C. R. Acad. Sci. Paris, Ser. I 354 (2016) 729–734

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Statistics

Estimating jump intensity and detecting jump instants in the context of p derivatives

Estimation de l'intensité et des instants de sauts pour des processus à p dérivées

Denis Bosq

LSTA, Université Pierre-et-Marie-Curie (Paris-6), France

ARTICLE INFO

Article history:
Received 3 February 2016
Accepted after revision 31 March 2016
Available online 5 May 2016

Presented by Paul Deheuvels

ABSTRACT

In this paper we consider the ARMAD$^{(p)}(q,r)$ process where $D[0,1]$ is the space of the càdlàg function and the p-th derivative has a possible jump. One envisages to detect the intensity and position of the jumps in the context of p derivatives. Asymptotic results are derived.

RÉSUMÉ

Dans cet article, nous considérons le processus ARMAD$^{(p)}(q,r)$, où $D = D[0,1]$ est l'espace des fonctions càdlàg et où la p^e dérivée a un saut éventuel. Nous envisageons de détecter l'intensité et la position des sauts. Des résultats asymptotiques sont obtenus.

1. Introduction

In the previous papers, the problem was to estimate the intensity of the jumps in a continuous or discrete context. More precisely, Bosq [6] and Blanke and Bosq [2] estimated the jumps of an observed continuous-time process. Also, some parts in Bosq [6] are generalized by Blanke and Bosq [3] in the case of sampled and functional autoregressive processes with jumps. In this note, we propose to generalize these results by considering observations with jumps in derivative (JD); namely, we observe $X^{(p)}$ and estimate the jumps of $X^{(p)}$ and $X^{(p+1)}$ for some integer $p \geq 0$. Applications appear in physics, finance, seismology, shocks, avalanches, wave propagation, etc. A lot of papers appear in this context; among many examples, we may cite Dmowska and Kostrov [9], Scherzer [12], Takahashi [13], Cates and Gelb [7], Tanushev [14], Joo and Qiu [11], Çetin and Sheynzon [8], Blanke and Vial [4,5], Horváth and Kokoszka [10, p. 208], etc. In the next Section, we begin with the case of one observation in continuous time. In Proposition 2.1, we obtain a result that allows us to distinguish the intensity of JD on the right and on the left, with clear notations, we put

E-mail address: denis.bosq@upmc.fr.

http://dx.doi.org/10.1016/j.crma.2016.03.016

$$J_X^{(p)}(t) = X^{(p)}(t+\alpha) - X^{(p)}(t-\beta),\ 0 < t-\beta < t < t+\alpha < 1\ \ p \in \mathbb{N}$$

where $X = (X(t),\ 0 < t < 1)$ has a p-th derivative on the right and on the left. Actually, one observes a "window". Note that it is also possible to observe discrete data of the form $(X^{(p)}(j\delta),\ 0 \le j \le k)$ where $k\delta = 1$. The observation of the window is somewhat similar since we may write it under the frame $\left(X^{(p)}(j\delta) - X^{(p)}(j-1)\delta),\ 1 \le j \le k\right)$.

In order to detect a jump for the $(p+1)$-th derivative, one may set

$$D_{t,\alpha,\beta}^{p,X} = \frac{X^{(p)}(t+\alpha) - X^{(p)}(t+)}{\alpha} - \frac{X^{(p)}(t-\beta) - X^{(p)}(t-)}{-\beta},$$

$0 < t-\beta < t < t+\alpha < 1$, where $X^{(p)}(t+)$ (resp. $X^{(p)}(t-)$), has a possible jump on the right (resp. on the left).

In Section 2, we study the ARMA$D^{(p)}(q,r)$ process in the context of p derivatives with some jump. Sections 3 and 4 are devoted to the estimation and the position of the jumps for derivatives. The main result is Proposition 4.1, since it allows us to detect the jump and to obtain the intensity of the jump's derivative simultaneously. In Section 5, we obtain a limit in the distribution.

