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Dynamical systems

Invariant measures for piecewise continuous maps

Mesures invariantes pour les applications continues par morceaux

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A R T I C L E I N F O

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A B S T R A C T

We say that $f:[0,1]\to[0,1]$ is a *piecewise continuous interval map* if there exists a partition $0=x_0<x_1<\cdots<x_d<x_{d+1}=1$ of $[0,1]$ such that $f|_{(x_{i-1},x_i)}$ is continuous and the lateral limits $w_0^+=\lim_{x\to 0^+} f(x)$, $w_{d+1}^-=\lim_{x\to 1^-} f(x)$, $w_i^-=\lim_{x\to x_i^-} f(x)$ and $w_i^+=\lim_{x\to x_i^+} f(x)$ exist for each i. We prove that every piecewise continuous interval map without connections admits an invariant Borel probability measure. We also prove that every injective piecewise continuous interval map with no connections and no periodic orbits is topologically semiconjugate to an interval exchange transformation.

R É S U M É

On dit que $f:[0,1]\to[0,1]$ est une *application d'intervalle continue par morceaux* s'il existe une partition $0=x_0<x_1<\cdots<x_d<x_{d+1}=1$ de $[0,1]$ telle que $f|_{(x_{i-1},x_i)}$ est continue et telle que les limites latérales $w_0^+=\lim_{x\to 0^+} f(x)$, $w_{d+1}^-=\lim_{x\to 1^-} f(x)$, $w_i^-=\lim_{x\to x_i^-} f(x)$ et $w_i^+=\lim_{x\to x_i^+} f(x)$ existent pour chaque i. On prouve que toute application d'intervalle continue par morceaux sans connexion admet une mesure de probabilité invariante. On prouve également que toute application injective d'intervalle continue par morceaux sans connexion et sans orbite périodique est topologiquement semiconjuguée à un échange d'intervalles.

1. Introduction

Much information about the long-term behaviour of the iterates of a map is revealed by its invariant measures. Regarding piecewise continuous interval maps, the presence of a non-atomic invariant Borel probability measure can be used to construct topological conjugacies or semiconjugacies with interval exchange transformations (IETs).

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Transfer operators have proved to be an important tool to obtain absolutely continuous invariant probability measures for piecewise smooth piecewise monotone interval maps (see [1,3–5,9]). In general, these types of results assume that each branch of the piecewise continuous map is C^r-smooth ($r \geq 1$), monotone and has derivative greater than 1.

The aim of this article is to prove the existence of invariant Borel probability measures for piecewise continuous interval maps not embraced by the transfer operator approach. In this way, our result includes gap maps, piecewise contractions and generalised interval exchange transformations (GIETs). No monotonicity and no smoothness assumptions, beyond the uniform continuity of each branch of the map, are assumed. Our result is the natural version of the Kryloff–Bogoliouboff Theorem (see [8]) for piecewise continuous interval maps.

We are also interested in constructing topological semiconjugacy between injective piecewise continuous interval maps and interval exchange transformations, possibly with flips. In this regard, it is worth mentioning the result by J. Milnor and W. Thurston (see [12]), which states that any continuous piecewise monotone interval map of positive entropy htop is topologically semiconjugate to a map with constant slope equal to $\pm e^{\mathrm{htop}}$. This result was generalised by L. Alsedà and M. Misiurewicz in [2] to piecewise continuous piecewise monotone interval maps of positive entropy. Concerning countably piecewise continuous piecewise monotone interval maps, a necessary and sufficient condition for the existence of a non-decreasing semiconjugacy to a map of constant slope was provided by M. Misiurewicz and S. Roth in [13]. The author and A. Nogueira proved in [14] that every injective piecewise contraction is topologically conjugate to a map with constant slope equal to $\pm\frac{1}{2}$.

