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Bourgain–Brézis–Mironescu formula for magnetic operators

Formule de Brézis–Bourgain–Mironescu pour des opérateurs magnétiques

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ABSTRACT

We prove a Bourgain–Brézis–Mironescu-type formula for a class of nonlocal magnetic spaces, which builds a bridge between a fractional magnetic operator recently introduced and the classical theory.

RÉSUMÉ

On démontre une formule du type Bourgain–Brézis–Mironescu pour une classe d'espaces magnétiques non locaux, qui jette un pont entre un opérateur magnétique fractionnaire récemment introduit et la théorie classique.

1. Introduction

Let $s \in (0,1)$ and $N > 2s$. If $A : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function, the nonlocal operator

$$(-\Delta)^s_A u(x) = c(N,s) \lim_{\varepsilon \searrow 0} \int_{B^c_\varepsilon(x)} \frac{u(x) - \mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \qquad x \in \mathbb{R}^N,$$

has been recently introduced in [6], where the ground-state solutions to $(-\Delta)^s_A u + u = |u|^{p-2}u$ in the three-dimensional setting have been obtained via concentration compactness arguments. If $A = 0$, then the above operator is consistent with the usual notion of fractional Laplacian. The motivations that led to its introduction are carefully described in [6] and rely essentially on the Lévy–Khintchine formula for the generator of a general Lévy process. We point out that the normalization constant $c(N,s)$ satisfies

$$\lim_{s \nearrow 1} \frac{c(N,s)}{1-s} = \frac{4N\Gamma(N/2)}{2\pi^{N/2}},$$

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where Γ denotes the Gamma function. For the sake of completeness, we recall that different definitions of nonlocal magnetic operator are viable, see, e.g., [8,9]. All these notions aim to extend the well-known definition of the magnetic Schrödinger operator

$$-\big(\nabla - \mathrm{i}A(x)\big)^2 u = -\Delta u + 2\mathrm{i}A(x)\cdot\nabla u + |A(x)|^2 u + \mathrm{i}u\,\mathrm{div}A(x),$$

namely the differential of the energy functional

$$\mathcal{E}_A(u) = \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2 \mathrm{d}x,$$

for which we refer the reader to [1,2,11] and the included references. In order to corroborate the justification for the introduction of $(-\Delta)^s_A$, in this note, we prove that a well-known formula due to Bourgain, Brézis and Mironescu (see [3,4,10]) for the limit of the Gagliardo semi-norm of $H^s(\Omega)$ as $s \nearrow 1$ extends to the magnetic setting. As a consequence, in a suitable sense, from the nonlocal to the local regime, it holds

$$(-\Delta)^s_A u \rightsquigarrow \big(\nabla - \mathrm{i}A(x)\big)^2 u, \qquad \text{for } s \nearrow 1.$$

We consider

$$[u]_{H^1_A(\Omega)} := \sqrt{\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2 \mathrm{d}x},$$

and define $H^1_A(\Omega)$ as the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H^1_A(\Omega)} < \infty$ endowed with the norm

$$\|u\|_{H^1_A(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + [u]^2_{H^1_A(\Omega)}}.$$

Our main results are the following.

Theorem 1.1 *(Magnetic Bourgain–Brézis–Mironescu). Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A \in C^2(\bar{\Omega})$. Then, for every $u \in H^1_A(\Omega)$, we have*

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x) - \mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y = K_N \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2 \mathrm{d}x,$$

where

$$K_N = \frac{1}{2}\int_{\mathbb{S}^{N-1}} |\omega\cdot\mathbf{e}|^2 \mathrm{d}\mathcal{H}^{N-1}(\omega), \tag{1.1}$$

being $\mathbb{S}^{N-1}$ the unit sphere and $\mathbf{e}$ any unit vector in $\mathbb{R}^N$.

As a variant of Theorem 1.1, if $H^1_{0,A}(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ in $H^1_A(\Omega)$, we get the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded and $A \in C^2(\bar{\Omega})$. Then, for every $u \in H^1_{0,A}(\Omega)$, we have:*

$$\lim_{s\nearrow 1}(1-s)\int_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y = K_N \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2 \mathrm{d}x.$$

Notations. Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $L^2(\Omega, \mathbb{C})$ the Lebesgue space of complex valued functions with summable square. For $s \in (0,1)$, the magnetic Gagliardo semi-norm is

$$[u]_{H^s_A(\Omega)} := \sqrt{\int_\Omega\int_\Omega \frac{|u(x) - \mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y}.$$

We denote by $H^s_A(\Omega)$ the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H^s_A(\Omega)} < \infty$ endowed with

$$\|u\|_{H^s_A(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + [u]^2_{H^s_A(\Omega)}}.$$

We denote by $B(x_0, R)$ the ball in $\mathbb{R}^N$ of center x_0 and radius $R > 0$. For any set $E \subset \mathbb{R}^N$, we will denote by E^c the complement of E. For $A, B \subset \mathbb{R}^N$ open and bounded, $A \Subset B$ means $\bar{A} \subset B$.

