Algebraic geometry

Remarks on minimal rational curves on moduli spaces of stable bundles

Remarques sur les courbes rationnelles minimales sur les espaces des modules de faisceaux stables

Liu Min
School of Mathematics and Statistics, Qingdao University, Qingdao 266071, PR China

A R T I C L E   I N F O

Article history:
Received 19 July 2016
Accepted after revision 31 August 2016
Available online 9 September 2016
Presented by Claire Voisin

A B S T R A C T

Let \( C \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field of characteristic zero, and \( M \) be the moduli space of stable bundles of rank 2 and with fixed determinant \( L \) of degree \( d \) on the curve \( C \). When \( g = 3 \) and \( d \) is even, we prove that, for any point \([W] \in M\), there is a minimal rational curve passing through \([W]\), which is not a Hecke curve. This complements a theorem of Xiaotao Sun.

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R É S U M É

Soient \( C \) une courbe projective lisse de genre \( g \geq 2 \) et \( M \) l'espace des modules de faisceaux stables de rang 2 et de déterminant fixe \( L \) de degré \( d \) sur \( C \). Nous prouvons que, lorsque \( g = 3 \) et \( d \) est pair, il existe, pour tout point \([W] \in M\), une courbe rationnelle minimale passant par \([W]\), qui n'est pas une courbe de Hecke. Cela complète un théorème de Xiaotao Sun.

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1. Introduction

Throughout this paper, we assume that \( C \) is a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field of characteristic zero. Let \( M := \text{SU}_C(r, L) \) be the moduli space of stable vector bundles of rank \( r \geq 2 \) and with the fixed determinant \( L \) of degree \( d \), which is a smooth quasi-projective Fano variety with \( \text{Pic}(M) = \mathbb{Z} \cdot \Theta \) and \( -K_M = 2(r, d) \Theta \), where \( \Theta \) is an ample divisor \([9,1]\). By a rational curve of \( M \), we mean a nontrivial proper morphism \( \phi : \mathbb{P}^1 \rightarrow M \) and its degree is defined to be \( \deg \phi^*(-K_M) \) (with respect to the ample anti-canonical line bundle \(-K_M\)).

In \([10]\), Xiaotao Sun has determined all rational curves of minimal degree passing through generic points of \( M \) except in the case where \( g = 3 \), \( r = 2 \), and \( d \) is even.
**Theorem 1.1.** *(Theorem 1 of [10])* If \( g \geq 3 \), then any rational curve \( \phi : \mathbb{P}^1 \to M \) passing through the generic point has degree at least \( 2r \). It has degree \( 2r \) if and only if it is a Hecke curve **unless** \( g = 3, r = 2, \text{ and } d \) is even.

This implies that all the rational curves of \((-K_M\))-degree smaller than \(2r\), called small rational curves, must lie in a proper closed subset \([3,4]\). In this note, we remark that the condition in Sun’s Theorem is necessary:

**Theorem 1.2.** If \( g = 2, r = 2 \) and \( d \) is odd, then, for any \([W] \in M\), there exists a rational curve passing through it, which has degree 2.

If \( g = 3, r = 2 \) and \( d \) is even, then, for any point \([W] \in M\), there exists a rational curve of degree 4 passing through it, which is not a Hecke curve.

Recall that, by Lemma 2.1 of [10], any rational curve \( \phi : \mathbb{P}^1 \to M \) is defined by a vector bundle \( E \) on \( f : X = C \times \mathbb{P}^1 \to C \).

**If** \( E \) **is semi-stable on generic fiber** \( X_p = f^{-1}(\xi) \) *(tensoring a pullback of line bundle on \( \mathbb{P}^1 \), we can assume the restriction of \( E \) to a generic fiber is of the form \( \mathcal{O}_{X_p}^{d} \), according to the arguments of section 2 in [10], there is a finite set \( S \subset C \) of points and a vector bundle \( V \) on \( C \) such that \( E \) just suits in the exact sequence

\[
0 \to f^*V \to E \to \bigoplus_{p \in S} Q_p \to 0
\]

where \( Q_p \) is a vector bundle on \( X_p = \{p\} \times \mathbb{P}^1 \). The curves defined by such \( E \) were said to be of Hecke type in [8,11] (since a Hecke curve by definition is defined by a vector bundle \( E \) suited in \( 0 \to f^*V \to E \to \mathcal{O}_{X_p}(-1) \to 0 \). **If** \( E \) **is not semi-stable on the generic fiber** \( X_p \) *(curves defined by such \( E \) were said of split type in [11])** and the curve has minimal degree \( 2(r, d) \), then \( E \) must suit in

\[
0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0
\]

where \( \pi : X \to \mathbb{P}^1 \) is the projection and \( V_1, V_2 \) are stable vector bundles on \( C \) of rank \( r_1, r_2 \), and degrees \( d_1, d_2 \) satisfy \( r_1d - d_1 = (r, d) \). Note that rational curves of degree \( 2(r, d) \) have degree 1 with respect to \( \Theta \) because of \(-K_M = 2(r, d) \Theta \), which will be called lines in \( M \).