The proof of the Kryloff–Bogoliouboff Theorem fails for discontinuous maps. In this article, we present a variation of this proof that overcomes such limitation. The hypothesis of no connections cannot be removed since there are examples of piecewise continuous maps that have connections and admit no Borel invariant measure. The proof presented here does not hold for countably piecewise continuous maps since for such maps the lateral limits might not exist at all points of $[0,1]$.

2. Statement of the results

Throughout this article, assume that $f:[0,1]\to[0,1]$ is a piecewise continuous interval map. Hence, there exists a partition $0=x_0<x_1<\cdots<x_d<x_{d+1}=1$ of $[0,1]$ such that $f|_{(x_{i-1},x_i)}$ is continuous and the lateral limits $w_0^+=\lim_{x\to 0^+} f(x)$, $w_{d+1}^-=\lim_{x\to 1^-} f(x)$, $w_i^-=\lim_{x\to x_i^-} f(x)$ and $w_i^+=\lim_{x\to x_i^+} f(x)$ exist for each i. Let

$$D=\{x_0,\ldots,x_{d+1}\},\quad W=\{w_0^+,w_1^-,w_1^+,\ldots,w_d^-,w_d^+,w_{d+1}^-\}.$$

We say that f has *no connections* if

$$\bigcup_{w\in W}\bigcup_{k=0}^{\infty}\{f^k(w)\}\cap D=\emptyset. \tag{1}$$

We say that $x\in[0,1]$ is a *periodic point* of f if there exists an integer $k\geq 1$ such that $f^k(x)=x$.

Our first result turns out to be a version of the Kryloff–Bogoliouboff Theorem [8] for piecewise continuous interval maps.

Theorem 2.1. *Let $f:[0,1]\to[0,1]$ be a piecewise continuous map with no connections, then f admits an invariant Borel probability measure μ. Moreover, if f has no periodic points, then the measure μ is non-atomic.*

The hypothesis of no connections in the statement of Theorem 2.1, although more readily checkable, may sound a bit restrictive because, for instance, it prohibits that a left-continuous map f takes one discontinuity into another. Indeed, what needs to be avoided for the existence of the invariant measure is the presence of *closed connections*, a more technical notion given in Section 3.

In the world of generalised interval exchange transformations, the hypothesis of no connections corresponds to the notion of having an ∞-complete path. As remarked in [11, p. 1586], every GIET with such property is topologically semiconjugate to an IET. The next result extends this claim to piecewise continuous maps. It can also be considered a generalisation of the item (a) of the Structure Theorem by Gutierrez [6, p. 18].

Corollary 2.2. *Let $f:[0,1]\to[0,1]$ be an injective piecewise continuous map with no connections and no periodic points, then f is topologically semiconjugate to an interval exchange transformation, possibly with flips.*

Now we present a class of piecewise continuous interval maps for which having no connections is a generic (in the measure-theoretical sense) property. We recall that an irrationality criterion for the absence of connections in IETs without flips was provided by M. Keane in [7].

Theorem 2.3. *Let $\phi_1,\ldots,\phi_{d+1}:[0,1]\to(0,1)$ be continuous maps and let $\Omega\subset\mathbb{R}^d$ be the open set $\Omega=\{(x_1,\ldots,x_d)\in\mathbb{R}^d \mid 0<x_1<\cdots<x_d<1\}$, then for Lebesgue almost every $(x_1,\ldots,x_d)\in\Omega$, the piecewise continuous map $f:[0,1]\to(0,1)$ defined by $f(x)=\phi_i(x)$ if $x\in I_i$, where $I_1=[0,x_1)$, $I_2=[x_1,x_2),\ldots,I_d=[x_{d-1},x_d)$, $I_{d+1}=[x_d,1]$, has no connections and hence admits an invariant Borel probability measure.*

3. Proof of Theorem 2.1

Henceforth, assume that the map f has no connections and no periodic orbits.

