C. R. Acad. Sci. Paris, Ser. I 354 (2016) 701–706

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic geometry

Logarithmic geometry and the Milnor fibration

Géometrie logarithmique et fibration de Milnor

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ARTICLE INFO

Article history:
Received 15 January 2016
Accepted after revision 8 April 2016
Available online 10 May 2016

Presented by Claire Voisin

ABSTRACT

Inspired by a description of the logarithmic space of Kato and Nakayama in terms of real oriented blowups, we describe Milnor fibrations and related constructions used by A'Campo in the language of logarithmic geometry.

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RÉSUMÉ

Inspiré par une description de l'espace logarithmitique de Kato et Nakayama à l'aide des éclatements réels orientés, nous décrivons la fibration de Milnor et des constructions utilisées par A'Campo en termes de géométrie logarithmique.

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1. Introduction

We fix $f \in \mathbb{C}[x_1, \ldots, x_{d+1}]$ and we denote its zero locus by $V(f)$. The singularity of f in a point $x \in V(f)$ can be studied in many ways. A classical approach is to consider the map

$$f : X \to D : x \mapsto f(x),$$

where $X = B(x, \varepsilon) \cap f^{-1}(D)$, $B(x, \varepsilon)$ is the open ball centered at x with radius ε in $\mathbb{C}^{d+1}$ and D is the open disc in $\mathbb{C}$ of radius η around 0. Above $D \setminus \{0\}$, this map f is a locally trivial fibration if $0 < \eta \ll \varepsilon \ll 1$. It is called the Milnor fibration of f at x, and its fibre is called the Milnor fibre of f at x. Its cohomology is equipped with a canonical monodromy action, which is induced by the generator of the fundamental group of $D \setminus \{0\}$.

In order to study this fibration more closely, A'Campo used in [1] a locally trivial fibration that has the same homotopy type as the Milnor fibration. He starts from an embedded resolution of $V(f)$ and uses the notion of real oriented blowups to construct several new spaces. One of these is actually a locally trivial fibration over S^1 and the associated monodromy action can be described explicitly, which makes it easier to study the Milnor fibration.

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[1] Supported by the KU Leuven grant OT/11/069.

http://dx.doi.org/10.1016/j.crma.2016.04.005
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On the other hand, Kato and Nakayama constructed in [2] a logarithmic space $X^{\log}$ associated with a complex analytic log variety. This construction can also be expressed by real oriented blowups if we are in the strict normal crossing case.

Using the language of log geometry, we construct a new continuous map $f_{MF}: X_{MF} \to S^1$. This X_{MF} is a closed subspace of the extended logspace X^{extLog}, which will be defined in Section 4 and is an extension of $X^{\log}$. The continuous map f_{MF} is a locally trivial fibration and we show that the fibres of f_{MF} are homotopic to the fibres of the Milnor fibration by relating this to the constructions used by A'Campo.

Finally, we apply this construction to special formal $\mathbb{C}[[t]]$-schemes instead of $f: X \to D$. This leads us to our main application and result. It states that the analytic Milnor fibre, introduced by Nicaise and Sebag in [4], determines the homotopy type of the Milnor fibration.

From now on, we will assume that $X_0 = f^{-1}(0)$ has strict normal crossings, as A'Campo did in [1]. We will comment on what to do if this is not the case in Remark 1.

2. Preliminaries

By **variety**, we will mean a complex analytic variety. For every morphism of varieties $f: X \to Y$, we will denote by $f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$ the corresponding morphism of sheaves. The underlying topological space of a variety X will also be denoted by X.

Given a sheaf $\mathcal{F}$ on X and $x \in X$, we denote by $\mathcal{F}_x$ the stalk of the sheaf $\mathcal{F}$ at x. Moreover, given $a \in \mathcal{F}(U)$ for some open U and $x \in U$, we denote by a_x the germ corresponding to a.

2.1. Monoids

We denote by $\mathbb{R}^{\geq 0}$ the set of non-negative real numbers, by $\mathbb{N}$ the set of non-negative integers and we set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We will consider the following monoids $\mathbb{R} = (\mathbb{R}, +)$, $\mathbb{N} = (\mathbb{N}, +)$, $\mathbb{R}^{\geq 0}_{\times} = (\mathbb{R}^{\geq 0}, \times)$, $\mathbb{R}^{\geq 0}_{+} = (\mathbb{R}^{\geq 0}, +)$ and $S^1 = (S^1, \cdot)$. A monoid is called **fine** if it is integral and finitely generated, and it is called **fs** if it is fine and saturated.