The rational curves we constructed in **Theorem 1.2** are of split type (thus they are not Hecke curves). We have in fact a more general result. Let \( M = SU_C(2, \mathcal{L}) \) be the moduli space of rank-two stable bundles with fixed determinant \( \mathcal{L} \) on a smooth projective curve \( C \) of genus \( g \geq 3 \). Let \( M_1 \subset M \) be the locus of stable bundles \([W] \in M\) with the Segre invariant \( s(W) = s \) *(refer to Section 3 for the definition of Segre invariant)*. Then we have the following theorem.

**Theorem 1.3.** When \( d \) is even, for any \([W] \in M_2\), there is a rational curve of split type passing through it, which has degree 4. If \( d \) is odd, for any \([W] \in M_1\), there is a rational curve of split type passing through it, which has degree 2.

When \( g = 3 \) and \( d \) is even, we have \( M_2 = M \) *(see Lemma 3.1)*. Thus **Theorem 1.2** is a corollary of **Theorem 1.3**.

### 2. Rational curves of split type

Let \( C \) be a smooth projective curve with genus \( g \geq 2 \) over an algebraically closed field of characteristic zero, \( W \) be a stable bundle of rank \( r \) and of degree \( d \) with determinant \( \mathcal{L} \) over \( C \). Assume that there is a stable subbundle \( V_1 \) of \( W \) such that

\[
r_1d - d_1 = (r, d),
\]

where \( r_1 = \text{rank} \, V_1, d_1 = \deg \, V_1 \) and \( d = \deg \, W \). Let \( V_2 := V/V_1 \) be the quotient bundle, then \( W \) fits a non-trivial extension

\[
0 \to V_1 \to W \to V_2 \to 0.
\]

It is known that there is a family of vector bundles \( \{E_p\}_{p \in P} \) on \( C \) parametrized by \( P = \mathbb{P} \text{Ext}^1(V_2, V_1) \) so that for each \( p \in P, \ E_p \) is isomorphic to the bundle obtained as the extension of \( V_2 \) by \( V_1 \) given by \( p \) *(see Lemma 2.3 of [9])**. Let \( l \) be a line in \( P = \mathbb{P} \text{Ext}^1(V_2, V_1) \) passing through the point \( p_0 \), where \( p_0 \) is the point in \( P \) given by \( \xi \). If it happens that \( E_p \) is stable for each \( p \in l \), then

\[
\{E_p\}_{p \in l}
\]

will define a rational curve of degree \( 2(r, d) \) *(with respect to \( -K_M \)) passing through \([W] \in SU_C(r, \mathcal{L}) \) *(10,4)*. Such a rational curve in \( SU_C(r, \mathcal{L}) \) will be called a **rational curve of split type**.

It is known that an extension \( 0 \to E \to W \to F \to 0 \), where \( E, W, F \) are vector bundles on \( C \), gives rise to an element \( \delta(W) \in H^1(C, Hom(F, E)) \), which is the image of the identity homomorphism in \( H^0(C, Hom(F, F)) \) by the connecting homomorphism \( H^0(C, Hom(F, F)) \to H^1(C, Hom(F, E)) \). This gives a one-one correspondence between the set of equivalent classes of extensions of \( F \) by \( E \) and \( H^1(C, Hom(F, E)) \) *(refer to section 2 in [9]).*
Lemma 2.1. Let $d$ be an even number, and $0 \to L_1 \to W \to L_2 \to 0$ be any non-trivial extension of $L_2$ by $L_1$, where $L_1$ (resp. $L_2$) is a line bundle of degree $\frac{d}{2} - 1$ (resp. $\frac{d}{2} + 1$). Then

(i) $W$ is semi-stable;
(ii) $W$ is non-stable if and only if the element $\delta(W) \in H^1(C, L_2^{-1} \otimes L_1)$ corresponding to $W$ is in the kernel of the map

$$H^1(C, L_2^{-1} \otimes L_1) \to H^1(C, L_2^{-1} \otimes L_1 \otimes L_2),$$

for some $x \in C$, where $L_x = \mathcal{O}_C(x)$ is the line bundle defined by $x$. In this case, $W$ is $S$-equivalent to $L_2 \otimes L_x^{-1} \otimes L_1 \otimes L_x$ (refer to section 2 of [7] for the definition of $S$-equivalent).