Lemma 3.1. *Given $x \in [0,1]$ and an integer $r \geq 1$, there exists an open subinterval J_x of $[0,1]$ containing x such that*

$$\{f(y), \ldots, f^r(y)\} \cap J_x = \emptyset \quad \textit{for every} \quad y \in J_x. \tag{2}$$

Proof. First let us prove the result for $x = x_i$, where $1 \leq i \leq d$. Let

$$\gamma = \bigcup_{k=1}^{r} \{f^{k-1}(w_i^-), f^k(x_i), f^{k-1}(w_i^+)\}.$$

By the uniform continuity of $f|_{(x_{j-1}, x_j)}$, $1 \leq j \leq d+1$, together with the hypothesis of no connections, we have that for every $\epsilon > 0$, there exist $0 < \delta < \epsilon$ and an interval $J_{x_i} = (x_i - \delta, x_i + \delta) \subset [0,1]$ such that

$$d\left(f^k(y), \gamma\right) < \epsilon \quad \text{for every} \quad y \in J_{x_i} \quad \text{and} \quad 1 \leq k \leq r, \tag{3}$$

where $d\left(f^k(y), \gamma\right) = \min_{z \in \gamma} |f^k(y) - z|$.

Let $\epsilon = \frac{1}{2} d(x_i, \gamma)$, then $\epsilon > 0$, otherwise f would have a connection or a periodic orbit. This together with (3) implies that $|f^k(y) - x_i| > \epsilon > \delta$ for all $y \in J_{x_i}$ and $1 \leq k \leq r$. Hence, (2) holds for every $x = x_i \in D$.

The cases in which $x = x_0 = 0$ or $x = x_{d+1} = 1$ follows likewise, by considering intervals of the form $J_{x_0} = [0, \delta)$ and $J_{x_{d+1}} = (1 - \delta, 1]$, respectively.

It remains to consider the case in which $x \notin \{x_0, \ldots, x_{d+1}\}$. Due to the hypothesis of no connections, there are only two possibilities: either $\{f^k(x) : k \geq 0\} \cap \{x_0, \ldots, x_{d+1}\} = \emptyset$ or there exist $k \geq 1$ (take the least value) and $0 \leq i \leq d+1$ such that $f^k(x) = x_i$. As for the first possibility, take $\gamma = \{f(x), \ldots, f^r(x)\}$, then f is continuous on $\{x\} \cup \gamma$. Moreover, since f has no periodic points, we have that $x \notin \gamma$. Therefore, for every $\epsilon > 0$, there exist $0 < \delta < \epsilon$ and an interval $J_x = (x - \delta, x + \delta)$ such that (3) holds for J_x in the place of J_{x_i}. To conclude the proof, proceed as before. Concerning the second possibility, let $J_{x_i} = (x_i - \delta, x_i + \delta)$ be as in the beginning of the proof, then, as already proved,

$$\{f(y), \ldots, f^r(y)\} \cap J_{x_i} = \emptyset \quad \text{for every} \quad y \in J_{x_i}. \tag{4}$$

Moreover, since k is the least value, f is locally continuous around $\{x, f(x), \ldots, f^{k-1}(x)\}$, thus there exists an interval $J_x = (x - \eta, x + \eta)$ such that $J_x, f(J_x), \ldots, f^k(J_x)$ are pairwise disjoint intervals and $f^k(J_x) \subset J_{x_i}$. Now (4) implies that (2) holds for every $y \in J_x$, which concludes the proof. □

Let $q \in [0,1]$ be given. Since f has no periodic orbits, there exists $\ell \geq 0$ such that $\left\{f^k(q) : k \geq \ell\right\} \cap D = \emptyset$. Hereafter, set $p = f^\ell(q)$, then

$$\{p, f(p), f^2(p), \ldots\} \cap D = \emptyset. \tag{5}$$

Denote by $(\mu_n)_{n=1}^{\infty}$ the sequence of Borel probability measures on $[0,1]$ defined by

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(p)},$$

where $\delta_{f^k(p)}$ is the Dirac probability measure on $[0,1]$ concentrated at $f^k(p)$.

