



Complex analysis

On h -extendible domains and associated models[☆]*Sur les domaines h -extensibles et les modèles associés*

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ABSTRACT

A boundary point of a smooth pseudoconvex domain in \mathbb{C}^n is said to be h -extendible if its Catlin's multi-type coincides with its D'Angelo's multi-type. There is a local model defined by Catlin's multi-weight. In this paper, we show that a domain in \mathbb{C}^n with a noncompact automorphism group is biholomorphically equivalent to its associated model if there exists a sequence of automorphisms of the domain that has an orbit converging to an h -extendible boundary point non-tangentially in a cone region.

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R É S U M É

Un point frontière d'un domaine pseudo-convexe lisse de \mathbb{C}^n est dit h -extensible si son multi-type de Catlin coïncide avec son multi-type de D'Angelo. Le multi-poids de Catlin définit un modèle local. Nous montrons ici qu'un domaine de \mathbb{C}^n avec un groupe d'automorphismes non compact est bi-holomorphiquement équivalent à son modèle associé s'il existe une suite d'automorphismes du domaine ayant une orbite convergeant non tangentiellement dans un cône, vers un point frontière h -extensible.

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1. Introduction

One of the main problems in the study of weakly pseudoconvex domains is to understand the properties of domains of finite type. There are several classes of domains of finite type that have been relatively better understood: domains of finite type in \mathbb{C}^2 , convex domains of finite type, and decoupled domains of finite type. All these domains are contained in a class of domains called h -extendible domains [14] or pseudoconvex domains of semiregular type [4]. In this paper, we study the h -extendible domains with a noncompact automorphism group.

Let $\Omega \subset \mathbb{C}^n$ be a domain, and denote by $\text{Aut}(\Omega)$ the group of holomorphic automorphisms equipped with the compact-open topology. If the automorphism group is not compact, then by a theorem of Cartan (see, e.g., [9]), there are points

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$p \in \partial\Omega$, $q \in \Omega$, and automorphisms $f_j \in \text{Aut}(\Omega)$ such that $f_j(q)$ converges to p as j goes to infinity. The point p is called a boundary orbit accumulation point. If Ω is bounded and strongly pseudoconvex near p , then Ω is biholomorphic to the unit ball in \mathbb{C}^n [12,13]. When the boundary of Ω is real analytic and convex near p , Kim gave a complete description of such domains [8]. In [1], Bedford and Pinchuk obtained the following result:

Theorem 1.1. [1, Theorem 2] *Any convex smoothly bounded domain of finite type in \mathbb{C}^{n+1} , having non-compact automorphism group, is biholomorphically equivalent to a domain of the form $\{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \text{Re } w + P(z) < 0\}$, where $P(z)$ is a weighted homogeneous polynomial.*

If Ω is smooth, convex, and of finite type $2m$ near p , then Gaussier proved the following result:

Theorem 1.2. [5, Theorem 1] *Let Ω be a domain in \mathbb{C}^{n+1} and $p \in \partial\Omega$. Assume that p is an accumulating point for a sequence of automorphisms of Ω . If $\partial\Omega$ is smooth, convex, and of finite type $2m$ near p , then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \text{Re } w + P(z) < 0\},$$

where $P(z)$ is a real nondegenerate convex polynomial of degree less than or equal to $2m$.

In the above statement, “finite type” means that there is no non-trivial analytic set tangent to arbitrarily high order to the boundary of Ω at p . Accordingly, the nondegeneracy of $P(z)$ is given by the condition that $\{z \in \mathbb{C}^n : P(z) = 0\}$ contains no nontrivial analytic set.

