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Topology

The hyperelliptic mapping class group of a nonorientable surface of genus $g \geq 4$ has a faithful representation into $\mathrm{GL}(g^2-1,\mathbb{R})$

Le groupe modulaire hyperelliptique d'une surface non orientable de genre $g \geq 4$ a une représentation fidèle dans $\mathrm{GL}(g^2-1,\mathbb{R})$

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A B S T R A C T

We prove that the hyperelliptic mapping class group of a nonorientable surface of genus $g \geq 4$ has a faithful linear representation of dimension g^2-1 over $\mathbb{R}$.

R É S U M É

Nous démontrons que le groupe modulaire hyperelliptique d'une surface non orientable de genre $g \geq 4$ a une représentation fidèle linéaire de dimension g^2-1 sur $\mathbb{R}$.

1. Introduction

Let $N_{g,n}$ be a smooth, nonorientable, compact surface of genus g with n punctures. If n is zero, then we omit it from the notation. Recall that N_g is a connected sum of g projective planes and $N_{g,n}$ is obtained from N_g by specifying the set Σ of n distinguished points in the interior of N_g.

Let $\mathrm{Diff}(N_{g,n})$ be the group of all diffeomorphisms $h\colon N_{g,n} \to N_{g,n}$ such that $h(\Sigma)=\Sigma$. By $\mathcal{M}(N_{g,n})$ we denote the quotient group of $\mathrm{Diff}(N_{g,n})$ by the subgroup consisting of maps isotopic to the identity, where we assume that maps and isotopies fix the set Σ. $\mathcal{M}(N_{g,n})$ is called the *mapping class group* of $N_{g,n}$.

The mapping class group $\mathcal{M}(S_{g,n})$ of an orientable surface $S_{g,n}$ of genus g with n punctures is defined analogously, but we consider only orientation-preserving maps. If we include orientation reversing maps, we obtain the so-called *extended mapping class group* $\mathcal{M}^{\pm}(S_{g,n})$. Suppose that the closed orientable surface S_{g-1}, where $g-1 \geq 2$, is embedded in $\mathbb{R}^3$ as shown in Fig. 1, in such a way that it is invariant under reflections across xy, yz, xz planes. Let $j\colon S_{g-1} \to S_{g-1}$ be the

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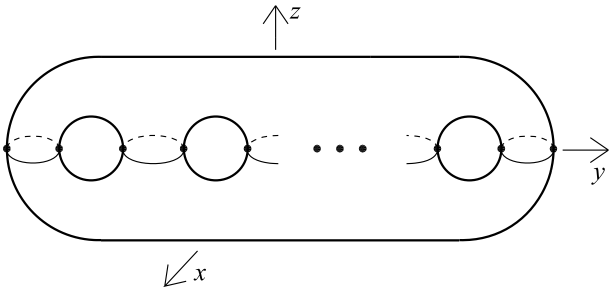

Fig. 1. Surface S_g embedded in $\mathbb{R}^3$.

symmetry defined by $j(x,y,z)=(-x,-y,-z)$. Denote by $C_{\mathcal{M}^{\pm}(S_{g-1})}(j)$ the centraliser of j in $\mathcal{M}^{\pm}(S_{g-1})$. The orbit space $S_{g-1}/\langle j\rangle$ is a nonorientable surface N_g of genus g and it is known (Theorem 1 of [3]) that the orbit space projection induces an epimorphism

$$\pi_j\colon C_{\mathcal{M}^{\pm}(S_{g-1})}(j)\to \mathcal{M}(N_g)$$

with kernel $\ker\pi_j=\langle j\rangle$. In particular

$$\mathcal{M}(N_g)\cong C_{\mathcal{M}^{\pm}(S_{g-1})}(j)/\langle j\rangle.$$

As was observed in the proof of Theorem 2.1 of [10], projection π_j has a section

$$i_j\colon \mathcal{M}(N_g)\to C_{\mathcal{M}(S_{g-1})}(j)\subset \mathcal{M}(S_{g-1}).$$

In fact, for any $h\in\mathcal{M}(N_g)$, we can define $i_j(h)$ to be an orientation preserving lift of h.

Let $\varrho\in C_{\mathcal{M}^{\pm}(S_{g-1})}(j)$ be the *hyperelliptic involution*, i.e. the half turn about the y-axis. The *hyperelliptic mapping class group* $\mathcal{M}^h(S_{g-1})$ is defined to be the centraliser of ϱ in $\mathcal{M}(S_{g-1})$. The hyperelliptic mapping class group turns out to be a very interesting and important subgroup, in particular its finite subgroups correspond to automorphism groups of hyperelliptic Riemann surfaces – see for example [9] and references therein.

Recently, we extended the notion of the hyperelliptic mapping class group to nonorientable surfaces [10], by defining $\mathcal{M}^h(N_g)$ to be the centraliser of $\pi_j(\varrho)$ in the mapping class group $\mathcal{M}(N_g)$. This definition is motivated by the notion of hyperelliptic Klein surfaces – see for example [4,5]. We say that $\pi_j(\varrho)$ is the *hyperelliptic involution* of N_g and by abuse of notation we write ϱ for $\pi_j(\varrho)$.

