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Functional analysis

On connected Lie groups and the approximation property

Sur les groupes de Lie connexes et la propriété d'approximation

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ABSTRACT

Recently, a complete characterization of connected Lie groups with the Approximation Property was given. The proof used the newly introduced property (T*). We present here a short proof of the same result avoiding the use of property (T*). Using property (T*), however, the characterization is extended to almost connected locally compact groups. We end with some remarks about the difficulty of going beyond the almost connected case.

RÉSUMÉ

Une caractérisation complète des groupes de Lie connexes avec la propriété d'approximation a été obtenue récemment. La preuve utilisait la propriété (T*), nouvellement introduite. Nous présentons ici une preuve courte du même résultat sans utiliser la propriété (T*). En utilisant (T*), cependant, la caractérisation est étendue aux groupes localement compacts presque connexes. Nous concluons avec quelques remarques sur la difficulté d'aller au-delà du cas presque connexe.

The Fourier algebra $A(G)$ of a locally compact group G was introduced by Eymard [5] and can be defined as the coefficient space of the left regular representation of G on $L^2(G)$,

$$A(G) = \{f * \widetilde{g} \mid f, g \in L^2(G)\},$$

where $\widetilde{g}(x) = \overline{g(x^{-1})}$. One can identify $A(G)$ with the predual of the group von Neumann algebra associated with the left regular representation. A multiplier φ of $A(G)$ is called completely bounded if its transposed operator M_φ is a completely bounded operator on the group von Neumann algebra. The completely bounded multiplier norm of φ is defined as

$$\|\varphi\|_{M_0A} = \|M_\varphi\|_{\text{cb}}.$$

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The space of completely bounded multipliers is denoted $M_0A(G)$. One has an inclusion $A(G) \subseteq M_0A(G)$. It was shown in [4, Proposition 1.10] that the space $M_0A(G)$ has an isometric predual obtained as the completion of $L^1(G)$ in the norm

$$\|f\|_{M_0A(G)_*} = \sup\left\{ \left| \int_G f(x)\varphi(x)\,\mathrm{d}x \right| : \varphi \in M_0A(G),\ \|\varphi\|_{M_0A} \le 1 \right\}.$$

Remark 1. In general, $M_0A(G)$ does not have a unique predual nor a unique weak* topology. The predual always refers to the one constructed explicitly above, and the weak* topology on $M_0A(G)$ is the weak* topology coming from this explicit predual.

Following Haagerup and Kraus [9], a locally compact group G has the Approximation Property (short: AP) if there exists a net in $A(G)$ converging to 1 in the weak* topology of $M_0A(G)$. Haagerup and Kraus showed that many groups have this property, e.g., all weakly amenable groups. This includes solvable groups, compact groups, and simple Lie groups of real rank one [3,11]. They showed moreover that the AP is preserved by passing to closed subgroups and preserved under group extensions (as opposed to weak amenability). It is also well-known (and routine to verify) that if K is a compact normal subgroup in G, then G has the AP if and only if G/K has the AP. We refer to [3,4,9] for details on completely bounded multipliers and the AP.

Haagerup and Kraus conjectured that the group $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, but the conjecture was settled only much later by Lafforgue and de la Salle [10]. Not long after, Haagerup and de Laat showed more generally that any simple Lie group of real rank at least two does not have the AP [6,7]. Finally, a complete characterization of connected Lie groups with the AP was given in [8] (see also Theorem 2 below). The proof in [8] involved among other things the property (T*), introduced in [8]. We give below a much shorter proof of the characterization of connected Lie groups with the AP that avoids the use of property (T*).

To state the characterization of connected Lie groups with the AP, we first recall the Levi decomposition of such a group (see [15, Section 3.18] for details). For a connected Lie group G, one can decompose its Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{r}$ is the solvable radical and $\mathfrak{s}$ is a semisimple subalgebra. One can further decompose $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ into simple summands $\mathfrak{s}_i$ $(i = 1,\ldots,n)$. Let R, S, and S_i denote the corresponding connected Lie subgroups of G. One has $G = RS$ as a set. This is called a Levi decomposition of G.

The subgroup R is solvable, normal, and closed. The subgroup S is semisimple and locally isomorphic to the direct product $S_1 \times \cdots \times S_n$ of simple factors, but it need not be closed in G.