In this paper we extend the above results to the h -extendible domains under a cone convergence condition. For $q \in \Omega$, $p \in \partial\Omega$ and $f_j \in \text{Aut}(\Omega)$, we say that $f_j(q)$ converges to p non-tangentially in a cone region if

$$f_j(q) \in \Gamma_\alpha(p) := \{z \in \Omega : |z - q| < \alpha \delta_p(z)\}$$

for all j large enough for some $\alpha > 1$. Here $\delta_p(z) = \min\{\text{dist}(z, \partial\Omega), \text{dist}(z, T_p\partial\Omega)\}$. If Ω is convex near p , then $\delta_p(z) = \text{dist}(z, \partial\Omega)$. By [10, Lemma 3], the cone region can be described as

$$\Gamma_\alpha(p) \subset \{z \in \Omega : 0 < \angle zpq' < \arccos(1/\alpha)\}. \tag{1.1}$$

We write Γ for $\Gamma_\alpha(p)$ from now on because the value of α is not important in our proof. The following is our main result.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a smoothly pseudoconvex domain, $p \in \partial\Omega$ is h -extendible with Catlin’s multi-type $(1, m_1, \dots, m_n)$. If there is a point $q \in \Omega$ and $f_j \in \text{Aut}(\Omega)$ such that $f_j(q)$ converges to $p \in \partial\Omega$ non-tangentially in a cone region as j goes to infinity, then Ω is biholomorphically equivalent to a domain of the form:*

$$\{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \text{Re } w + P(z) < 0\}.$$

Here $P(z)$ is a $(1/m_1, \dots, 1/m_n)$ -homogeneous polynomial with no pluriharmonic terms.

In section 2, we recall some basic definitions and preparatory results. In section 3, we give the proof of our main result.

2. Preliminaries

Let Ω be a domain in \mathbb{C}^n and $p \in \Omega$. We first recall the definition of Catlin’s multi-type (see e.g. [3]).

Let \mathbf{L}_n denote the set of all n -tuples $\mu = (\mu_1, \dots, \mu_n)$ with $1 \leq \mu_i \leq \infty$ such that

- (i) $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \infty$;
- (ii) For each k , either $\mu_k = \infty$ or there is a set of nonnegative integers a_1, \dots, a_k with $a_k > 0$ such that $\sum_{j=1}^k a_j / \mu_j = 1$.

An element of \mathbf{L}_n will be referred to as a list. The set of lists can be ordered lexicographically, i.e. if $\mu' = (\mu'_1, \dots, \mu'_n)$ and $\mu'' = (\mu''_1, \dots, \mu''_n)$, then $\mu' < \mu''$ if for some k , $\mu'_j = \mu''_j$ for all $j < k$, but $\mu'_k < \mu''_k$. A list $\mu \in \mathbf{L}_n$ with rational components is called distinguished if there exist holomorphic coordinates (z_1, \dots, z_n) centered at p such that

$$\text{If } \sum_{i=1}^n \frac{\alpha_i + \beta_i}{\mu_i} < 1, \text{ then } D^\alpha \bar{D}^\beta r(p) = 0.$$

Here D^α and \bar{D}^β denote the partial differential operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \dots \partial \bar{z}_n^{\beta_n}}.$$

Definition 2.1. [2] The multi-type $\mathcal{M}(\partial\Omega, p) = (m_1, \dots, m_n)$ is defined to be the least list \mathcal{M} in \mathbf{L}_n such that $\mathcal{M} \geq \mu$ for every distinguished list μ .

The multi-type is an upper semicontinuous holomorphic invariant with rational components. If the multi-type $\mathcal{M}(\partial\Omega, p)$ of p is finite, i.e. $m_n < \infty$, then it was shown in [2] that $\mathcal{M}(\partial\Omega, p) = \mu$ for some distinguished element μ of \mathbf{L}_n . Since p is a smooth point, it is easy to see that the first entry $m_1 = 1$ in \mathcal{M} . Note that if Ω is pseudoconvex near p , then each m_k , $2 \leq k \leq n$, is an even number. We call $\Lambda = (\lambda_1, \dots, \lambda_n) = (1/m_1, \dots, 1/m_n)$ the multi-weight of p .