2. The ARMA$D^{(p)}(q,r)$ process

For some $p \in \mathbb{N}$, we define the ARMA$D^{(p)}(q,r)$ process in the space $D = D[0,1]$ (see [1]), by setting

$$X_n - l_1(X_{n-1}) - \ldots - l_q(X_{n-q}) = Z_n - \lambda_1(Z_{n-1}) - \ldots - \lambda_r(Z_{n-r})$$

where $l_j,\ 1 \le j \le q$ and $\lambda_i,\ 1 \le i \le r$ are continuous linear operators in D and $(Z_n^{(p)})$ are i.i.d. square integrable and $E(Z_n^{(p)}(t_0) - Z_n^{(p)}(t_0-)) \neq 0$. Moreover, we suppose that $X_n^{(p)}$ and $Z_n^{(p)}$ do exist and are càdlàg for this integer p.

Now we set $C_p = C_p[0,1]$ where C_p is p times continuously derivable (on 0 and on 1), and we make the following assumption:

A1 – $l_j(D) \subset C_p,\ 1 \le j \le q,\ \lambda_i(D) \subset C_p,\ 1 \le i \le r.$

As an example, we may suppose that

$$l_j(x)(t) = \int_0^1 a_j(s,t)\, x(s) \mathrm{d}s,\ 0 \le t \le 1,\ 1 \le j \le q,\ x \in D_p$$

where "$x \in D_p$" means that the p-th derivative does exist and is (possibly) càdlàg. Now, we consider the following assumption:

A'1 – $\left| \frac{\partial a_j^{(p)}(s,t_1)}{\partial t^{(p)}} - \frac{\partial a_j^{(p)}(s,t_2)}{\partial t^{(p)}} \right| \le M\ |t_2 - t_1|^{\gamma}\ ,\ 1 \le j \le q,\ M > 0,\ 0 \le t_1 < t_2 \le 1,\ ,\ 0 \le s \le 1,\ 0 < \gamma \le 1.$

The case of $\lambda_i,\ 1 \le i \le r$ may be treated with a similar assumption, say **A"1**. Then, A'1 and A"1 imply A1. The following statement is simple, but crucial:

Proposition 2.1. *If A1 holds we have for* $0 < t < 1$, $n \in \mathbb{Z}$*:*

$$X_n^{(p)}(t) - X_n^{(p)}(t-) = Z_n^{(p)}(t) - Z_n^{(p)}(t-).$$

Hence, if t has a jump, $(X_n^{(p)}(t) - X_n^{(p)}(t-),\ n \in \mathbb{Z})$ *are i.i.d.*

Proof. By using the ARMA$D^{(p)}(q,r)$ process we get two formulas

$$X_n^{(p)}(t) - (l_1(X_{n-1}))^{(p)}(t) - \ldots - (l_q(X_{n-q}))^{(p)}(t) = Z_n^{(p)}(t) - (\lambda_1(Z_{n-1}))^{(p)}(t) - \ldots - (\lambda_r(Z_{n-r}))^{(p)}(t),$$

$$X_n^{(p)}(t-) - (l_1(X_{n-1}))^{(p)}(t-) - \ldots - (l_q(X_{n-q}))^{(p)}(t-) = Z_n^{(p)}(t-) - (\lambda_1(Z_{n-1}))^{(p)}(t-) - \ldots - (\lambda_r(Z_{n-r}))^{(p)}(t-).$$

Subtracting them and taking into account A1, one obtains $X_n^{(p)}(t) - X_n^{(p)}(t-) = Z_n^{(p)}(t) - Z_n^{(p)}(t-)$ and, if there is a jump at t_0, $(X_n^{(p)}(t_0) - X_n^{(p)}(t_0-))$ is i.i.d. □

3. Estimating the intensity of the jump for derivatives

For the sake of simplicity, we suppose that there is only one jump for derivatives, say t_0, for the ARMA$D^{(p)}(q,r)$ process; then we have

$$X_i^{(p)}(t_0) - X_i^{(p)}(t_0-) \neq 0,\ a.s.,\ i = 1,\ldots,n.$$

For the moment, we suppose that t_0 is known. Now, in order to estimate intensity of jump, we set

$$\bar{J}_n^{(p)}(t_0) = \frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t_0+\alpha_n) - X_i^{(p)}(t_0-\beta_n)\right]$$

and

$$\bar{D}_n^{(p)}(t_0) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{X_i^{(p)}(t_0+\alpha_n) - X_i^{(p)}(t_0)}{\alpha_n} - \frac{X_i^{(p)}(t_0-\beta_n) - X_i^{(p)}(t_0-)}{-\beta_n}\right]$$

with $0 < t_0 - \beta_n < t_0 < t_0 + \alpha_n < 1$ and $(\alpha_n) \downarrow 0(+)$ and $(\beta_n) \downarrow 0(+)$.