By the Banach–Alaoglu Theorem, the space of Borel probability measures on a compact metric space is compact with respect to the weak* topology. Hence, there exist a Borel probability measure on $[0,1]$, denoted henceforth by μ, and a subsequence of $\{\mu_n\}$, denoted henceforth by $\{\mu_{n_j}\}_{j=1}^{\infty}$, that converges to μ in the weak* topology.

The next result is going to be used twice, in Lemma 3.3 as well as in Lemma 3.5.

Lemma 3.2. *Let $x \in [0,1]$. Given $\epsilon > 0$, there exist an open subinterval J_x of $[0,1]$ containing x, and an integer $j_0 \geq 1$ such that*

$$\mu_{n_j}(J_x) < \epsilon \quad \textit{for every} \quad j \geq j_0. \tag{6}$$

Proof. Let $r \geq 1$ be an integer so great that $\dfrac{2}{r} < \epsilon$. Since $\{n_j\}_{j=1}^{\infty}$ is a subsequence of $\{1, 2, \ldots\}$, there exists $j_0 \geq 1$ such that $n_j > r$ for every $j \geq j_0$. Let J_x be as in the statement of Lemma 3.1. Let $j \geq j_0$ and $\ell = \#\{0 \leq k \leq n_j - 1 \mid f^k(p) \in J_x\}$, where # denotes cardinality. By (2), we have that $(\ell - 1)r \leq n_j$, thus

$$\mu_{n_j}(J_x) = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} \delta_{f^k(p)}(J_x) = \frac{\#\{0 \leq k \leq n_j - 1 \mid f^k(p) \in J_x\}}{n_j} \leq \frac{2}{r} < \epsilon \text{ for every } j \geq j_0. \qquad \square$$

Lemma 3.3. *The measure μ is non-atomic.*

Proof. Let $x \in (0,1)$. Given $\epsilon > 0$, let J_x be an open subinterval of $[0,1]$ containing x as in the statement of Lemma 3.2. Since the set $S = \{z \in [0,1] : \mu(\{z\}) > 0\}$ is at most countable, there exists a subinterval $J'_x \subset J_x$ containing x such that $\mu\left(\partial J'_x\right) = 0$, where $\partial J'_x$ denotes the endpoints of the interval J'_x. By [15, Theorem 6.1, p. 40] and by (6),

$$\mu(\{x\}) \le \mu(J'_x) = \lim_{j\to\infty} \mu_{n_j}(J'_x) \le \limsup_{j\to\infty} \mu_{n_j}(J_x) \le \epsilon.$$

The fact that ϵ is arbitrary yields $\mu(\{x\}) = 0$.

Now let $A_1 \subset A_2 \subset \cdots$ be a sequence of subsets of $[0,1]$ such that $\bigcup_{k\ge 1} A_k = (0,1)$ and $\partial A_k \cap S = \emptyset$ for every $k \ge 1$. By (5), we have that $\mu_{n_j}(A_k) = 1$ for every $j,k \ge 1$. By [15, Theorem 6.1, p. 40] once more, we have that

$$\mu(A_k) = \lim_{j\to\infty} \mu_{n_j}(A_k) = 1 \quad \text{for every} \quad k \ge 1.$$

In this way,

$$\mu\left((0,1)\right) = \lim_{k\to\infty} \mu(A_k) = 1, \quad \text{thus} \quad \mu\left(\{0\}\right) = \mu\left(\{1\}\right) = 0. \qquad \square$$

The convergence of $\{\mu_{n_j}\}_{j=1}^{\infty}$ to μ in the weak* topology implies that $\lim_{j\to\infty} \int \phi \, d\mu_{n_j} = \int \phi \, d\mu$ for every continuous function $\phi : [0,1] \to \mathbb{R}$. The next lemma extends this claim for the piecewise continuous map $\phi = \varphi \circ f$.

Remark 3.4. As pointed out by C. Liverani in [10, p. 4], the point where the proof of the Kryloff–Bogoliouboff Theorem fails is Lemma 3.5, which is automatic for continuous functions.

