: for $k = 1, \dots, n$, we put $\mathfrak{F}_k$ the σ-field generated by $((X_1, Y_1), \dots (X_k, Y_k))$, we set $\mathfrak{G}_k$ the σ-field generated by $((X_1, Y_1), \dots (X_k, Y_k), X_{k+1})$, we assume that the conditional distribution of Y given $\mathfrak{G}_k$ depends only on X_{k+1}, and we suppose that the strictly stationary ergodic process $(X_i, Y_i)_{i \in \mathbb{N}^*}$ satisfies

(H1)

$$\begin{cases} \text{(i) the function } \phi(x,r) := \mathbb{P}(X \in B(x,r)) \text{ is such that } \phi(x,r) > 0, \ \forall r > 0 \\ \quad \text{where } B(x,h) = \left\{x' \in \mathcal{F} / d(x',x) < h\right\}; \\ \text{(ii) for all } i = 1, \dots, n \text{ there exists a deterministic function } \phi_i(x,\cdot) \text{ such that} \\ \quad \text{almost surely } 0 < \mathbb{P}(X_i \in B(x,r) | \mathfrak{F}_{i-1}) \le \phi_i(x,r), \ \forall r > 0, \text{ and } \phi_i(x,r) \to 0 \text{ as } r \to 0; \\ \text{(iii) for all sequence } (r_i)_{i=1,\dots n} > 0, \qquad \dfrac{\sum_{i=1}^n \mathbb{P}(X_i \in B(x,r_i) | \mathfrak{F}_{i-1})}{\sum_{i=1}^n \phi(x,r_i)} \to 1 \qquad \text{a.co.} \end{cases}$$

We point out that this assumption is quite milder than that considered by Laib and Louani [12]. Indeed, unlike in Laib and Louani [12], it is not necessary to write (approximatively) the concentration function $\mathbb{P}(X_i \in B(x,r))$ and the conditional concentration function $\mathbb{P}(X_i \in B(x,r)|\mathfrak{F}_{i-1})$ as a product of two independent nonnegative functions of the center and of the radius. Recall that this asymptotic decomposition of these small-ball probability functions requires the boundedness of its associated Onsager–Machlup function (see [2]), but here it is not necessary to employ this function.

3. Hypotheses and results

From now on, x will stand for a fixed point in $\mathcal{F}$ and C or C' denote some generic constant in $\mathbb{R}^{*+}$. In order to establish the almost complete convergence (a.co.) of $\widehat{t_\alpha}(x)$ to $t_\alpha(x)$, we fix the neighborhood N_x of x, we assume that the regular version $F^{x'}$ of the conditional distribution function of Y given $X = x'$ exists for all $x' \in N_x$, we suppose that F^x has a continuous density f^x with respect to (w.r.t.) Lebesgue's measure over $\mathbb{R}$ and we consider the following assumptions:

(H2) there exists $\delta > 0$, such that $\forall (t_1, t_2) \in [t_\alpha(x) - \delta, t_\alpha(x) + \delta]^2$, $\forall (x_1, x_2) \in N_x^2$,

$$|F^{x_1}(t_1) - F^{x_2}(t_2)| \le C' \left(d(x_1,x_2)^{\beta_1} + |t_1 - t_2|^{\beta_2}\right) \text{ and } \inf_{y \in [t_\alpha(x) - \delta, \, t_\alpha(x) + \delta]} f^x(y) > C$$

with $C > 0$, $C' > 0$, $\beta_1 > 0$, $\beta_2 > 0$;

(H3) K is a function with support $(0,1)$ such that $C \mathbb{1}_{(0,1)} < K(t) < C' \mathbb{1}_{(0,1)}$;

(H4)

$$\lim_{n \to \infty} \frac{\varphi_n(x) \log n}{n^2 \psi_n^2(x)} = 0 \text{ where } \varphi_n(x) = \sum_{i=1}^n \phi_i(x, a_i) \text{ and } \psi_n(x) = n^{-1} \sum_{i=1}^n \phi(x, a_i).$$

We obtain the following theorem which deals with the pointwise a. co. convergence of $\widetilde{t}_\alpha(x)$ to $t_\alpha(x)$.