2. Preliminary results

We start with the following Lemma.

Lemma 2.1. *Assume that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded. Then, for any compact $V \subset \mathbb{R}^N$ with $\Omega \Subset V$, there exists $C = C(A, V) > 0$ such that*

$$\int_{\mathbb{R}^N} |u(y+h) - \mathrm{e}^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)} u(y)|^2 \mathrm{d}y \le C|h|^2 \|u\|^2_{H^1_A(\mathbb{R}^N)},$$

for all $u \in H^1_A(\mathbb{R}^N)$ such that $u = 0$ on V^c and any $h \in \mathbb{R}^N$ with $|h| \le 1$.

Proof. Assume first that $u \in C^\infty_0(\mathbb{R}^N)$ with $u = 0$ on V^c. Fix $y, h \in \mathbb{R}^N$ and define

$$\varphi(t) := \mathrm{e}^{\mathrm{i}(1-t)h\cdot A\left(y+\frac{h}{2}\right)} u(y+th), \quad t \in [0,1].$$

Then we have

$$u(y+h) - \mathrm{e}^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)} u(y) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\mathrm{d}t,$$

and since

$$\varphi'(t) = \mathrm{e}^{\mathrm{i}(1-t)h\cdot A\left(y+\frac{h}{2}\right)}\, h \cdot \Big(\nabla_y u(y+th) - \mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big),$$

by Hölder inequality we get

$$|u(y+h) - \mathrm{e}^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)} u(y)|^2 \le |h|^2 \int_0^1 \Big|\nabla_y u(y+th) - \mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big|^2 \mathrm{d}t.$$

Therefore, integrating with respect to y over $\mathbb{R}^N$ and using Fubini's Theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(y+h) - \mathrm{e}^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)} u(y)|^2 dy &\le |h|^2 \int_0^1 \mathrm{d}t \int_{\mathbb{R}^N} \Big|\nabla_y u(y+th) - \mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big|^2 \mathrm{d}y \\
&= |h|^2 \int_0^1 \mathrm{d}t \int_{\mathbb{R}^N} \Big|\nabla_z u(z) - \mathrm{i}A\Big(z+\frac{1-2t}{2}h\Big)u(z)\Big|^2 \mathrm{d}z \\
&\le 2|h|^2 \int_{\mathbb{R}^N} |\nabla_z u(z) - \mathrm{i}A\,(z)\,u(z)|^2 \mathrm{d}z \\
&\quad + 2|h|^2 \int_V \Big|A\Big(z+\frac{1-2t}{2}h\Big) - A(z)\Big|^2 |u(z)|^2 \mathrm{d}z.
\end{aligned}$$

Then, since A is bounded on the set V, we have for some constant $C > 0$

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(y+h) - \mathrm{e}^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)} u(y)|^2 \mathrm{d}y &\le C|h|^2 \left(\int_{\mathbb{R}^N} |\nabla_z u(z) - \mathrm{i}A\,(z)\,u(z)|^2 \mathrm{d}z + \int_{\mathbb{R}^N} |u(z)|^2 \mathrm{d}z \right) \\
&= C|h|^2 \|u\|^2_{H^1_A(\mathbb{R}^N)}.
\end{aligned}$$

When dealing with a general u we can argue by a density argument. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $V \subset \mathbb{R}^N$ a compact set with $\Omega \Subset V$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded. Then there exists $C(\Omega, V, A) > 0$ such that for any $u \in H^1_A(\Omega)$ there exists $Eu \in H^1_A(\mathbb{R}^N)$ such that $Eu = u$ in Ω, $Eu = 0$ in V^c and*

$$\|Eu\|_{H^1_A(\mathbb{R}^N)} \le C(\Omega, V, A)\|u\|_{H^1_A(\Omega)}.$$

Proof. Observe that, for any bounded set $W \subset \mathbb{R}^N$ there exist $C_1(A, W), C_2(A, W) > 0$ with

$$C_1(A, W)\|u\|_{H^1(W)} \le \|u\|_{H^1_A(W)} \le C_2(A, W)\|u\|_{H^1(W)}, \quad \text{for any } u \in H^1(W).$$