Lemma 3.5. *For every continuous function $\varphi : [0,1] \to \mathbb{R}$,*

$$\lim_{j\to\infty} \int \varphi \circ f \, d\mu_{n_j} = \int \varphi \circ f \, d\mu.$$

Proof. Let $\epsilon > 0$ be arbitrarily small. By Lemma 3.3, we have that $\mu(\{x_i\}) = 0$ for every $1 \le i \le d$. Hence, there exists an open interval J'_{x_i} containing x_i such that $\mu(J'_{x_i}) < \epsilon$ for every $1 \le i \le d$. By Lemma 3.2, there exist an open interval J''_{x_i} containing x_i, and an integer $j_0 \ge 1$ such that

$$\mu_{n_j}\left(J''_{x_i}\right) < \epsilon \quad \text{for every} \quad j \ge j_0 \quad \text{and} \quad 1 \le i \le d.$$

Set $J_{x_i} = J'_{x_i} \cap J''_{x_i}$. The function $\varphi \circ f$ is bounded by some constant M and continuous on each interval (x_{i-1}, x_i) for every $1 \le i \le d+1$. In this way, there exists a continuous function $h : [0,1] \to [-M, M]$ such that $h(x) = \varphi \circ f(x)$ for every $x \in [0,1] \setminus \bigcup_{i=1}^{d} J_{x_i}$. Putting it all together yields

$$\left| \int \varphi \circ f \, d\mu_{n_j} - \int h \, d\mu_{n_j} \right| \le \int |\varphi \circ f - h| \, d\mu_{n_j} \le 2M d\epsilon \quad \text{for every} \quad j \ge j_0, \tag{7}$$

and

$$\left| \int \varphi \circ f \, d\mu - \int h \, d\mu \right| \le 2M d\epsilon. \tag{8}$$

Finally, since h is continuous on $[0,1]$ and μ_{n_j} converges to μ in the weak* topology, there exists $j_1 \ge j_0$ such that

$$\left| \int h \, d\mu_{n_j} - \int h \, d\mu \right| \le \epsilon \quad \text{for every} \quad j \ge j_1. \tag{9}$$

It follows from the equations (7), (8) and (9) that

$$\left| \int \varphi \circ f \, d\mu_{n_j} - \int \varphi \circ f \, d\mu \right| \le (4Md + 1)\epsilon \quad \text{for every} \quad j \ge j_1,$$

which concludes the proof. $\square$

Lemma 3.6 *([16, Theorem 6.2, p. 147]). Let m_1 and m_2 be two Borel probability measures on $[0,1]$. If $\int \varphi \, dm_1 = \int \varphi \, dm_2$ for every continuous function $\varphi : [0,1] \to \mathbb{R}$, then $m_1 = m_2$.*

Given a Borel probability measure m on $[0,1]$ and an integer $k \geq 1$, let $m \circ f^{-k}$ denote the Borel measure defined by $(m \circ f^{-k})(B) = m\left(f^{-k}(B)\right)$ for any Borel set B. In particular, for $m = \delta_p$ we have that $\delta_p \circ f^{-k} = \delta_{f^k(p)}$.

Lemma 3.7 *([16, Lemma 6.6, p. 150]). Let $\psi : [0,1] \to \mathbb{R}$ be a Borel-measurable function, $k \geq 1$ an integer, and m a Borel probability measure on $[0,1]$, then*

$$\int \psi \circ f^k \, \mathrm{d}m = \int \psi \, \mathrm{d}(m \circ f^{-k}).$$