We call a monoid P **regular** if it is isomorphic to $\mathbb{N}^n$ for some n. A regular monoid is automatically fs. Under the isomorphism $\operatorname{Hom}\left(\mathbb{N}^{n+1}, \mathbb{R}^{\geq 0}_{+}\right) \cong \left(\mathbb{R}^{\geq 0}_{+}\right)^{n+1}$, the standard simplex

$$\Delta_n = \left\{ (x_0, \ldots, x_n) \in (\mathbb{R}^{\geq 0})^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}$$

corresponds to $\left\{ \varphi \in \operatorname{Hom}\left(\mathbb{N}^{n+1}, \mathbb{R}^{\geq 0}_{+}\right) \mid \varphi(1, 1, \ldots, 1) = 1 \right\}$.

Assume P is a monoid generated by finitely many elements $e_1, \ldots, e_n$. Let $\operatorname{Hom}(P, S^1)$ be endowed with the pointwise convergence topology. We can consider it to be a subset of $\left(S^1\right)^n$ by mapping φ in $\operatorname{Hom}(P, S^1)$ to $(\varphi(e_1), \ldots, \varphi(e_n))$ and this topology is exactly the induced topology. Hence, $\operatorname{Hom}(P, S^1)$ is a compact topological space since it is a closed subspace of $\left(S^1\right)^n$.

2.2. Log geometry

Fix a variety X. A **pre-log structure** on X is a morphism of sheaves of monoids $\alpha: \mathcal{M} \to \mathcal{O}_X$, where we consider the multiplication as the operation on $\mathcal{O}_X$. A **log structure** on X is a pre-log structure $\alpha: \mathcal{M} \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^{\times}) \to \mathcal{O}_X^{\times}$ is an isomorphism.

A **log variety** is a variety endowed with a log structure. Given log varieties X and Y, a **log morphism** $f: X \to Y$ consists of a morphism of varieties $f: X \to Y$ and a morphism of sheaves of monoids $f^{\#}: \mathcal{M}_Y \to f_*\mathcal{M}_X$ such that $f^{\#} \circ \alpha_Y = f_*\alpha_X \circ f^{\#}$.

Example 1. Let Z be a closed subset of a variety X. Define the sheaf of monoids $\mathcal{M}_Z$ on X by

$$\mathcal{M}_Z(V) = \{ f \in \mathcal{O}_X(V) \mid f_{|V \setminus Z} \text{ is invertible} \}$$

for every open V in X. We call this the log structure induced by Z.

Example 2. We will denote by T the log variety whose underlying variety is a point and whose log structure is given by

$$\mathbb{R}^{\geq 0}_{\times} \times S^1 \to \mathbb{C} : (r, s) \to rs.$$

We now introduce the notion of a chart. The log structure $\mathcal{M}^a$ induced by the pre-log structure $\alpha: \mathcal{M} \to \mathcal{O}_X$ is defined by $\mathcal{M} \oplus_{\mathcal{O}_X} \mathcal{O}_X^{\times} \to \mathcal{O}_X$. We have $\mathcal{M}^a_x = \mathcal{M}_x \oplus_{\mathcal{O}_{X,x}} \mathcal{O}_x^{\times}$ for all $x \in X$. Let X be a log variety with log structure $\alpha: \mathcal{M} \to \mathcal{O}_X$. A **chart** on an open U of X is a monoid morphism $P \to \mathcal{M}_X(U)$ such that the log structure on U induced by $\underline{P} \to \mathcal{O}_U$ is the log structure $\mathcal{M}$ restricted to U. Here is $\underline{P}$ the constant sheaf with values in P and the morphism is obtained by considering $P \to \mathcal{M}_X(U) \xrightarrow{\alpha} \mathcal{O}_X(U)$.

Example 3. Let X be the affine line $\mathbb{A}^1_{\mathbb{C}}$ with coordinate z and set $Z = \{0\}$. Then $\mathcal{M}_Z$ is a log structure and $\mathbb{N} \to \mathcal{M}_Z(X) : n \mapsto z^n$ is a chart.