Definition 2.2. Let $f(z)$ be a function on \mathbb{C}^n and $\Lambda = (\lambda_1, \dots, \lambda_n)$ is a multi-weight. For any real number $t \geq 0$, set

$$\pi_t(z) = (t^{\lambda_1} z_1, \dots, t^{\lambda_n} z_n) \quad \forall z \in \mathbb{C}^n.$$

We say that f is Λ -homogeneous with weight α if $f(\pi_t(z)) = t^\alpha f(z)$ for every $t > 0$ and $z \in \mathbb{C}^n \setminus \{0\}$. If $\alpha = 1$, then f is simply called Λ -homogeneous.

Let Ω be a domain in \mathbb{C}^{n+1} , $n \geq 1$. Suppose $p \in \partial\Omega$ is of finite multi-type $(1, m_1, \dots, m_n)$. Then in suitable local coordinates, the defining function of Ω near p has the form:

$$r(w, z) = \operatorname{Re} w + P(z) + R(w, z),$$

where $w \in \mathbb{C}$, $z \in \mathbb{C}^n$, P is a $(1/m_1, \dots, 1/m_n)$ -homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, and R is smooth and satisfies

$$|R(w, z)| \leq C(|w| + \sum_{i=1}^n |z_i|^{m_i})^\gamma,$$

for some constant $\gamma > 1$ and $C > 0$. We call $D = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} w + P(z) < 0\}$ an associated model for Ω at p .

Now we give the definition of h -extendible domains.

Definition 2.3. Let $\Omega \in \mathbb{C}^{n+1}$ be a domain and suppose that the Catlin’s multi-type of a boundary point p is $(1, m_1, \dots, m_n)$ with $m_n < \infty$. Set $\Lambda = (1/m_1, \dots, 1/m_n)$. If $D_\Lambda = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} w + P(z) < 0\}$ is an associated model of Ω near p , then D_Λ is called h -extendible at p if there is a C^1 function $a(z)$ on $\mathbb{C}^n \setminus \{0\}$ satisfying the following conditions:

- (i) $a(z) > 0$ whenever $z \neq 0$;
- (ii) $a(z)$ is Λ -homogeneous;
- (iii) $P(z) - \epsilon a(z)$ is strictly plurisubharmonic on $\mathbb{C}^n \setminus \{0\}$ when $0 < \epsilon \leq 1$.

We say that Ω is h -extendible at p if its associated model D_Λ is h -extendible at p . And Ω is called h -extendible if each one of its boundary points is h -extendible.

We call $a(z)$ a bumping function for $P(z)$. These conditions state that the model domain for Ω at p can be approximated from the outside by the pseudoconvex domains $\{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re}(w) + P(z) - \epsilon a(z) < 0\}$ having the same homogeneity as Ω . The key geometric property to the applications of h -extendible domains is the following relationship between h -extendible domains and h -extendible models.

Theorem 2.4. [15, Theorem 4.7] Let Ω be a smooth domain in \mathbb{C}^n and p an h -extendible boundary point of $\partial\Omega$. Then there are local holomorphic coordinates (z, w) and an h -extendible model Q (with the same multi-weight as p) such that in these coordinates $p = 0$ and $\bar{\Omega} \setminus \{p\} \subset Q$ near p .

For any integer $n \geq 1$, let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a fixed n -tuple of positive numbers and $\mu > 0$. Denote by $\mathcal{O}(\mu, \Lambda)$ the set of smooth functions f defined near the origin of \mathbb{C}^n such that

$$D^\alpha \bar{D}^\beta f(0) = 0, \quad \text{whenever} \quad \sum_{i=1}^n (\alpha_i + \beta_i) \lambda_i \leq \mu.$$

If $n = 1$ and $\Lambda = (1)$, then we use $\mathcal{O}(\mu)$ to denote the functions vanishing to order at least μ at the origin. Then we have the following lemma.