This follows easily, via simple computations, by the definition of the norm of $H^1_A(W)$ and in view of the local boundedness assumption on the potential A. Now, by the standard extension property for $H^1(\Omega)$ (see, e.g., [7, Theorem 1, p. 254]), there exists $C(\Omega, V) > 0$ such that, for any $u \in H^1(\Omega)$, there exists a function $Eu \in H^1(\mathbb{R}^N)$ such that $Eu = u$ in Ω, $Eu = 0$ in V^c and $\|Eu\|_{H^1(\mathbb{R}^N)} \le C(\Omega, V)\|u\|_{H^1(\Omega)}$. Then, for any $u \in H^1_A(\Omega)$, we get

$$\begin{aligned}\|Eu\|_{H^1_A(\mathbb{R}^N)} &= \|Eu\|_{H^1_A(V)} \le C_2(A, V)\|Eu\|_{H^1(V)} = C_2(A, V)\|Eu\|_{H^1(\mathbb{R}^N)} \\ &\le C(\Omega, V)C_2(A, V)\|u\|_{H^1(\Omega)} \le C(\Omega, V)C_2(A, V)C_1^{-1}(A, \Omega)\|u\|_{H^1_A(\Omega)},\end{aligned}$$

which concludes the proof. □

We can now prove the following result:

Lemma 2.3. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Let $u \in H^1_A(\Omega)$ and $\rho \in L^1(\mathbb{R}^N)$ with $\rho \ge 0$. Then*

$$\int_\Omega \int_\Omega \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho(x-y)\,dx\,dy \le C\|\rho\|_{L^1}\|u\|^2_{H^1_A(\Omega)}$$

where C depends only on Ω and A.

Proof. Let $V \subset \mathbb{R}^N$ be a fixed compact set with $\Omega \Subset V$. Given $u \in H^1_A(\Omega)$, by Lemma 2.2, there exists a function $\tilde{u} \in H^1_A(\mathbb{R}^N)$ with $\tilde{u} = u$ on Ω and $\tilde{u} = 0$ on V^c. By Lemma 2.1 and 2.2,

$$\int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{ih\cdot A\left(y+\frac{h}{2}\right)}\tilde{u}(y)|^2 dy \le C|h|^2\|\tilde{u}\|^2_{H^1_A(\mathbb{R}^N)} \le C|h|^2\|u\|^2_{H^1_A(\Omega)}, \tag{2.1}$$

for some positive constant C depending on Ω and A. Then, in light of (2.1), we get

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho(x-y)\,dx\,dy &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho(h)\frac{|\tilde{u}(y+h) - e^{ih\cdot A\left(y+\frac{h}{2}\right)}\tilde{u}(y)|^2}{|h|^2}dy\,dh \\ &= \int_{\mathbb{R}^N} \frac{\rho(h)}{|h|^2}\Big(\int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{ih\cdot A\left(y+\frac{h}{2}\right)}\tilde{u}(y)|^2 dy\Big)dh \\ &\le C\|\rho\|_{L^1}\|u\|^2_{H^1_A(\Omega)},\end{aligned}$$

which concludes the proof. □

Lemma 2.4. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded and let $u \in H^1_{0,A}(\Omega)$. Then, we have*

$$(1-s)\int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dx\,dy \le C\|u\|^2_{H^1_A(\Omega)}$$

where C depends only on Ω and A.

Proof. Given $u \in C^\infty_c(\Omega)$, by Lemma 2.1 we have

$$\int_{\mathbb{R}^N} |u(y+h) - e^{ih\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2 dy \le C|h|^2\|u\|^2_{H^1_A(\Omega)},$$

for some $C>0$ depending on Ω and A and all $h\in\mathbb{R}^N$ with $|h|\le 1$. Then, we get

$$(1-s)\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \le (1-s)\int\limits_{\mathbb{R}^{2N}} \frac{|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2}{|h|^{N+2s}}\mathrm{d}y\,\mathrm{d}h$$

$$=(1-s)\int\limits_{\{|h|\le 1\}} \frac{1}{|h|^{N+2s}}\Big(\int\limits_{\mathbb{R}^N}|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2\mathrm{d}y\Big)\mathrm{d}h$$

$$+4(1-s)\int\limits_{\{|h|\ge 1\}}\frac{1}{|h|^{N+2s}}\mathrm{d}h\|u\|^2_{L^2(\Omega)}$$

$$\le (1-s)\int\limits_{\{|h|\le 1\}}\frac{1}{|h|^{N+2s-2}}\mathrm{d}h\|u\|^2_{H^1_A(\Omega)}+C\|u\|^2_{L^2}\le C\|u\|^2_{H^1_A(\Omega)}.$$