Now, we need to make the following assumptions:

A2 – $X_n^{(p)}$, $n \in \mathbb{Z}$, admits a Taylor expansion with Lagrange remainder of order $(p+2)$ on $[t,1]$ (respectively on [0,t]), $t \in]0,1[$.

A3 – $\left\|Z_n^{(p)}\right\|_\infty \le m_{(p)}$, and $\left\|X_n^{(p)}\right\|_\infty \le m_{(p)}$ a.s., $n \in \mathbb{Z}$.

Then:

Proposition 3.1. *Under A1, A2, and A3 one obtains*

$$\bar{J}_n^{(p)}(t_0) \to E(X_1^{(p)}(t_0) - X_1^{(p)}(t_0-))\ a.s.$$

and

$$\left|\bar{J}_n^{(p)}(t)\right| \le (\alpha_n + \beta_n)\, m_{(p+1)} \to 0\ a.s.,\ t \ne t_0.$$

Proof. For $i = 1, \ldots, n$, A2 gives:

$$X_i^{(p)}(t+\alpha_n) = X_i^{(p)}(t+) + \alpha_n X_i^{(p+1)}(t + \theta_{n1}\alpha_n)$$

and

$$X_i^{(p)}(t-\beta_n) = X_i^{(p)}(t-) - \beta_n X_i^{(p+1)}(t - \theta_{n2}\beta_n),$$

with $0 < \theta_{n1} < 1$, $0 < \theta_{n2} < 1$, $0 < t - \beta_n < t < t + \alpha_n < 1$. Now, from A3 we get

$$\left|X_i^{(p)}(t+\alpha_n) - X_i^{(p)}(t+) - X_i^{(p)}(t-\beta_n) + X_i^{(p)}(t-)\right| \le (\alpha_n + \beta_n)\, m_{(p+1)}$$

hence

$$\left|\frac{1}{n}\sum_{i=1}^{n}(-X_i^{(p)}(t_0) + X_i^{(p)}(t_0-)) + X_i^{(p)}(t_0+\alpha_n) - X_i^{(p)}(t_0-\beta_n)\right| \le (\alpha_n+\beta_n)\, m_{(p+1)},\ 1 \le j \le k.$$

A1 and A3 give consistency. Finally, since there is no jump, the second result is clear. □

Proposition 3.2. *A1, A2 and A3 entail*

$$\bar{D}_n^{(p)}(t_0) \to E(X_1^{(p+1)}(t_0) - X_1^{(p+1)}(t_0-)),\ a.s.,$$

and

$$\left|\bar{D}_n^{(p)}(t)\right| \le (\alpha_n + \beta_n)\, m_{(p+2)} \to 0\ a.s.,\ t \ne t_0.$$

Proof. For $i = 1, \ldots, n$, A2 gives

$$\frac{X_i^{(p)}(t+\alpha_n) - X_i^{(p)}(t+)}{\alpha_n} = X_i^{(p+1)}(t+) + \frac{\alpha_n}{2!} X_i^{(p+2)}(t + \theta_{1n}\alpha_n)$$

and similarly

$$\frac{X_i^{(p)}(t-\beta_n) - X_i^{(p)}(t-)}{\beta_n} = X_i^{(p+1)}(t-) - \frac{\beta_n}{2!} X_i^{(p+2)}(t - \theta_{2n}\beta_n).$$

Taking the empirical mean, it follows that

$$\bar{D_n}^{(p)}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p+1)}(t+) - X_i^{(p+1)}(t-)\right] + \frac{1}{2n}\sum_{i=1}^{n}\left[\alpha_n X_i^{(p+2)}(t+\theta_{1n}\alpha_n) - \beta_n X_i^{(p+2)}(t-\theta_{2n}\beta_n)\right]$$
$$=: U_n + V_n.$$

Since A3 is almost surely bounded, we have $|V_n| \le (\alpha_n + \beta_n)\, m_{(p+2)}$, a.s. Now, if t has a jump at t_0, Proposition 2.1 and A3 entails boundedness and we obtain $U_n \to E(X_1^{(p+1)}(t_0) - X_1^{(p+1)}(t_0-))$.