Lemma 2.5. [15, Lemma 4.11] Let Ω be a domain in \mathbb{C}^{n+1} and $p \in \partial\Omega$ h -extendible. Suppose that the Catlin’s multi-type of p is $(1, m_1, \dots, m_n)$ with $m_n < \infty$ and let $\Lambda = (1/m_1, \dots, 1/m_n)$. Then there are local holomorphic coordinates (\tilde{z}, \tilde{w}) , such that in these coordinates $p = 0$ and Ω can be described near p as follows:

$$\Omega = \{(\tilde{w}, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} \tilde{w} + \tilde{P}(\tilde{z}) + \tilde{R}_1(\tilde{z}) + \tilde{R}_2(\operatorname{Im} \tilde{w}) + (\operatorname{Im} \tilde{w})\tilde{R}(\tilde{z}) < 0\}.$$

Here $\tilde{P}(\tilde{z})$ is a Λ -homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic terms, $\tilde{R}_1 \in \mathcal{O}(1, \Lambda)$, $\tilde{R} \in \mathcal{O}(1/2, \Lambda)$ and $\tilde{R}_2 \in \mathcal{O}(2)$.

Remark 2.6. Assume that Ω is given by $\{(w, z) \in \mathbb{C} \times \mathbb{C}^n : r(w, z) < 0\}$ near $p = (0', 0)$ and $\Gamma \subset \Omega$ is a cone with vertex at p . Convexity is not a biholomorphic invariant, but from the proof of [15, Main Theorem] and [15, Lemma 4.10, Lemma 4.11], one easily sees that cones are preserved in both Theorem 2.4 and Lemma 2.5.

3. Proof of the Main Theorem

We first prove the main theorem in the special case where Ω is described as in Lemma 2.5. The process can be divided into three steps by the standard scaling method (see, e.g., [7]). First we show that any compact set in Ω can be mapped into some neighborhood U of p . Then we move $f_j(q)$ to the origin by complex linear maps and stretching the coordinates at the origin. We prove that the images of $U \cap \Omega$ under the stretching map have a nontrivial limit. The limit is biholomorphic to the associated model. Composing the automorphisms of Ω with the stretching maps, we have a sequence from any compact set of Ω to its associated model, and we call this sequence the scaling sequence. Finally, we show that the limit of the scaling sequence gives a biholomorphic mapping between Ω and its associated model. For the general case, we construct biholomorphic maps of coordinates that keep the cone convergence. In the above process, we have used several different local coordinates, which will result in different model domains. However, in [11], Nikolov showed, using a scaling method, that all model domains in different coordinates are biholomorphically equivalent. Therefore, we can reduce the general case to the special case.

First we need a localization lemma.

Lemma 3.1. [7, Proposition 9.2.8] For any neighborhood U of p in \mathbb{C}^n , let K be any compact subset of Ω , then there is a $N > 0$, such that $f_j(K) \subset \Omega \cap U$ for all $j \geq N$.

Now we define

$$\Pi_t(w, z) = (tw, t^{1/m_1}z_1, \dots, t^{1/m_n}z_n) = (tw, \pi_t(z)), t \in \mathbb{R}.$$

Let Ω be as in Lemma 2.5 and U a neighborhood of $0 \in \partial\Omega$. For any $\xi = (\xi_0, \xi') \in \Omega \cap U$, where $\xi' \in \mathbb{C}^n$ and $\xi_0 \in \mathbb{C}$, set $t = |\operatorname{Re}(\xi)|$ and $\hat{\xi} = (\hat{\xi}_0, \hat{\xi}') := \Pi_{t^{-1}}(\xi)$. As $\xi \in \Gamma$, it is easy to see that $|\hat{\xi}| \lesssim |\operatorname{Re} \xi_0| \approx t = |\operatorname{Re}(\xi)|$. Here $|A| \lesssim |B|$ means that there is a constant $C > 0$ which only depends on Ω, U and Γ such that $|A| \leq C|B|$, and $|A| \approx |B|$ if $|A| \lesssim |B|$ and $|B| \lesssim |A|$. So if $\xi \in \Omega \cap \Gamma$ and $|\xi|$ is small, then we have

$$|\hat{\xi}'| = |\pi_{1/t}(\xi')| \lesssim \sum_{i=1}^n t^{-\lambda_i} |\xi_i| \lesssim \sum_{i=1}^n t^{-\lambda_i} t \rightarrow 0 \tag{3.1}$$

since $0 < \lambda_i < 1$ for all $1 \leq i \leq n$, and

$$|\hat{\xi}_0| \lesssim t^{-1} |\xi_0| \lesssim t^{-1} |\xi| \lesssim 1. \tag{3.2}$$

Thus we get that $|\hat{\xi}| \lesssim 1$.