Example 4. Let X be a variety and let E be a strict normal crossing divisor on X. Fix a point x in X and consider an open neighborhood U of x and local coordinates $x_1, \ldots, x_n$ at x such that $E \cap U = \cup_{i=1}^r V(x_i)$. Then the morphism $\mathbb{N}^r \to \mathcal{M}_E(U) : (n_1, \ldots, n_r) \mapsto \prod_{i=1}^r x_i^{n_i}$ is a chart on U for $\mathcal{M}_E$. This implies that $\mathcal{M}_{E,y}$ is the submonoid of $\mathcal{O}_{X,y}$ generated by $\mathcal{O}^\times_{X,y}$ and $x_1, \ldots, x_r$ for all $y \in U$.

A **fs log variety** is a log variety X with log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$ such that the stalks of the sheaf $\mathcal{M}/\mathcal{O}^\times$ are fs at all points of X and such that for all $x \in X$ there exist a neighborhood U of x and a chart $P \to \mathcal{M}(U)$ such that P is fs.

3. Log spaces

We briefly review the definition of the log space $X^{\log}$ of Kato and Nakayama from [2]. Fix a fs log variety X. Define the **logarithmic space** $X^{\log}$ to be the set

$$X^{\log} = \left\{ (x, \varphi) \;\middle|\; x \in X, \varphi \in \mathrm{Hom}(\mathcal{M}_x, S^1) \text{ such that } \varphi(f) = \frac{f(x)}{|f(x)|} \text{ for all } f \in \mathcal{O}^\times_{X,x} \right\}.$$

This log space $X^{\log}$ can also be viewed as

$$X(T) = \mathrm{Hom}(T, X),$$

where T is the log variety introduced in Example 2. Note that it has a natural map $\pi_X : X^{\log} \to X : (x, \varphi) \mapsto x$.

Assume there exists a chart $\beta : P \to \mathcal{M}(U)$ on an open U of X such that P is fs. Then we have a natural map $i : U^{\log} \to U \times \mathrm{Hom}\left(P, S^1\right) : (x, \varphi) \mapsto (x, \varphi \circ \beta_x)$ in $U \times \mathrm{Hom}(P, S^1)$, where $\beta_x : P \to \mathcal{M}_x$. We endow $X^{\log}$ with the weakest topology such that π_X is continuous and such that i is continuous for every open U for which there exists a chart $\beta : P \to \mathcal{M}_X(U)$ on U with P fs. It is sufficient to check this condition on any atlas of charts.

An element $p \in P$ can be evaluated in a point $x \in U$ by composing the maps $\beta_x : P \to \mathcal{M}_x$ and $\mathcal{M}_x \to \mathcal{O}_{X,x}$. The map i is injective and the image is the closed subset

$$\{(x, \varphi) \in U \times \mathrm{Hom}(P, S^1) \mid p(x) = \varphi(p)|p(x)| \text{ for all } p \in P\}.$$

An easy consequence then is that the map $\pi_X : X^{\log} \to X$ is proper and continuous.

Example 5. We continue Example 3. Using the chart $\mathbb{N} \to \mathcal{M}_Z(X) : n \mapsto z^n$, we obtain

$$X^{\log} = \{(z, \varphi) \in \mathbb{C} \times \mathrm{Hom}(\mathbb{N}, S^1) \mid \varphi(1)|z| = z\} = \mathbb{R}^{\geq 0} \times S^1,$$

where $\pi_X : X^{\log} \to X : (r, s) \mapsto rs$.

Example 6. Consider the affine n-dimensional space $X = \mathbb{A}^n_{\mathbb{C}}$ with coordinates $z_1, \ldots, z_n$ and the log structure $\mathcal{M}_E$ where $E = \cup_{i=1}^n V(z_i)$. This is the log structure induced by the map $\beta : \mathbb{N}^n \to \mathcal{O}_X(X) : (a_1, \ldots, a_n) \mapsto \prod_{i=1}^n z_i^{a_i}$. In this case, $X^{\log}$ is homeomorphic to $\left(S^1\right)^n \times \left(\mathbb{R}^{\geq 0}\right)^n$ and $\pi_X : X^{\log} \to X$ is

$$\left(S^1\right)^n \times \left(\mathbb{R}^{\geq 0}\right)^n \to X : (r_1, \ldots, r_n, s_1, \ldots, s_n) \mapsto (r_1 s_1, \ldots, r_n s_n).$$