If there is no jump, the result is clear. □

4. Estimating the intensity and the position of the jump

We now need to make the following assumption:

A4 – $\left|X_n^{(p)}(t) - X_n^{(p)}(s)\right| \le M_p\ |t-s|^\gamma$, $0 < \gamma \le 1$, $0 < s < t < 1$, $n \in \mathbb{Z}$ and M_p is almost surely bounded.

In order to simplify the exposition, we suppose that there exists only one unknown jump say t_0, satisfying $0 < t_0 < 1$. We begin with one data and set $J^{(p)}(t, \alpha_n, \beta_n) = X_1^{(p)}(t+\alpha_n) - X_1^{(p)}(t-\beta_n)$, $0 < t-\beta_n < t < t+\alpha_n < 1$, where $(\alpha_n) \downarrow 0(+)$ and $(\beta_n) \downarrow 0(+)$.

Now we put

$$\hat{t}_{0,n} = \arg \max_{0<t-\beta_n<t<t+\alpha_n<1} \left|J^{(p)}(t,\alpha_n,\beta_n)\right|,$$

then we get Proposition 4.1.

Proposition 4.1. *If A1, A2, A3 hold, we have*

$$\left|\hat{t}_{0,n} - t_0\right| \le \max(\alpha_n,\ \beta_n) \to 0\ a.s.$$

Proof. By using a variant of Proposition 3.1, we obtain

$$\sup_{t \ne t_0}\left|J^{(p)}(t,\alpha_n,\beta_n)\right| \le (\alpha_n + \beta_n)\, m_{(p+1)} \to 0\ a.s.$$

and

$$\left|J^{(p)}(t_0,\alpha_n,\beta_n)\right| \to \left|X_1^{(p)}(t_0) - X_1^{(p)}(t_0-)\right| \ne 0\ a.s.$$

thus, a.s. for n large enough we obtain

$$\hat{t}_{0,n} = \arg\max |X_1^{(p)}(t_0+\alpha_n) - X_1^{(p)}(t_0-\beta_n)|,$$

hence $t_0 - \beta_n < \hat{t}_{0,n} < t_0 + \alpha_n$ almost surely for n large enough, hence the result since

$$t_0 - \max(\alpha_n,\beta_n) < \hat{t}_{0,n} < t_0 + \max(\alpha_n,\beta_n). \quad \square$$

Then it is possible to estimate the intensity of the jump even if t_0 is unknown:

Proposition 4.2. *Under A1, A2, A3, A4, one obtains*

$$\frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(\hat{t}_{0,n}+\alpha_n) - X_i^{(p)}(\hat{t}_{0,n}-\beta_n)\right] \to E(X_1^{(p)}(t_0) - X_1^{(p)}(t_0-))$$

almost surely.

Proof. From A4 we get

$$\left|X_i^{(p)}(\hat{t}_{0,n}+\alpha_n) - X_i^{(p)}(t_0+)\right| \le M_p\ \left|\hat{t}_{0,n}+\alpha_n - t_0(+)\right|^\gamma,\ 1 \le i \le n$$

and

$$\left|X_i^{(p)}(\hat{t}_{0,n}-\beta_n - X_i^{(p)}(t_0(-))\right| \le M_p\ \left|\hat{t}_{0,n}-\beta_n - t_0(-)\right|^\gamma,\ 1 \le i \le n.$$

Then

$$\frac{1}{n}\sum_{i=1}^{n}\left|X_i^{(p)}(\hat{t}_{0,n}+\alpha_n)-X_i^{(p)}(t_0+)-X_i^{(p)}(\hat{t}_{0,n}-\beta_n)+X_i^{(p)}(t_0(-)\right|$$
$$\leq M_p\left[\left|\hat{t}_{0,n}-\beta_n-t_0(-)\right|^{\gamma}+\left|\hat{t}_{0,n}+\alpha_n-t_0(+)\right|^{\gamma}\right]$$

which tends to zero almost surely from Proposition 4.1. Finally Proposition 2.1 entails

$$\frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t_0+)-X_i^{(p)}(t_0-)\right]\to E(X_1^{(p)}(t_0+)-X_1^{(p)}(t_0-))$$