Theorem 1. *Under the hypotheses of Proposition 1, we have*

$$\widehat{t}_\alpha(x) - t_\alpha(x) = O_{\text{a.co.}}\left(\frac{1}{\varphi_n(x)}\sum_{i=1}^{n} a_i^{\beta_1}\phi_i(x, a_i) + \sqrt{\frac{\varphi_n(x)\log n}{n^2\psi_n^2(x)}}\right).$$

The detail of proof is omitted due to space limitation. However, it is based on the following proposition.

Proposition 1. *Assume that (H1)–(H4) are satisfied, then, we have*

$$\sup_{y\in[t_\alpha(x)-\delta,\, t_\alpha(x)+\delta]} |\widehat{F}^x(y) - F^x(y)| = O_{\text{a.co.}}\left(\frac{1}{\varphi_n(x)}\sum_{i=1}^{n} a_i^{\beta_1}\phi_i(x, a_i) + \sqrt{\frac{\varphi_n(x)\log n}{n^2\psi_n^2(x)}}\right).$$

Proof of Proposition 1. We start by writing

$$\widehat{F}^x(y) - F^x(y) = \widehat{B}_n(x, y) + \frac{\widehat{R}_n(x, y)}{\widehat{F}_D(x)} + \frac{\widehat{Q}_n(x, y)}{\widehat{F}_D(x)}$$

where

$$\widehat{Q}_n(x, y) := (\widehat{F}_N^x(y) - \bar{F}_N^x(y)) - F^x(y)(\widehat{F}_D(x) - \bar{F}_D(x))$$

$$\widehat{B}_n(x, y) := \frac{\bar{F}_N^x(y)}{\bar{F}_D(x)} - F^x(y), \quad \text{and} \quad \widehat{R}_n(x, y) := -\widehat{B}_n(x, y)(\widehat{F}_D(x) - \bar{F}_D(x))$$

with

$$\widehat{F}_N^x(y) := \frac{1}{n\psi_n(x)}\sum_{i=1}^{n} K(a_i^{-1}d(x, X_i))\mathbb{1}_{\{Y_i\le y\}}, \quad \bar{F}_N^x(y) := \frac{1}{n\psi_n(x)}\sum_{i=1}^{n} \mathbb{E}\left[K(a_i^{-1}d(x, X_i)))\mathbb{1}_{\{Y_i\le y\}}|\mathfrak{F}_{i-1}\right],$$

$$\widehat{F}_D(x) := \frac{1}{n\psi_n(x)}\sum_{i=1}^{n} K(a_i^{-1}d(x, X_i)), \qquad \bar{F}_D(x) := \frac{1}{n\psi_n(x)}\sum_{i=1}^{n} \mathbb{E}\left[K(a_i^{-1}d(x, X_i))|\mathfrak{F}_{i-1}\right].$$

Thus, Proposition 1 is a consequence of the following intermediate results, whose proofs are given in the Appendix.

Lemma 1. *Under Hypotheses (H1), (H3) and (H4), we have* $\widehat{F}_D(x) - \bar{F}_D(x) = O_{\text{a.co.}}\left(\sqrt{\frac{\varphi_n(x)\log n}{n^2\psi_n^2(x)}}\right)$.

Lemma 2. *Under the hypotheses of Lemma 1, we have* $\exists C > 0 \quad \sum_{n=1}^{\infty} \mathbb{P}\left(\widehat{F}_D(x) \le C\right) < \infty$.

Lemma 3. *Under Hypotheses (H1)–(H3), we have* $\sup_{y\in[t_\alpha(x)-\delta,\, t_\alpha(x)+\delta]} |\widehat{B}_n(x, y)| = O\left(\frac{1}{\varphi_n(x)}\sum_{i=1}^{n} a_i^{\beta_1}\phi_i(x, a_i)\right)$.