Recall our morphism $f : X \to D$ from the introduction. We equip X with the log structure $\mathcal{M} = \mathcal{M}_{X_0}$ induced by $X_0 = f^{-1}(0)$. On D we consider the log structure $\mathcal{M}_{\{0\}}$ induced by the origin. Hence $f : X \to D$ is a log morphism. This morphism is determined by a global section of $\mathcal{M}$, which we will also denote by f. We now can define the tube around $X_0 = f^{-1}(0)$ in X.

Definition 3.1. The **tube** $\mathcal{X}$ **of** $f : X \to D$ around the special fibre X_0 is defined to be

$$\mathcal{X} = \left\{ (x, \varphi) \in X^{\log} \mid f(x) = 0 \right\} \subseteq X^{\log}.$$

Remark that $X^{\log}$ is equipped with a continuous map $X^{\log} \to S^1 : (x, \varphi) \mapsto \varphi(f_x)$. Both this map and the restriction $f^{\mathrm{tube}} : \mathcal{X} \to S^1 : (x, \varphi) \mapsto \varphi(f_x)$ are locally trivial fibrations, which follows from the fact that X_0 has strict normal crossings.

4. Extending the log space

Recall that there exists a monodromy action on $X^{\log}$ by lifting a generator of the fundamental group of $D^{\log} = S^1 \times [0, \eta)$ since $f^{\log}$ is a locally trivial fibration. Our goal is to describe this action more easily. To do so, we will introduce an intermediate space X^{extLog} and consider its closed subspace X_{MF}, which turns out to be homotopic to $\mathcal{X}$.

Define $\tilde{T}$ to be the log variety whose underlying space is a point and log structure

$$\mathbb{R}^{\geq 0}_{\times} \times S^1 \times \mathbb{R}^{\geq 0}_{+} \to \mathbb{C} : (r, s, p) \mapsto rs.$$

Definition 4.1. Let X be a fs log variety. The **extended logarithmic space** X^{extLog} is defined as the set $X(\tilde{T}) = \text{Hom}(\tilde{T}, X)$. This is the same as the set

$$\left\{(x, \varphi, \psi) \,\middle|\, x \in X, \varphi \in \text{Hom}(\mathcal{M}_x, S^1), \psi \in \text{Hom}(\mathcal{M}_x, \mathbb{R}^{\geq 0}_{+}) \text{ such that } (x, \varphi) \in X^{\log}\right\}.$$

The natural morphism of log varieties $\tilde{T} \to T$ induces maps $\rho_X : X^{\text{extLog}} \to X$ and $\sigma_X : X^{\text{extLog}} \to X^{\log}$ with the property $\rho_X = \pi_X \circ \sigma_X$.

We endow X^{extLog} with the weakest topology such that ρ_X is continuous and such that the map

$$j : U^{\text{extLog}} \to U \times \text{Hom}(P, S^1) \times \text{Hom}(P, \mathbb{R}^{\geq 0}_{+}) : (x, \varphi, \psi) \mapsto (x, \varphi \circ \beta_x, \psi \circ \beta_x)$$

is continuous for every open U for which there exists a chart $\beta : P \to \mathcal{M}(U)$ with P fs.

Remark that

$$\mathcal{M}_x/\mathcal{O}^{\times}_{X,x} \cong (P \oplus_{\mathcal{O}_{X,x}} \mathcal{O}^{\times}_{X,x})/\mathcal{O}^{\times}_{X,x} \cong P/\beta_x^{-1}(\mathcal{O}^{\times}_{X,x}).$$

A morphism $\psi : \mathcal{M}_x \to \mathbb{R}^{\geq 0}_{+}$ is thus fully determined by the induced map $P/\beta_x^{-1}(\mathcal{O}^{\times}_{X,x}) \to \mathbb{R}^{\geq 0}_{+}$ since $\mathcal{O}^{\times}_{X,x}$ is a subgroup of the monoid $\mathcal{M}_x$ and must be send to a subgroup of $\mathbb{R}^{\geq 0}_{+}$, which can only by $\{0\}$. Hence the map j is injective and the image is the closed subspace defined by $p(x) = \varphi(p)|p(x)|$ and $\psi(p)p(x) = 0$ for all $p \in P$ where $(x, \varphi, \psi) \in X \times \text{Hom}(P, S^1) \times \text{Hom}(P, \mathbb{R}^{\geq 0}_{+})$.

