Dynamical systems

On the Anosov character of the Pappus–Schwartz representations

Sur le caractère Anosov des représentations de Pappus–Schwartz

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A B S T R A C T

In the paper Pappus’s Theorem and The Modular Group (1993) [4], R.E. Schwartz observed that the classical Pappus theorem gives rise to an action of the modular group on the space of marked boxes. He inferred from this a 2-dimensional family of faithful representations of the modular group into the group of projective symmetries. These representations have a dynamical behavior very similar to the one of Anosov representations, even if they are never Anosov themselves. In this note, we announce the main result of V. Pardini Valério (2016) [3], which elucidates this Anosov character of the Schwartz representations by proving that their restrictions to the index-2 subgroup are limits of Anosov representations.

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R É S U M É


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1. Pappus theorem and marked boxes

Let $V$ be a 3-dimensional vector space and $\mathbb{P}(V)$ the associated projective spaces with $V$.

**Theorem 1.1 (Pappus).** If the points $a_1, a_2, a_3$ are colinear and the points $b_1, b_2, b_3$ are colinear in $\mathbb{P}(V)$, then the points $c_3 = a_1b_2 \cap a_2b_1$, $c_2 = a_1b_3 \cap a_3b_1$, $c_1 = a_2b_3 \cap a_3b_2$ are also colinear in $\mathbb{P}(V)$.

An important fact is that the Pappus Theorem, on certain conditions, can be iterated infinitely many times (see Fig. 1).

![Fig. 1. Iteration of the Pappus Theorem; marked box $\Theta$ in $\mathbb{P}(V)$.](image)

A **marked box** $\Theta$ is a special pair of 6-tuples having the incidences relatives shown in Fig. 1. If $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$, then $p, q, r, s, t, b \in \mathbb{P}(V)$, $P, Q, R, S, T, B \in \mathbb{P}(V^*)$, $T \cap B \notin \{p, q, r, s, t, b\}$, $S = bp, R = bq, P = ts, Q = tr, T = pq$ and $B = rs$. Let $CM$ be the set of marked boxes.

The marked box $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$ is **convex** if the following two conditions hold: $p$ and $q$ separate $t$ and $T \cap B$ on the line $T$, and $r$ and $s$ separate $b$ and $T \cap B$ on the line $B$. The **convex interior** of $\Theta$ is the open convex quadrilateral whose vertices, in cyclic order, are $p, q, r$ and $s$ (for more details, see [3, section 2.2]). We denote it by $\hat{\Theta}$.

**1.1. The action of the group of projective symmetries on $CM$**

Let $V$ be a 3-dimensional vector space and $V^*$ its dual vector space. Projective transformations and dualities generate the group $G$ of projective symmetries of the flag variety $\mathcal{F}$. Projective transformations alone define an index-2 subgroup $\mathcal{H} \cong PGL(3, \mathbb{R})$ of $G$.

Given a projective transformation $T$, and using the notation $x = T(x)$ for every point or line $x$ in $\mathbb{P}(V)$, and for any marked box $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$, define (see Fig. 1):

$T(\Theta) = ((p, q, r, s; t, b), (P, Q, R, S; T, B)) \in CM$.

Similarly, given a duality $D$, and denoting $x^* = D(x)$ for $x \in \mathbb{P}(V)$, and $X^* = D^*(X)$ for $X$ being a projective line, define (pay attention to the maybe surprising Schwartz re-ordering):

$D(\Theta) = ((P^*, Q^*, S^*, R^*; T^*, B^*), (q^*, p^*, r^*, s^*; t^*, b^*)) \in CM$.

**1.2. The group of elementary transformations of marked boxes**

Let $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B)) \in CM$. Pappus’ Theorem gives us two new elements of $CM$ that are images of $\Theta$ by two special permutations $\tau_1$ and $\tau_2$ on $CM$ (see Fig. 2). These permutations are defined by

$\tau_1(\Theta) = ((p, q, Q, R, P; s, t, (qs)(pr)), (P, Q, Q, P, S; T, (Q)(P, S)))$,

$\tau_2(\Theta) = ((Q, R, P, S, r, (qs)(pr), b), (p, q, s, Q, R, (P, S, B))$.

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1. In this brief note, we abusively do not distinguish overmarked boxes from marked boxes as in [3] and [4].
Theorem 3.2. \[
\rho/\Theta_1(\gamma)(/\Theta_1(\gamma)) = (\rho/\Theta_1(\gamma)(\Theta(\gamma)) \quad (\rho_\Theta-\text{equivariant property}).
\]

Proof. The proof follows basically from the fact that the actions of PSL\(2, \mathbb{Z}\) and \(\Theta\) on \(L_0\) commute with each other (Remark 1), even if the actions of \(\Theta\) and \(G\) on CM commute with each other (see [4, Theorem 2.4] and, for more details, [3, Lemma 3.1, Theorem 3.2]). Already the fact that \(\rho_\Theta : \text{PSL}(2, \mathbb{Z}) \rightarrow G\) is a faithful representation follows from the fact that the action of PSL\(2, \mathbb{Z}\) on \(L_0\) is free. \(\square\)
2.2. The Schwartz map