Since $\varrho\in C_{\mathcal{M}^{\pm}(S_{g-1})}(j)$, we have restrictions of π_j and i_j to the maps

$$\pi_j\colon C_{\mathcal{M}^{\pm}(S_{g-1})}(\langle j,\varrho\rangle)\to \mathcal{M}^h(N_g)$$
$$i_j\colon \mathcal{M}^h(N_g)\to C_{\mathcal{M}(S_{g-1})}(\langle j,\varrho\rangle)\subset \mathcal{M}^h(S_{g-1}).$$

2. Linear representations of the hyperelliptic mapping class group

Mapping class groups of projective plane N_1 and of Klein bottle N_2 are finite, hence the first nontrivial case is the group $\mathcal{M}(N_3)$. This is an interesting case, because it is well known [3,8] that

$$\mathcal{M}^h(N_3)=\mathcal{M}(N_3)\cong \mathrm{GL}(2,\mathbb{Z}).$$

In particular, $\mathcal{M}^h(N_3)$ has a faithful linear representation of real dimension 2.

For $g\geq 4$, we can produce a faithful linear representation of the hyperelliptic mapping class group $\mathcal{M}^h(N_g)$ as a composition of the section

$$i_j\colon \mathcal{M}^h(N_g)\to C_{\mathcal{M}(S_{g-1})}(\langle j,\varrho\rangle)\subset \mathcal{M}^h(S_{g-1})$$

and a faithful linear representation of $\mathcal{M}^h(S_{g-1})$ obtained by Korkmaz [6] or by Bigelow and Budney [2]. Recall that both of these representations of $\mathcal{M}^h(S_{g-1})$ are obtained form the Lawrence–Krammer representation of the braid group [1,7].

The above argument is immediate, but the resulting representation of $\mathcal{M}^h(N_g)$ is far from being optimal. In fact, if we use the Bigelow–Budney representation of $\mathcal{M}^h(S_{g-1})$ (which has much smaller dimension than the one obtained by Korkmaz), the dimension of the obtained representation of $\mathcal{M}^h(N_g)$ is equal to

$$2g\cdot\binom{2g-1}{2}+2(g-1)=2(g-1)(2g^2-g+1).$$

Theorem 1. *If $g \geq 4$, then the hyperelliptic mapping class group $\mathcal{M}^h(N_g)$ has a faithful linear representation of real dimension $g^2 - 1$.*

Proof. Let $\mathcal{M}^{\pm}(S_{0,g+1})$ be the extended mapping class group of a sphere with $g+1$ punctures $\{p_1, \dots, p_{g+1}\}$, and let $\mathcal{M}^{\pm}(S_{0,g,1})$ be the stabiliser of p_{g+1} with respect to the action of $\mathcal{M}^{\pm}(S_{0,g+1})$ on the set of punctures. By Theorem 2.1 of [10], the orbit space projection $\mathcal{M}^h(N_g) \to \mathcal{M}^h(N_g)/\langle \varrho \rangle$ induces an epimorphism

$$\pi_\varrho \colon \mathcal{M}^h(N_g) \to \mathcal{M}^{\pm}(S_{0,g,1})$$

with $\ker \pi_\varrho = \langle \varrho \rangle$. Moreover, by rescaling the Lawrence–Krammer representation of the braid group [1], Bigelow and Budney constructed in the proof of Theorem 2.1 of [2] a faithful linear representation

$$\mathcal{L}' \colon \mathcal{M}(S_{0,g,1}) \to \mathrm{GL}\left(\binom{g}{2}, \mathbb{R}\right).$$

To be more precise, they obtained a representation over $\mathbb{C}$; however, their argument works without any changes over $\mathbb{R}$.

Since $\mathcal{M}(S_{0,g,1})$ is a subgroup of index 2 in $\mathcal{M}^{\pm}(S_{0,g,1})$, the latter group has an induced faithful linear representation of dimension $2 \cdot \binom{g}{2} = g^2 - g$. This gives us a linear representation

$$\mathcal{L}_1 \colon \mathcal{M}^h(N_g) \to \mathrm{GL}\left(g^2 - g, \mathbb{R}\right)$$

with kernel $\ker \mathcal{L}_1 = \langle \varrho \rangle$. It is straightforward to check that if

$$\mathcal{L}_2 \colon \mathcal{M}^h(N_g) \to \mathrm{H}_1(N_g; \mathbb{R}) \subset \mathrm{GL}(g-1, \mathbb{R})$$

is a standard homology representation then $\mathcal{L}_1 \oplus \mathcal{L}_2$ is a required faithful linear representation of $\mathcal{M}^h(N_g)$ of dimension $g^2 - g + g - 1 = g^2 - 1$. □

Remark 1. The above theorem gives an upper bound $g^2 - 1$ on the minimal dimension of a faithful linear representation of the hyperelliptic mapping class group $\mathcal{M}^h(N_g)$. As we mentioned in the introduction, the hyperelliptic mapping class group $\mathcal{M}^h(N_3)$ has a faithful linear representation of real dimension 2, hence it seems very unlikely that the obtained bound is sharp.

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