Now, for any $\xi \in \Omega \cap U \cap \Gamma$, define

$$L_{\hat{\xi}}(w, z) = (w, z) - \hat{\xi}.$$

It is a holomorphic automorphism of \mathbb{C}^n that moves ξ to the origin and $L_{\hat{\xi}}^{-1} = L_{-\hat{\xi}}$. Then consider the holomorphic mappings $D_{\hat{\xi}} := \Pi_{t^{-1}} \circ L_{\hat{\xi}}(\Omega \cap U)$ and $U_{\hat{\xi}} := \Pi_{t^{-1}} \circ L_{\hat{\xi}}(U)$. Set

$$r_{\hat{\xi}}(w, z) = t^{-1}r(L_{-\hat{\xi}} \circ \Pi_t(w, z)) \quad \text{and} \quad r_0(w, z) = \operatorname{Re} w - 1 + P(z). \tag{3.3}$$

Then locally $D_{\hat{\xi}}$ is defined by

$$\begin{aligned} D_{\hat{\xi}} &= \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : r(L_{-\hat{\xi}} \circ \Pi_t(w, z)) < 0\} \cap U_{\hat{\xi}} \\ &= \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : t^{-1}r(L_{-\hat{\xi}} \circ \Pi_t(w, z)) < 0\} \cap U_{\hat{\xi}} \\ &= \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : r_{\hat{\xi}}(w, z) < 0\} \cap U_{\hat{\xi}}. \end{aligned}$$

Then by the estimates (3.1) and (3.2) it is easy to see that $U_{\hat{\xi}}$ converges normally to \mathbb{C}^{n+1} as $\hat{\xi}$ tends to zero. One readily checks that

$$\lim_{\Omega \cap U \cap \Gamma \ni \xi \rightarrow 0} r_\xi(w, z) = r_0(w, z) \tag{3.4}$$

where the convergence is uniform on compact subsets of \mathbb{C}^{n+1} , which means that D_ξ converges normally to $D_0 := \{(w, z) \in \mathbb{C}^{n+1} : r_0(w, z) < 0\}$. (For the definition of the normal convergence, see, e.g., [7, Definition 9.2.2].)

On the other hand, by Theorem 2.4, there is an h -extendible model

$$Q = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \rho(w, z) = \operatorname{Re} w + Q(z) < 0\}, \tag{3.5}$$

where $Q(z)$ is a Λ -homogeneous function on $\mathbb{C}^n \setminus \{0\}$, such that if U is a small neighborhood of p then $\bar{\Omega} \cap U \setminus \{0\} \subset Q \cap U$. Furthermore, shrinking U if necessary, there is a constant C , only depending on Ω, U and Γ , such that, for all small $\xi \in \Omega \cap U \cap \Gamma$, we have

$$D_\xi(Q \cap U) \subset Q_0 := \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} w + Q(z) - C < 0\}. \tag{3.6}$$

Now we prove a simplified version of the main theorem.

Proposition 3.2. *Let Ω be a smooth pseudoconvex domain in \mathbb{C}^{n+1} . Assume that $p \in \partial\Omega$ is h -extendible with multi-type $(1, m_1, \dots, m_n)$, $m_n < \infty$ and let $\Lambda = (1/m_1, \dots, 1/m_n)$. Suppose that, near $p = (0', 0)$, Ω has a defining function $r(z, w)$ of the form:*

$$r(w, z) = \operatorname{Re} w + P(z) + R_1(z) + R_2(\operatorname{Im} w) + (\operatorname{Im} w)R(z) < 0.$$

Here $P(z)$ is a Λ -homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic terms, and $R_1 \in \mathcal{O}(1, \Lambda)$, $R \in \mathcal{O}(1/2, \Lambda)$ and $R_2 \in \mathcal{O}(2)$. If there is a point $q \in \Omega$ and $f_j \in \operatorname{Aut}(\Omega)$ such that $f_j(q)$ converges to p non-tangentially in a cone region Γ , then Ω is biholomorphic to a domain of the form:

$$D := \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} w + P(z) < 0\}. \tag{3.7}$$