In addition, we have a continuous map $X^{\text{extLog}} \to X^{\log} : (x, \varphi, \psi) \mapsto (x, \varphi)$ and a natural injection $X^{\log} \to X^{\text{extLog}} : (x, \varphi) \mapsto (x, \varphi, 0)$. Via this map, $X^{\log}$ is a closed subspace of X^{extLog} and the space $X^{\log}$ is a retract of X^{extLog}. This is also a consequence of the following stronger result.

Proposition 4.2. *There exists a strong deformation retraction of X^{extLog} onto $X^{\log}$.*

Proof. Define

$$X^{\text{extLog}} \times [0, 1] \to X^{\text{extLog}} : ((x, \varphi, \psi), \lambda) \mapsto (x, \varphi, (1 - \lambda) \cdot \psi) \tag{1}$$

where $(1 - \lambda) \cdot \psi : \mathcal{M}_x \to (\mathbb{R}_{\geq 0,+}) : m \mapsto (1 - \lambda)\psi(m)$. $\square$

We will now introduce our (candidate) monodromy action on X^{extLog}.

Definition 4.3. Consider $\lambda \in [0, 1]$. Define

$$g_\lambda : X^{\text{extLog}} \to X^{\text{extLog}} : (x, \varphi, \psi) \mapsto (x, \varphi \cdot \varphi_{\psi,\lambda}, \psi)$$

where $\varphi_{\psi,\lambda} : \mathcal{M}_x \mapsto S^1 : a \mapsto \exp(2\pi i \lambda \psi(a))$.

This definition is functorial in nature: given a morphism of fs log varieties $h : X \to Y$, we have an induced map $h^{\text{extLog}} : X^{\text{extLog}} \to Y^{\text{extLog}}$ and

$$h^{\text{extLog}} \circ g_\lambda = g_\lambda \circ h^{\text{extLog}}. \tag{2}$$

Example 7. Let $X = \mathbb{A}^1_{\mathbb{C}}$ as in Example 5. In this case we have

$$X^{\text{extLog}} = \left\{(z, s, q) \in \mathbb{C} \times S^1 \times \mathbb{R}^{\geq 0} \mid z = s|z|, zq = 0\right\} = \left\{(r, s, q) \in \mathbb{R}^{\geq 0} \times S^1 \times \mathbb{R}^{\geq 0} \mid rq = 0\right\}$$

and

$$g_\lambda : X^{\text{extLog}} \to X^{\text{extLog}} : (z, s, q) \mapsto (z, s \exp(2\pi i \lambda q), q).$$

Example 8. We continue Example 6. We find that

$$X^{\text{extLog}} = \left\{(r_1, \dots, r_n, s_1, \dots, s_n, q_1, \dots, q_n) \in \left(\mathbb{R}_+^{\geq 0}\right)^n \times \left(S^1\right)^n \times \left(\mathbb{R}_+^{\geq 0}\right)^n \,\middle|\, \forall i \in \{1, \dots, n\} : r_i q_i = 0\right\}$$

such that

$$g_\lambda : X^{\text{extLog}} \to X^{\text{extLog}} : (r_1, \dots, r_n, s_1, \dots, s_n, q_1, \dots, q_n)$$
$$\mapsto (r_1, \dots, r_n, s_1 \exp(2\pi i \lambda q_i), \dots, s_n \exp(2\pi i \lambda q_n), q_1, \dots, q_n).$$

5. Milnor fibration

We now continue looking at the log morphism $f : X \to D$. The induced continuous map on the corresponding extended log spaces can be described as follows:

$$X^{\text{extLog}} \to D^{\text{extLog}} : (x, \varphi, \psi) \mapsto (|f(x)|, \varphi(f_x), \psi(f_x)). \tag{3}$$

Definition 5.1. Define

$$X_{MF} := \{(x, \varphi, \psi) \in X^{\text{extLog}} \mid \psi(f) = 1\}$$

and we call

$$f_{MF} : X_{MF} \to S^1 : (x, \varphi, \psi) \mapsto \varphi(f),$$

the **A'Campo extension of the Milnor fibration** associated with f.