almost surely. Hence the desired result. □

Now, we may slightly modify position of jump by setting

$$t_{0,n}^{*}=\arg\max\left|\bar{J}_n^{(p)}(t,\alpha_n,\beta_n)\right|$$

where $0<t-\beta_n<t<t+\alpha_n<1$. Then we get

Proposition 4.3. *If A1, A2, A3, A4 hold, we have again*

$$\left|t_{0,n}^{*}-t_0\right|\leq\max(\alpha_n,\ \beta_n)\to 0\ a.s.$$

Proof. The proof of Proposition 4.1 is similar; we obtain

$$\sup_{t\neq t_0}|\bar{J}_n^{(p)}(t,\alpha_n,\beta_n)|\leq(\alpha_n+\beta_n)\,m_{(p+1)}\to 0\ a.s.$$

Now, from Proposition 4.2, we get

$$|\frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t_{0,n}^{*}+\alpha_n)-X_i^{(p)}(t_{0,n}^{*}-\beta_n)\right]|\to_{a.s.}|E(X_1^{(p)}(t_0)-X_1^{(p)}(t_0-))|\neq 0,$$

then we have $t_{0,n}^{*}=\arg\max\left|\frac{1}{n}\sum_{i=1}^{n}(X_i^{(p)}(t_0+\alpha_n)-X_i^{(p)}(t_0-\beta_n))\right|$, hence the result. □

5. Limit in distribution

Proposition 5.1. *If A1, A2, A3, A4 hold and if* $(\alpha_n{+}\beta_n)\sqrt{n}\to 0$, *then*

$$\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[X_i^{(p)}(t+\alpha_n)-X_i^{(p)}(t-\beta_n)\right]\right|\leq m_{(p+1)}\,(\alpha_n{+}\beta_n)\sqrt{n}\to 0\ a.s.$$

with $0<t-\beta_n<t<t+\alpha_n<1$. *If there exists a jump at* t_0 *we obtain*

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left[X_i^{(p)}(t_0+\alpha_n)-X_i^{(p)}(t_0-\beta_n)\right]-E(X_1^{(p)}(t_0+)-X_1^{(p)}(t_0-))\right)$$
$$\Longrightarrow N\sim\mathcal{N}(0,\ V(X_1^{(p)}(t_0+)-X_1^{(p)}(t_0-)).$$

Proof. For $i=1,\dots,n$ and under A2

$$X_i^{(p)}(t+\alpha_n)=X_i^{(p)}(t+)+\alpha_n\,X_i^{(p+1)}(t+\theta_{n1}\alpha_n)$$

and

$$X_i^{(p)}(t-\beta_n)=X_i^{(p)}(t-)-\beta_n\,X_i^{(p+1)}(t-\theta_{n2}\beta_n),$$

$0<\theta_{n1}<1,\ 0<\theta_{n2}<1,\ ,0<t-\beta_n<t<t{+}\alpha_n<1.$

Hence, if there is no jump and α_n and β_n are small enough

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left[X_i^{(p)}(t+\alpha_n)-X_i^{(p)}(t-\beta_n)\right]\right)$$
$$=\frac{\alpha_n}{\sqrt{n}}\sum_{i=1}^{n}X_i^{(p+1)}(t+\theta_{n1}\alpha_n)-\beta_n\sum_{i=1}^{n}X_i^{(p+1)}(t-\theta_{n2}\beta_n):=B_n$$

and, from A3 one obtains $|B_n|\le m_{(p+1)}(\alpha_n+\beta_n)\sqrt{n}\to 0$ *a.s.* hence the first result. Concerning the second result, we may write

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left[X_i^{(p)}(t+\alpha_n)-X_i^{(p)}(t-\beta_n)\right]-E(X_1^{(p)}(t_0+)-X_1^{(p)}(t_0-))\right)$$
$$=\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left[X_i^{(p)}(t_0+)-X_i^{(p)}(t_0-)\right]-E(X_1^{(p)}(t_0+)-X_1^{(p)}(t_0-))\right)+B_n.$$

Now, by using Proposition 2.1, we obtain the central limit theorem and the condition $B_n\to 0$ a.s. gives the result. □

Remark. It is also possible to detect more jumps, but the proof is rather intricate. Also, the boundedness of A3 is rather strong, but it will be possible to replace that assumption with the existence of an exponential moment.

Acknowledgement

I am grateful to the referee for fruitful suggestions.

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