Proof. Let U be a neighborhood of p and K be an arbitrary compact subset of Ω with $q \in K$. By the localization principle in Lemma 3.1, $f_j(K) \subset \Omega \cap U$ for all j large enough. Set

$$\begin{aligned} \Pi_j(w, z) &= \Pi_{|f_j(q)|^{-1}}(w, z), \quad L_j(w, z) = L_{f_j(q)}(w, z), \\ r_j(w, z) &= r_{f_j(q)}(w, z), \quad D_j = D_{f_j(q)}. \end{aligned}$$

By definition, $L_j(f_j(q)) = (0, '0)$ and $\Pi_j \circ L_j(f_j(q)) = (0, '0)$. Then for all j large enough, we define the following biholomorphic mappings:

$$\sigma_j := \Pi_j \circ L_j \circ f_j.$$

In order to show that Ω is biholomorphic to $D_0 = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : r_0(w, z) < 0\}$, first we show that there is a limit of σ_j , say σ , which defines a holomorphic map from Ω to D_0 and is biholomorphic near q . Second, we construct a holomorphic map from D_0 to Ω . Finally we show that they are holomorphic inverses to each other.

First we show that a limit of σ_j defines a holomorphic map from Ω to D_0 . Note that $\sigma_j|_K$ is a map from K to D_j and D_j converges normally to D_0 (by (3.4)), and $D_j \subset Q_0$ (by (3.6)) for all j large enough. As Q_0 is a taut domain, $\{\sigma_j\}$ form a normal family and let σ be one of the limits. Since D_j converges normally to D_0 , we have $\sigma|_K : K \rightarrow \bar{D}_0$. Since K is arbitrary, σ is defined on the whole Ω . Then by the tautness of D_0 , either $\sigma(\Omega) \subset \partial D_0$ or $\sigma(\Omega) \subset D_0$. On the other hand, $\sigma_j(q) = \Pi_j \circ L_j \circ f_j(q) = (0, '0)$. But $(0, '0) \in D_0$ and D_0 is open, so $\sigma(\Omega) \subset D_0$.

Next we show that σ is biholomorphic near $q \in \Omega$. Since D_0 is open, there is a constant $\delta > 0$ such that for all j large, we have the ball centered at $(0, '0)$ and with radius δ such that $B((0, '0), \delta) \subset D_j = \Pi_j \circ L_j(\Omega \cap U)$. Since $\partial\Omega$ is of finite type at p , it is of finite type at points p' in $\partial\Omega$ near p , since finite type is an open condition (see, e.g., [3]). As p is an accumulating point, by [6, Example 3.1.2], Ω is taut, hence hyperbolic. So there is a neighborhood W of q and a constant $\delta_1 > 0$ so that $|F_\Omega(\zeta, X)| \geq \delta_1|X|$ for any $\zeta \in W$ and $X \in T_\zeta\Omega$. Then we have

$$\delta_1|X| \leq F_\Omega(q, X) = F_{f_j(\Omega)}(f_j(q), df_j(q)X) \tag{3.8}$$

$$\leq F_{\Omega \cap U}(f_j(q), df_j(q)X) \tag{3.9}$$

$$= F_{\Pi_j \circ L_j(\Omega \cap U)}((0, '0), d\Pi_j \circ dL_j \circ df_j(q)X) \tag{3.10}$$

$$\leq F_{B((0, '0), \delta)}((0, '0), d\sigma_j(q)X) \tag{3.11}$$

$$\leq \delta_2|d\sigma_j(q)||X|, \tag{3.12}$$

for all $0 \neq X \in T_q\Omega$. Equations (3.8) and (3.10) hold because the Kobayashi metric is a biholomorphic invariant and inequalities (3.9) and (3.11) hold because Kobayashi metric decreases under holomorphic mappings. Inequality (3.12) holds by the explicit Kobayashi metric of the ball with radius δ at the origin. So we get

$$|d\sigma_j(q)| \geq \frac{\delta_1}{\delta_2}.$$

The constants δ_1 and δ_2 do not depend on j . So letting j go to infinity, and setting $\delta_3 = \delta_1/\delta_2$, we get

$$|d\sigma(q)| \geq \delta_3 > 0.$$

This shows that σ is biholomorphic near q .