The assertion then follows by a density argument. □

If $A|_\Omega$ is smooth (and extended if necessary to a locally bounded field on Ω^c), we get the following result.

Theorem 2.5. *Assume that $A\in C^2(\bar\Omega)$. Let $u\in H^1_A(\Omega)$ and consider a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of nonnegative radial functions in $L^1(\mathbb{R}^N)$ with*

$$\lim_{n\to\infty}\int_0^\infty \rho_n(r)r^{N-1}\mathrm{d}x=1, \tag{2.2}$$

and such that, for every $\delta>0$,

$$\lim_{n\to\infty}\int_\delta^\infty \rho_n(r)r^{N-1}\mathrm{d}r=0. \tag{2.3}$$

Then, we have

$$\lim_{n\to\infty}\int\limits_\Omega\int\limits_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,\mathrm{d}x\,\mathrm{d}y=2K_N\int\limits_\Omega|\nabla u-\mathrm{i}A(x)u|^2\,\mathrm{d}x \tag{2.4}$$

being K_N the constant introduced in (1.1).

Proof. Let us first observe that by (2.2) and (2.3) we easily obtain that, for every $\delta>0$,

$$\lim_{n\to\infty}\int_0^\delta \rho_n(r)r^N\mathrm{d}r=\lim_{n\to\infty}\int_0^\delta\rho_n(r)r^{N+1}\mathrm{d}r=0. \tag{2.5}$$

In fact, taken any $0<\tau<\delta$, we have

$$\int_0^\delta\rho_n(r)r^N\mathrm{d}r=\int_0^\tau\rho_n(r)r^N\mathrm{d}r+\int_\tau^\delta\rho_n(r)r^N\mathrm{d}r\le\tau\int_0^\tau\rho_n(r)r^{N-1}\mathrm{d}r+\delta\int_\tau^\infty\rho_n(r)r^{N-1}\mathrm{d}r,$$

from which formula (2.5) follows using (2.2), (2.3) and letting $\tau\searrow 0$. We follow the main lines of the proof in [3]. Setting

$$F^u_n(x,y):=\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|}\rho_n^{1/2}(x-y),\qquad x,y\in\Omega,\ n\in\mathbb{N},$$

by virtue of Lemma 2.3, for all $u,v\in H^1_A(\Omega)$, recalling (2.2) we have

$$\big|\|F^u_n\|_{L^2(\Omega\times\Omega)}-\|F^v_n\|_{L^2(\Omega\times\Omega)}\big|\le\|F^u_n-F^v_n\|_{L^2(\Omega\times\Omega)}\le C\|u-v\|_{H^1_A(\Omega)},$$

for some $C>0$ depending on Ω and A. This allows to reduce the proof of (2.4) to $u\in C^2(\bar\Omega)$. If we set

$$\varphi(y) := e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y),$$

since

$$\nabla_y \varphi(y) = e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} \Big(\nabla_y u(y) - iA\Big(\frac{x+y}{2}\Big)u(y) + \frac{i}{2}\, u(y)(x-y)\cdot \nabla_y A\Big(\frac{x+y}{2}\Big)\Big),$$

if $x \in \Omega$, a second-order Taylor expansion gives (since $u, A \in C^2$, then $\nabla_y^2 \varphi$ is bounded on $\bar{\Omega}$)

$$u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y) = \varphi(x) - \varphi(y) = (\nabla u(x) - iA(x)u(x)) \cdot (x-y) + \mathcal{O}(|x-y|^2).$$

Hence, for any fixed $x \in \Omega$,

$$\frac{\big|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)\big|}{|x-y|} = \left|(\nabla u(x) - iA(x)u(x)) \cdot \frac{x-y}{|x-y|}\right| + \mathcal{O}(|x-y|). \tag{2.6}$$