Lemma 4. *Under the hypotheses of Theorem 1, we have* $\sup_{y\in[t_\alpha(x)-\delta,\, t_\alpha(x)+\delta]} |\widehat{F}_N^x(y) - \bar{F}_N^x(y)| = O_{\text{a.co.}}\left(\sqrt{\frac{\varphi_n(x)\log n}{n^2\psi_n^2(x)}}\right)$.

4. Discussions and conclusion

In this section, we go back to discuss the practical interest of three structural axes of our study, such as the ergodicity assumption on the data, the recursivity of the estimate and the almost complete consistency.

– *On the ergodicity assumption*
In practice, the ergodicity assumption is a fundamental postulate of statistical physics in order to control the thermodynamic properties of gases, atoms, electrons, or plasmas. This hypothesis is also used in signal processing, for studying the evolution of a random signal. From a theoretical point of view, modeling a functional time series data with ergodicity condition allows us to include several usual cases not covered by the classical mixing dependency. Moreover, the ergodic assumption permits also to avoid the complicated probabilistic calculations of the mixing condition. So, we can say that the nonparametric analysis of functional time ergodic data has a great impact in practice as well as in theory.

- *On the recursivity of the estimate*
  The main advantage of the recursive estimate is that the smoothing parameter is linked to the observation (X_i, Y_i), which permits to update our estimator for each additional observation. Moreover, the recursive estimate is very fast in practice. Indeed, the latter can be obtained by a recursive formula (see [1]). So, the computation of the recursive estimator for an additional observation (X_{n+1}, Y_{n+1}) is based on its value computed with n observations, then its computational time is smaller than that of the classical one, for which we must recompute the estimate on the whole sample $(X_i, Y_i)_{i=1,\ldots n+1}$.
- *On the almost complete consistency*
  It is well known that the almost complete consistency is stronger than the two popular stochastic consistencies (almost sure convergence and the convergence probability). This is the main motivation to explore the asymptotic properties of the estimate by the almost complete convergence. Compared to the existing result in the classical case, our convergence rate is given in a general form. Indeed, on the one hand, the convergence rate of Theorem 1 exploits the structural axes of our study; in particular, the ergodicity assumption, which is controlled by the function φ_i. On the other hand, our convergence rate can be particularized for several usual cases, such as the classical kernel case, the independence case, and the finite-dimensional case.
  - *The classical kernel case:* as noted earlier, the classical kernel method studied by Laksaci et al. [13] is a particular case of our work with $a_i = a_n$, for all $1 \le i \le n$. Therefore, condition (H4) is replaced by

$$\frac{\varphi(x, a_n) \log n}{n^2 \phi^2(x, a_n)} \to 0 \tag{3}$$

where $\varphi(x, a_n) = \sum_{i=1}^{n} \phi_i(x, a_n)$. The convergence rate is given by the following corollary.

Corollary 1. *Under Hypotheses (H1)–(H3) and (3), we have*

$$\widehat{t_\alpha}(x) - t_\alpha(x) = O\left(a_n^{\beta_1}\right) + O_{\text{a.co.}}\left(\sqrt{\frac{\varphi(x, a_n) \log n}{n^2 \phi^2(x, a_n)}}\right).$$

Remark 1. We point out that, as far as we know, there is no work in the literature on the conditional quantile estimation in functional ergodic data. Therefore, we can advance that our results are also new in this area.

  - *The independence case:* in this situation, Condition (H1) reduces to $\phi(x, r) > 0$, and, for all $i = 1, \ldots n$, $\phi_i(x, r) = \phi(x, r)$. Thus, $\varphi_n(x) = n\psi_n(x)$ and our theorems lead to the next corollary.

Corollary 2. *Under Hypotheses (H1)–(H4), we have*

$$\widehat{t_\alpha}(x) - t_\alpha(x) = O\left(\frac{1}{\varphi_n(x)} \sum_{i=1}^{n} a_i^{\beta_1} \phi(x, a_i)\right) + O_{\text{a.co.}}\left(\sqrt{\frac{\log n}{\varphi_n(x)}}\right).$$

Remark 2. We note that we obtain the same convergence rate of Laksaci et al. [13] by considering both cases (classical kernel and independent data); it suffices to combine both corollaries (2 and 1).

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