Proof. (i) See Lemma 2.2 in [4] and [5].

(ii) Let $L'$ be a line bundle of degree $\frac{d}{2}$. Then, since $H^0(C, \text{Hom}(L', L_1)) = 0$, it is easy to see that $H^0(C, \text{Hom}(L', W)) \neq 0$ if and only if $L'$ is of the form $L_2 \otimes L_x^{-1}$ for some $x \in C$ and the natural map $L_2 \otimes L_x^{-1} \to L_2$ can be lifted into a map $L_2 \otimes L_x^{-1} \to W$.

Consider the commutative diagram of vector bundles

$$
\begin{array}{ccc}
0 & \to & \text{Hom}(L_2, L_1) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}(L_2 \otimes L_x^{-1}, L_1)
\end{array}
\quad
\begin{array}{ccc}
& \to & \text{Hom}(L_2, W) \\
& \downarrow & \downarrow \\
& \to & \text{Hom}(L_2 \otimes L_x^{-1}, W)
\end{array}
\quad
\begin{array}{ccc}
& & \text{Hom}(L_2, L_2) \\
& & \downarrow \\
& & \text{Hom}(L_2 \otimes L_x^{-1}, L_2)
\end{array}
\to 0,
$$

where the horizontal sequences are exact and the vertical maps are induced by the natural map $L_2 \otimes L_x^{-1} \to L_2$. From this, we deduce the commutative diagram

$$
\begin{array}{ccc}
0 & \to & H^0(C, \text{Hom}(L_2, W)) \\
\downarrow & & \downarrow \\
0 & \to & H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, W))
\end{array}
\quad
\begin{array}{ccc}
& \to & H^0(C, \text{Hom}(L_2, L_2)) \\
& \downarrow & \downarrow \\
& \to & H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_2))
\end{array}
\quad
\begin{array}{ccc}
& & H^1(C, \text{Hom}(L_2, L_1)) \\
& & \downarrow \\
& & H^1(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_1))
\end{array}
\to \cdots
$$

which implies the lemma. \qed

Remark 2.2. Lemma 2.1 (ii) asserts that the non-stable bundles in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ correspond precisely to the image of $C$ in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ under the map given by the linear system $K_C \otimes L_1^{-1} \otimes L_2$. Which implies that the dimension of the subset of non-stable bundles in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ is at most 1.

3. Proof of Theorem 1.3

Let $C$ be a smooth irreducible curve over an algebraically closed field of characteristic zero, $W$ a vector bundle of rank 2 over $C$, set

$$m(W) := \max \{\text{deg}(L) | L \subset W \text{ is a sub line bundle of } W\}.$$  \hfill (3)

A sub line bundle $L$ of $W$ of maximal degree $m(W)$ is called a maximal sub line bundle. The Segre invariant is defined by

$$s(W) := \text{deg}(W) - 2m(W).$$  \hfill (4)

Note that $s(W) \equiv \text{deg}(W)$ (mod 2) and that $W$ is stable (resp. semi-stable) if and only if $s(W) \geq 1$ (resp. $s(W) \geq 0$). Nagata proved in [6] that

$$s(W) \leq g.$$

It is easy to see that

Lemma 3.1. If $g = 3$, then, for any stable bundle $W$ over $C$ of rank 2 and with even degree $d$, we have $s(W) = 2$.

In general, the function $s : M \to \mathbb{Z}$ defined by $[W] \mapsto s(W)$ is lower semicontinuous and gives a stratification of $M$ into locally closed subsets $M_s$ according to the value of $s$. Then, by Proposition 3.1 in [2], we have

Proposition 3.2. ([2]) Suppose that $1 \leq s \leq g - 2$ and $s \equiv d$ (mod 2). Then $M_s$ is an irreducible algebraic variety of dimension $2g + s - 2$.

The proof of Theorem 1.3 follows the following two propositions.
**Proposition 3.3.** Suppose that $g \geq 3$, $r = 2$, $d$ is even and $M_2$ is non-empty. Then, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4.

**Proof.** For any $[W] \in M_2$, there is a sub line bundle $L_1$ of $W$ with deg $L_1 = \frac{d}{2} - 1$, where $d = \deg \mathcal{L}$. Let $L_2 := W / L_1$ be the quotient bundle, which has degree $\frac{d}{2} + 1$. It is easy to see that

$$1 \times d - \left( \frac{d}{2} - 1 \right) \times 2 = 2 = (2, d).$$

Let $i : L_1 \to W$ be the natural injection, then

$$0 \longrightarrow L_1 \overset{i}{\longrightarrow} W \overset{i}{\longrightarrow} L_2 \longrightarrow 0$$

is a non-trivial extension (otherwise, we have $W \cong L_1 \oplus L_2$, which contradicts the stability of $W$).