Lemma 3.8. *The measure μ is invariant by f.*

Proof. By Lemma 3.6 and Lemma 3.7 (taking $\psi = \varphi$, $k = 1$ and $m = \mu$), it suffices to show that

$$\int \varphi \circ f \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu \tag{10}$$

for every continuous function $\varphi : [0,1] \to \mathbb{R}$. By Lemma 3.5, for every continuous function $\varphi : [0,1] \to \mathbb{R}$,

$$\left| \int \varphi \circ f \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu \right| = \lim_{j \to \infty} \left| \int \varphi \circ f \, \mathrm{d}\mu_{n_j} - \int \varphi \, \mathrm{d}\mu_{n_j} \right|. \tag{11}$$

By Lemma 3.7 once more (now taking $\psi = \varphi \circ f$ and $m = \delta_p$), we reach

$$\int \varphi \circ f \, \mathrm{d}\mu_{n_j} = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} \int \varphi \circ f \, \mathrm{d}(\delta_p \circ f^{-k}) = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} \int \varphi \circ f^{k+1} \, \mathrm{d}\delta_p. \tag{12}$$

Likewise,

$$\int \varphi \, \mathrm{d}\mu_{n_j} = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} \int \varphi \, \mathrm{d}(\delta_p \circ f^{-k}) = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} \int \varphi \circ f^{k} \, \mathrm{d}\delta_p. \tag{13}$$

It follows from (11), (12) and (13) that

$$\begin{aligned} \left| \int \varphi \circ f \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu \right| &= \lim_{j \to \infty} \left| \frac{1}{n_j} \int \sum_{k=0}^{n_j - 1} \left(\varphi \circ f^{k+1} - \varphi \circ f^k \right) \mathrm{d}\delta_p \right| \\ &= \lim_{j \to \infty} \left| \frac{1}{n_j} \int \left(\varphi \circ f^{n_j} - \varphi \right) \mathrm{d}\delta_p \right| \\ &\leq \lim_{j \to \infty} \frac{2\|f\|}{n_j} = 0. \end{aligned}$$

Hence, (10) holds, which concludes the proof. □

Remark 3.9. The proof of Theorem 2.1 follows from Lemmas 3.3 and 3.8.

4. Proof of the other results

Corollary 2.2. *Let $f : [0,1] \to [0,1]$ be an injective piecewise continuous map with no connections and no periodic orbits, then f is topologically semiconjugate to an interval exchange transformation, possibly with flips.*

Proof. By Theorem 2.1, f admits a non-atomic Borel probability measure μ invariant by f. Let $h : [0,1] \to [0,1]$ be defined by $h(x) = \mu([0,x])$, then h is a continuous non-decreasing surjective map. Let $1 \leq i \leq d+1$ and $x, y \in (x_{i-1}, x_i)$ be such that $h(x) = h(y)$. We claim that $h(f(x)) = h(f(y))$. Assume that $x \leq y$ and $f(x) \leq f(y)$, then, the injectivity of f together with the continuity of $f|_{(x_{i-1}, x_i)}$ yields $[x, y] = f^{-1}([f(x), f(y)])$. Hence, since μ is non-atomic,

$$|h(f(y)) - h(f(x))| = \mu([f(x), f(y)]) = \mu\left(f^{-1}([f(x), f(y)])\right) = \mu([x, y]) = |h(y) - h(x)|. \tag{14}$$

As for the other cases, to proceed likewise to show that (14) still holds. Hence, the claim is true.

Let $T : [0,1] \to [0,1]$ be defined by $T(h(x)) = h(f(x))$. By the claim, T is well defined. Let $t_0, t_1, \ldots, t_{d+1}$ be defined by $t_0 = 0$, $t_{d+1} = 1$ and $t_i = h(x_i)$ for every $1 \leq i \leq d$. By (14), we have that for every $t, s \in (t_{i-1}, t_i)$, there exist $x, y \in (x_{i-1}, x_i)$ such that $t = h(x)$, $s = h(y)$ and

$$|T(t) - T(s)| = |h(f(x)) - h(f(y))| = |h(x) - h(y)| = |t - s| \quad \text{for every} \quad t, s \in (t_{i-1}, t_i).$$