Remark that if $(x, \varphi, \psi) \in X_{MF}$, the fact that $\psi(f) = 1$ implies that $|f(x)| = 0$ and thus $x \in X_0 = f^{-1}(0)$. Hence X_{MF} closely resembles $\mathcal{X}$.

Example 9. We continue Example 7 and we consider $f = z^N$. Then $X_{MF} = S^1$ and $f_{MF} : S^1 \to S^1 : z \mapsto z^N$. Under this isomorphism g_λ becomes

$$g_{x,\lambda} : S^1 \to S^1 : z \mapsto e^{\frac{2\pi i \lambda}{N}} z$$

and thus this coincides with lifting a generator of the fundamental group.

Example 10. Consider X as in Example 8 and $f = \prod_{i=1}^n x_i^{N_i}$ such that $N_i > 0$ for all $i \in \{1, \dots, n\}$. We see that

$$X_{MF} = \left\{ (r_1, \dots, r_n, s_1, \dots, s_n, q_1, \dots, q_n) \in \left(\mathbb{R}^{\geq 0}\right)^n \times (S^1)^n \times \left(\mathbb{R}^{\geq 0}\right)^n \,\middle|\, \sum_{i=1}^n N_i q_i = 1, \forall i \in \{1, \dots, n\} : r_i q_i = 0 \right\}$$

and

$$f_{MF} : X_{MF} \to S^1 : (r_1, \dots, r_n, s_1, \dots, s_n, q_1, \dots, q_n) \mapsto \prod_{i=1}^n s_i^{N_i}.$$

We now can make the following observation: consider an element $x \in X$ in which $i > 0$ irreducible components of X_0 meet. If we compare $\mathcal{X}$ to X_{MF} above x, we have added a standard simplex Δ_{i-1}. A'Campo described X_{MF} in [1] exactly in terms of these standard simplices and defined g_λ similarly. He used however a different normalization of Δ_{i-1}.

We formulate now some properties concerning the map f_{MF} and X_{MF}.

Proposition 5.2.

(i) The map f_{MF} is a locally trivial fibration.
(ii) Under the homotopy defined in (1), X_{MF} is deformed into $\mathcal{X}$.
(iii) The space X_{MF} is invariant under g_λ for all $\lambda \in [0, 1]$.
(iv) Moreover, we have that

$$(f_{MF} \circ g_\lambda)(x, \varphi, \psi) = \exp(2\pi i \lambda) f_{MF}(x, \varphi, \psi)$$

for all $\lambda \in [0, 1]$ and $(x, \varphi, \psi) \in X_{MF}$.

Proof. (*i*): This is the construction described in [1] and the g_λ coincide. (*ii*): Remark that there exists $(x, \varphi, \psi) \in X_{MF}$ if and only if $f(x) = 0$. (*iii*) and (*iv*): This follows from equation (2). □

We call $M_{X_{MF}} := g_{1|X_{MF}} : X_{MF} \to X_{MF}$ **the (geometric) monodromy on** X_{MF}. It is a lift of a generator of the fundamental group of S^1. We conclude this section by remarking that this is exactly what we wanted.

Theorem 5.3. *The fibers of $f_{MF} : X_{MF} \to S^1$ are homotopic to the fibers of the Milnor fibration. Moreover, both can be deformed into $f^{\text{tube}} : \mathcal{X} \to S^1$.*

Proof. Remark for this that $X \setminus X_0 \subset X^{\log} \supset \mathcal{X}$ and that $\mathcal{X}$ can be deformed into X_{MF}. Since they behave well with respect to f, $f^{\log}$, f^{tube} and f_{MF}, the theorem follows. □

Remark 1. In the case where X_0 does not have strict normal crossings, we are still able to define X_{MF} and $\mathcal{X}$.

Fix an embedded resolution of singularities $\pi : X' \to X$ and consider $f' = f \circ \pi : X' \to D$. We then define $f^{\text{tube}} : \mathcal{X} \to S^1$ to be $(f')^{\text{tube}} : \mathcal{X}' \to S^1$. This definition however depends on the choice of embedded resolution, but Theorem 5.3 implies that $(f')^{\text{tube}} : \mathcal{X}' \to S^1$ is homotopic to $f : X \setminus X_0 \to D \setminus \{0\}$, and thus this definition is well-defined up to homotopy.