Fix $x \in \Omega$. If we set $R_x := \mathrm{dist}(x, \partial\Omega)$, integrating with respect to y, we have

$$\int_\Omega \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^2} \rho_n(x-y)\, dy = \int_{B(x,R_x)} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^2} \rho_n(x-y)\, dy + \int_{\Omega \setminus B(x,R_x)} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^2} \rho_n(x-y)\, dy. \tag{2.7}$$

The second integral goes to zero by conditions (2.3), since

$$\lim_{n\to\infty} \int_{\Omega \setminus B(x,R_x)} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^2} \rho_n(x-y)\, dy \le C \lim_{n\to\infty} \int_{B^c(0,R_x)} \rho_n(z)\, dz = 0.$$

Now, in light of (2.6), following [3], we compute

$$\int_{B(x,R_x)} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^2} \rho_n(x-y)\, dy = Q_N |\nabla u(x) - iA(x)u(x)|^2 \int_0^{R_x} r^{N-1} \rho_n(r)\, dr + \mathcal{O}\left(\int_0^{R_x} r^N \rho_n(r)\, dr\right) + \mathcal{O}\left(\int_0^{R_x} r^{N+1} \rho_n(r) dr\right),$$

where we have set

$$Q_N = \int_{\mathbb{S}^{N-1}} |\omega \cdot \mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega),$$

being $\mathbf{e} \in \mathbb{R}^N$ a unit vector. Letting $n \to \infty$ in (2.7), the result follows by dominated convergence, taking into account formulas (2.5). □

3. Proofs of Theorem 1.1 and 1.2

3.1. Proof of Theorem 1.1

If $r_\Omega := \mathrm{diam}(\Omega)$, we consider a radial cut-off $\psi \in C_c^\infty(\mathbb{R}^N)$, $\psi(x) = \psi_0(|x|)$ with $\psi_0(t) = 1$ for $t < r_\Omega$ and $\psi_0(t) = 0$ for $t > 2r_\Omega$. Then, by construction, $\psi_0(|x-y|) = 1$, for every $x, y \in \Omega$. Furthermore, let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a sequence with $s_n \nearrow 1$ as $n \to \infty$ and consider the sequence of radial functions in $L^1(\mathbb{R}^N)$

$$\rho_n(|x|) = \frac{2(1-s_n)}{|x|^{N+2s_n-2}} \psi_0(|x|), \qquad x \in \mathbb{R}^N,\ n \in \mathbb{N}. \tag{3.1}$$

Notice that (2.2) holds, since

$$\lim_{n\to\infty}\int_0^{r_\Omega}\rho_n(r)r^{N-1}\mathrm{d}r=\lim_{n\to\infty}2(1-s_n)\int_0^{r_\Omega}\frac{1}{r^{2s_n-1}}\mathrm{d}r=\lim_{n\to\infty}r_\Omega^{2-2s_n}=1,$$

and

$$\lim_{n\to\infty}\int_{r_\Omega}^{2r_\Omega}\rho_n(r)r^{N-1}\mathrm{d}r=\lim_{n\to\infty}2(1-s_n)\int_{r_\Omega}^{2r_\Omega}\frac{\psi_0(r)}{t^{2s_n-1}}\mathrm{d}r=0.$$

In a similar fashion, for any $\delta>0$, there holds

$$\lim_{n\to\infty}\int_\delta^{\infty}\rho_n(r)r^{N-1}\mathrm{d}r\le\lim_{n\to\infty}2(1-s_n)\int_\delta^{2r_\Omega}\frac{1}{t^{2s_n-1}}\mathrm{d}t=0.$$

Then Theorem 1.1 follows directly from Theorem 2.5 using ρ_n as defined in (3.1). □

3.2. Proof of Theorem 1.2

In light of Theorem 1.1 and since $u=0$ on Ω^c, we have

$$\lim_{s\nearrow 1}(1-s)\int_{\mathbb{R}^{2N}}\frac{|u(x)-\mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y=K_N\int_\Omega|\nabla u-\mathrm{i}A(x)u|^2\mathrm{d}x+\lim_{s\nearrow 1}R_s,$$

where

$$R_s\le 2(1-s)\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}\frac{|u(x)|^2}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y.$$

On the other hand, arguing as in the proof of [5, Proposition 2.8], we get $R_s\to 0$ as $s\nearrow 1$ when $u\in C_c^\infty(\Omega)$ and, on account of Lemma 2.4, for general function in $H^1_{0,A}(\Omega)$ by a density argument. □

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