Now we construct a holomorphic map from D_0 to Ω . Set

$$\omega_j = f_j^{-1} \circ \Pi_j^{-1} \circ L_j^{-1}.$$

For any compact subset $\tilde{K} \subset D_0$, as D_j converges normally to D_0 , so for j large enough, $\tilde{K} \subset D_j = \Pi_j \circ L_j(\Omega \cap U)$, that is $L_j^{-1} \circ \Pi_j^{-1}(\tilde{K}) \subset \Omega \cap U$. Thus the sequence $\omega_j|_{\tilde{K}} : \tilde{K} \rightarrow \Omega$ is well defined. As Ω is taut, $\{\omega_j\}$ form a normally family and let ω be one of the limits. Since \tilde{K} is arbitrary, ω is defined on D_0 and $\omega(D_0) \subset \bar{\Omega}$. But $\omega_j(0, '0) = f_j^{-1} \circ \Pi_j^{-1} \circ L_j^{-1}(0, '0) = q$, so $\omega(D_0) \subset \Omega$.

Now we show that σ is a biholomorphism. As σ_j converges to σ uniformly on compact sets of Ω and σ is a local biholomorphism near q , there exists a constant $\delta_4 > 0$ and a compact set $N \subset \Omega$ with $q \in N$, such that $B((0, '0), \delta_4) \subset \sigma_j(N) \subset D_0$ for all large j . This implies that $\omega_j(B((0, '0), \delta_4)) \subset f_j^{-1}(f_j(N)) = N$. Thus for large j , the mapping

$$\sigma_j \circ \omega_j|_{B((0, '0), \delta_4)} = \text{id}_{B((0, '0), \delta_4)}$$

is well defined. Let j go to infinity, we have $\sigma \circ \omega|_{B((0, '0), \delta_4)} = \text{id}_{B((0, '0), \delta_4)}$, and hence $\sigma \circ \omega = \text{id}_{D_0}$. On the other hand, on any compact set $K \subset \Omega$ with $q \in K$, we have

$$\omega_j \circ \sigma_j|_K = \text{id}_K.$$

Let j go to infinity, we see that $\omega \circ \sigma|_K = \text{id}_K$, thus $\omega \circ \sigma = \text{id}_\Omega$. Hence σ is a biholomorphic mapping between D_0 and Ω .

By the defining functions of D_0 in (3.3) and D in (3.7), it is easy to see that Ω and its associated model D are biholomorphically equivalent. \square

Now we give the proof of the main theorem (Theorem 1.3):

Proof of Theorem 1.3. Suppose that Ω is given by $\{(w, z) \in \mathbb{C} \times \mathbb{C}^n : r(w, z) < 0\}$ in a neighborhood U of p . By assumption, there exist holomorphic automorphisms $f_j \in \text{Aut}(\Omega)$ and $q \in \Omega$ such that $f_j(q)$ converge to the boundary point p nontangentially in a cone Γ as j goes to infinity. Let $(\tilde{w}, \tilde{z}) = H_p(w, z)$ be the local coordinate change given in Lemma 2.5. Then, by Remark 2.6, the image of the cone Γ under H_p , defined by $\tilde{\Gamma} := H_p(\Gamma)$, is also a cone with vertex $p = 0$. So one gets $H_p(f_j(q)) \in \tilde{\Gamma}$. By Proposition 3.2, Ω is biholomorphic to the model $D = \{(\tilde{w}, \tilde{z}) : \text{Re } \tilde{w} + \tilde{P}(\tilde{z}) < 0\}$. By [11], all models in different coordinates are biholomorphically equivalent. This completes the proof. \square

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