Similarly, we can define X_{MF} and g_λ together with the map f_{MF} up to homotopy.

6. The analytic Milnor fiber

6.1. Special formal schemes

We can extend the definition of $f^{\text{tube}} : \mathcal{X} \to S^1$ and X_{MF} to the setting of special formal $\mathbb{C}[[t]]$-schemes. Given a fs log special formal $\mathbb{C}[[t]]$-scheme $\mathfrak{X}$, we can construct topological spaces $\mathfrak{X}^{\log}$ and $\mathfrak{X}^{\text{extLog}}$ generalizing the definitions of $X^{\log}$ and X^{extLog} given above.

We endow $\operatorname{Spf}\mathbb{C}[[t]]$ with its standard log structure induced by $\mathbb{N} \to \mathbb{C}[[t]] : n \mapsto t^n$. Then $(\operatorname{Spf}\mathbb{C}[[t]])^{\log} \cong S^1$. Now consider a generically smooth special formal $\mathbb{C}[[t]]$-scheme $\mathfrak{X}$ and fix a resolution of singularities $\mathfrak{X}' \to \mathfrak{X}$. See [3, Section 2] for more details on special formal $\mathbb{C}[[t]]$-schemes. We can equip $\mathfrak{X}'$ with the log structure $\mathcal{M}_\eta$, the log structure induced by the generic fibre of $\mathfrak{X}'$. We define $\mathfrak{X}^{\text{tube}}$ to be $\mathfrak{X}'^{\log}$. It comes equipped with a morphism $p : \mathfrak{X}^{\text{tube}} \to \operatorname{Spf}\mathbb{C}[[t]]^{\text{tube}} = S^1$ of topological spaces. Assuming a suitable weak factorization theorem for generically smooth special formal schemes, one can show that this definition is independent of the choice of $\mathfrak{X}'$ up to homotopy. The necessary weak factorization theorem is part of a work in progress by Abramovich and Temkin. It can also be shown that $p : \mathfrak{X}^{\text{tube}} \to S^1$ is a locally trivial fibration by exploiting the topology of the special fiber $\mathfrak{X}'_s$ of $\mathfrak{X}'$ and the structure of S^1.

6.2. The analytic Milnor fibre

This has interesting applications to the study of the analytic Milnor fibre, which was introduced by Nicaise and Sebag in [4]. Consider a smooth, irreducible algebraic $\mathbb{C}$-variety X and a dominant morphism $f : X \to \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t]$. Let $x \in V(f)$ and denote by F_x the topological Milnor fiber of f at x. The analytic Milnor fiber $\mathcal{F}_x$ of f at x is, by definition, the generic fibre of the special formal scheme $\operatorname{Spf}\widehat{\mathcal{O}}_{X,x} \to \operatorname{Spf}\mathbb{C}[[t]]$ obtained from f by completion at x. It is a smooth analytic space over the non-Archimedean field $\mathbb{C}((t))$.

Let $\mathbb{C}\{\{t\}\}$ be an algebraic closure of $\mathbb{C}((t))$. Theorem 9.2 in [4] states that there exist canonical isomorphisms

$$H^i_{sing}(F_x, \mathbb{Q}_l) \cong H^i\left(\mathcal{F}_x \widehat{\times} \widehat{\mathbb{C}\{\{t\}\}}, \mathbb{Q}_l\right),$$

such that the monodromy action on the left-hand side corresponds to the action of the canonical topological generator of $\operatorname{Gal}(\mathbb{C}\{\{t\}\}/\mathbb{C}((t)))$ on the right-hand side. Hence, $\mathcal{F}_x$ captures the cohomology of the Milnor fibre F_x, together with the monodromy action.

Our results imply the following theorem.

Theorem 6.1. *The analytic Milnor fibre $\mathcal{F}_x$ determines the homotopy type of F_x.*

Indeed, it follows from our results that, for every special formal $\mathbb{C}[[t]]$-model $\mathfrak{X}$ of $\mathcal{F}_x$, the map $p : \mathfrak{X}^{\text{tube}} \to S^1$ is homotopic to the Milnor fibration of f at x.

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