Theorem 2 *([8]). Let G be a connected Lie group, let $G = RS$ denote a Levi decomposition, and suppose that S is locally isomorphic to the product $S_1 \times \cdots \times S_n$ of connected simple factors. Then the following are equivalent.*

(1) *The group G has the AP.*
(2) *The group S has the AP.*
(3) *The groups S_i, with $i = 1,\ldots,n$, have the AP.*
(4) *The real rank of the groups S_i, with $i = 1,\ldots,n$, is at most 1.*

Proof. The equivalence of (3) and (4) is given by [7, Theorem 5.1].

Suppose (4) holds. We show that (1) and (2) hold. The proof of this implication is the proof from [8]. For completeness, we include it: The group R is solvable, so it has the AP. Since the AP is preserved under group extensions [9, Theorem 1.15], it is therefore enough to prove that G/R has the AP. The Lie algebra of G/R is the Lie algebra of S, so to prove (i) and (ii), it suffices to prove that any connected Lie group S' locally isomorphic to $S_1 \times \cdots \times S_n$ has the AP. Using [9, Theorem 1.15] again, we may assume that the center of S' is trivial. If Z_i denotes the center of S_i, then $S' \simeq (S_1/Z_1) \times \cdots \times (S_n/Z_n)$. Each group S_i/Z_i has real rank at most 1 and finite center and hence has the AP [3]. Hence S' has the AP.

Suppose (4) does not hold. We show that (1) does not hold (which also proves that (2) does not hold). Fix some i. The closure $\bar{S}_i$ of S_i in G is of the form S_iC for some connected, compact, central subgroup of G (see [13, p. 614]). Let $G' = G/C$, and let $\pi\colon G \to G'$ be the quotient homomorphism. As C is compact, π is a closed map, and $\pi(S_i) = \pi(\bar{S}_i)$ is a closed subgroup of G'. Since $\ker\pi \cap S_i$ is contained in the center of S_i and therefore is discrete (in the topology of S_i), the group $\pi(S_i)$ is locally isomorphic to S_i. If now the real rank of S_i is at least 2, then $\pi(S_i)$ does not have the AP [7, Theorem 5.1]. Hence G' does not have the AP, and it follows that G does not have the AP. □

A characterization of almost connected groups with the AP

Let G be a locally compact group. Recall from [8] that there is a unique left invariant mean on $M_0A(G)$, and we say that G has property (T*) if this mean is weak* continuous. It is clear that non-compact groups with property (T*) do not have the AP. We establish the following converse for almost connected groups. Recall that G is almost connected if the quotient group G/G_0 is compact, where G_0 denotes the connected component of the identity in G.

Theorem 3. *For an almost connected locally compact group G, the following are equivalent.*

(1) *The group G has the AP.*
(2) *No closed non-compact subgroup of G has property (T*).*

Proof. (1) $\Longrightarrow$ (2) is clear and holds also without the assumption of almost connectedness. We prove (2) $\Longrightarrow$ (1). Suppose G does not have AP. We prove the existence of a closed non-compact subgroup of G with property (T*).

By [9, Theorem 1.15], we may assume that G is connected. Then there is a compact normal subgroup K of G such that G/K is a Lie group (see [12, Theorem 4.6]). Then G/K is a connected Lie group without the AP. It follows from the remark after Theorem C in [8] that we can find a closed, non-compact subgroup $H_0 \leq G/K$ with property (T*). Let H be the inverse image of H_0 in G. By [8, Proposition 5.13], H has property (T*), and H is clearly a closed non-compact subgroup of G. □

Remark 4. Theorem 3 is not true in general without the assumption of almost connectedness. In fact, there are discrete groups without the AP and without infinite subgroups with property (T*). This is not a new result, but simply a compilation of known results, the last ingredient being the recent result of Arzhantseva and Osajda (see [14, Theorem 2] and [1]) about the existence of discrete groups without property A but with the Haagerup property. Let G be such a group.

As property (T*) implies property (T) (see [8, Proposition 5.3]), and property (T) is an obstruction to the Haagerup property, it is clear that G has no infinite subgroups with property (T*).

On the other hand, if the discrete group G had the AP then its reduced group C*-algebra would have the strong operator approximation property. This is known to imply exactness of the group C*-algebra, which implies that G is an exact group. Finally, exactness for groups is equivalent to property A. This shows that G cannot have the AP. Proofs of all the claimed statements can be found in [2, Chapters 5 and 12].

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