It is known that there is a family of vector bundles $\mathcal{E}$ on $C$ parametrized by $P_{(L_1, L_2)} = \mathbb{P} \text{Ext}^1(L_2, L_1)$ so that for each $p \in P_{(L_1, L_2)}$, the $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of $L_2$ by $L_1$ given by $p$ (see Lemma 2.3 of [9]). More precisely, there is a universal extension

$$0 \to f^* L_1 \otimes \mathcal{O}_{P_{(L_1, L_2)}}(1) \to \mathcal{E} \to f^* L_2 \to 0 \tag{5}$$

on $C \times P_{(L_1, L_2)}$, where $f : C \times P_{(L_1, L_2)} \to C$ and $\pi : C \times P_{(L_1, L_2)} \to P_{(L_1, L_2)}$ are projections. Then $\mathcal{E}$ is a family of semi-stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$ (Lemma 2.1). Thus, the universal extension (5) defines a morphism

$$\Phi_{(L_1, L_2)} : P_{(L_1, L_2)} \longrightarrow UC(2, \mathcal{L}), \tag{6}$$

where $UC(2, \mathcal{L})$ denotes the moduli space of semi-stable bundles of rank 2 and with fixed determinant $\mathcal{L}$, which is a projective compactification of $M$.

It is easy to see that $P_{(L_1, L_2)}$ is a projective space of dimension $g \geq 3$. By Lemma 2.1 and Remark 2.2, there is a line $l$ in $P_{(L_1, L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \overset{i}{\longrightarrow} W \longrightarrow L_2 \longrightarrow 0]$$

such that $\mathcal{E}_p$ is stable for each $p \in l$. Thus, $\Phi_{(L_1, L_2)}(l) \subset M = SU_C(2, \mathcal{L})$ and

$$\Phi_{(L_1, L_2)}|l : l \to M = SU_C(2, \mathcal{L}) \tag{7}$$

is a rational curve of split type passing through the point $[W] \in M$. 

**Proposition 3.4.** Suppose $g \geq 2$, $r = 2$, $d$ is odd and $M_1$ is non-empty. Then, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.

**Proof.** Let $[W]$ be a point in $M_1$, then we have $s(W) = 1$ and there is a sub line bundle $L_1$ of $W$ with deg $L_1 = \frac{d-1}{2}$, where $d = \deg \mathcal{L}$. Let $L_2 := W / L_1$, which is a line bundle of degree $\frac{d+1}{2}$. It is easy to see that

$$1 \times d - \frac{d-1}{2} \times 2 = 1 = (2, d).$$

Let $i : L_1 \to W$ be the natural injection, then

$$0 \longrightarrow L_1 \overset{i}{\longrightarrow} W \overset{i}{\longrightarrow} L_2 \longrightarrow 0$$

is a non-trivial extension because $W$ is a stable bundle.

It is known that there is a family of vector bundles $(\mathcal{E}_p)$ on $C$ parametrized by $P_{(L_1, L_2)} = \mathbb{P} \text{Ext}^1(L_2, L_1)$ such that for each $p \in P_{(L_1, L_2)}$, $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of $L_2$ by $L_1$ given by $p$ (Lemma 2.3 of [9]). By Lemma 3.1 of [10], $(\mathcal{E}_p)$ is a family of stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$, which defines a morphism

$$\Psi_{(L_1, L_2)} : P_{(L_1, L_2)} \longrightarrow SU_C(2, \mathcal{L}) = M. \tag{8}$$

Let $l$ be a line in $P_{(L_1, L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \overset{i}{\longrightarrow} W \longrightarrow L_2 \longrightarrow 0],$$

then

$$\Psi_{(L_1, L_2)}|l : l \to M = SU_C(2, \mathcal{L}) \tag{9}$$

is a rational curve of split type passing through the point $[W] \in M$, which has degree 2. 

When $g = 2$, the same as Lemma 3.1, we have:
Lemma 3.5. If \( g = 2, r = 2 \) and \( d \) is odd, for any \([W] \in M\), \( s(W) = 1\).

By Lemma 3.5 and Proposition 3.4, we have:

Proposition 3.6. If \( g = 2, r = 2 \) and \( d \) is odd, then, for any \([W] \in M\), there exists a rational curve of split type passing through it, which has degree 2.

Acknowledgements

The author is grateful to her supervisor Prof. Xiaotao Sun for his helpful suggestions in the preparation of this paper and to Prof. Meng Chen and Prof. Kejian Xu for their help.

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