This proves that $T|_{(t_{i-1},t_i)}$ is an isometry; therefore, T is an interval exchange transformation, possibly with flips. By definition, $T \circ h = h \circ f$, thus f is topologically semiconjugate to T. □

Theorem 2.3. *Let $\phi_1, \ldots, \phi_{d+1} : [0,1] \to (0,1)$ be continuous maps and let $\Omega \subset \mathbb{R}^d$ be the open set $\Omega = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \cdots < x_d < 1\}$, then for Lebesgue almost every $(x_1, \ldots, x_d) \in \Omega$, the piecewise continuous map $f : [0,1] \to (0,1)$ defined by $f(x) = \phi_i(x)$ if $x \in I_i$, where $I_1 = [0, x_1)$, $I_2 = [x_1, x_2), \ldots, I_d = [x_{d-1}, x_d)$, $I_{d+1} = [x_d, 1]$, has no connections and hence admits an invariant Borel probability measure.*

Proof. Denote by Id the identity map on $[0,1]$. Set $\mathscr{C}_0 = \{Id\}$. Let

$$\mathscr{C}_k = \{\phi_i \circ h \mid 1 \le i \le d+1, h \in \mathscr{C}_{k-1}\}, \quad k \ge 1.$$

For each $0 \le i \le d+1$, $1 \le j \le d$, $w_i \in \{w_i^-, w_i^+\}$ and $h \in \bigcup_{k \ge 0} \mathscr{C}_k$, the set $\{(x_1, \ldots, x_d) \in \Omega \mid x_j = h(w_i)\}$ is the graph of a continuous function defined on $[0,1]$, thus it is a Lebesgue null set. This together with the fact that $x_0 = 0$ and $x_{d+1} = 1$ do not belong to the range of any $h \in \bigcup_{k \ge 1} \mathscr{C}_k$ implies that the set of parameters $(x_1, \ldots, x_d) \in \Omega$ for which the map f has connections is a Lebesgue null set, denoted by N. Let $(x_1, \ldots, x_d) \in \Omega \setminus N$, then either f has a periodic point or f has no periodic points and no connections. In the first case, f has an invariant Borel probability measure supported on its periodic orbits, while in the second case, by Theorem 2.1, f admits an invariant non-atomic Borel probability measure. □

3. Final remarks

The claim of Theorem 2.1 keeps true if in its statement the term "no connections" is replaced by the term "no closed connections" defined below. Let $f : [0,1] \to [0,1]$ be as in (1) and let $\bar{f} : \mathscr{P}([0,1]) \to \mathscr{P}([0,1])$ be the map defined on each set $A \subset [0,1]$ by

$$\bar{f}(A) = \bigcup_{x \in A} \left\{ \lim_{\epsilon \to 0^+} f(x - \epsilon), \lim_{\epsilon \to 0^+} f(x + \epsilon) \right\},$$

where $\lim_{\epsilon \to 0^+} f(-\epsilon) := f(0)$ and $\lim_{\epsilon \to 0^+} f(1 + \epsilon) := f(1)$. We say that the map f has a *closed connection* if there exist $0 \le i \le d+1$ and $k \ge 1$ such that $x_i \in \bigcup_{k \ge 1} \bar{f}^k(\{x_i\})$.

The existence of connections neither implies nor is implied by the existence of periodic points. In fact, let $f_1, f_2 : [0,1] \to [0,1]$ be the piecewise continuous maps defined by

$$f_1(x) = \begin{cases} \dfrac{x}{2} + \dfrac{1}{8} & \text{if } 0 \le x < \dfrac{1}{2} \\ \dfrac{x}{2} + \dfrac{3}{8} & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}, \quad f_2(x) = \begin{cases} \dfrac{x}{2} + \dfrac{1}{4} & \text{if } 0 \le x < \dfrac{1}{2} \\ \dfrac{x}{2} & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}.$$

The map f_1 has two periodic points and no connections. The map f_2 has a closed connection but no periodic points. Moreover, it does not admit any invariant Borel probability measure.

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