Two Farey geodesics have the same tail in $\partial \mathbb{H}^2$ if and only if their labels are marked boxes with the same top point. Therefore, it defines a map $\psi : \mathbb{Q} \cup \{\infty\} \to \mathbb{P}(V)$ that can be extended to an injective $\rho_\Theta$-equivariant continuous map $\varphi_\Theta : \partial \mathbb{H}^2 \to \mathbb{P}(V)$ (see [4, Theorem 3.2]). Similarly, there is an injective $\rho_\Theta$-equivariant continuous map $\varphi_\Theta^* : \partial \mathbb{H}^2 \to \mathbb{P}(V^*)$. The maps $\varphi_\Theta$ and $\varphi_\Theta^*$ combine to form a $\rho_\Theta$-equivariant map:

$$\Phi := (\varphi_\Theta, \varphi_\Theta^* : \partial \mathbb{H}^2 \to \mathcal{F} \subset \mathbb{P}(V) \times \mathbb{P}(V^*)),$$

where $\mathcal{F}$ is the flag variety. We call the composition of $\Phi$ with the canonical projection of $\partial \text{PSL}(2, \mathbb{Z})$ into $(\partial \mathbb{H}^2)$ the Schwartz map, where $\partial \text{PSL}(2, \mathbb{Z})$ is the Gromov boundary.

3. Anosov representations

The Anosov representation theory was introduced by François Labourie in [2] for representations of closed surface groups. It does not apply directly to the modular group $\text{PSL}(2, \mathbb{Z})$. However $\text{PSL}(2, \mathbb{Z})$ is Gromov-hyperbolic. Hence we use here a formulation inspired from [1], in the simple case of convex cocompact subgroups of $\text{PSL}(2, \mathbb{R})$.

3.1. Definition of Anosov representations

Given $x \in \mathbb{P}(V)$, let $Q_x(V)$ be the space of norms on tangent space $T_x \mathbb{P}(V)$ at $x$. Similarly, given $x \in \mathbb{P}(V^*)$, let $Q_x(V^*)$ be the space of norms on tangent space $T_x \mathbb{P}(V^*)$ at $x$. We denote by $Q(V)$ the bundle of base $\mathbb{P}(V)$ with fiber $Q_x(V)$ on $x \in \mathbb{P}(V)$. Similarly, we denote by $Q(V^*)$ the bundle of base $\mathbb{P}(V^*)$ with fiber $Q_x(V^*)$ on $x \in \mathbb{P}(V^*)$. Let $\Omega(\phi^t)$ be the nonwandering set of the geodesic flow $\phi^t$ on $T^1(\Gamma \setminus \mathbb{H}^2)$.

**Definition 3.1.** Let $\Gamma$ be a convex cocompact discrete subgroup of $\text{PSL}(2, \mathbb{R})$ with limit set $\Lambda_\Gamma$. A homomorphism $\rho : \Gamma \to \mathcal{H} \cong \text{PSL}(3, \mathbb{R})$ is an **Anosov representation** if there are

(i) a $\Gamma$-equivariant map

$$\Phi = (\varphi, \varphi^* : \Lambda_\Gamma \to \mathcal{F} \subset \mathbb{P}(V) \times \mathbb{P}(V^*)),$$

(ii) two maps $\nu_+ : \Omega(\phi^t) \to Q(V)$ and $\nu_- : \Omega(\phi^t) \to Q(V^*)$ such that, for every nonwandering geodesic $c : \mathbb{R} \to \mathbb{H}^2$ with extremities $c_-, c_+ \in \Lambda_\Gamma$ we have that

- for all $v \in T_{\phi^t(c)} \mathbb{P}(V)$ the size of $v$ for the norm $\nu_+(c(t), c'(t))$, increases exponentially with $t$;
- for all $v \in T_{\phi^t(c)} \mathbb{P}(V^*)$ the size of $v$ for the norm $\nu_-(c(t), c'(t))$, decreases exponentially with $t$.

The group $\Gamma$ of this definition is a Gromov-hyperbolic group. Since it is convex cocompact, its Gromov boundary $\partial \Gamma$ is $\Gamma$-equivariantly homeomorphic to its limit set $\Lambda_\Gamma$.

In the sequel, we will consider Anosov representations of a finite index subgroup of $\text{PSL}(2, \mathbb{Z})$, which is not convex cocompact. But we replace simply $\text{PSL}(2, \mathbb{Z})$ by a convex cocompact discrete subgroup of $\text{PSL}(2, \mathbb{R})$ obtained by “opening the cusps”, thus we build an example on a 3-fold symmetric 3-punctured sphere having geodesic boundaries of small length.

3.2. Schwartz representations are not Anosov

The Schwartz representation $\rho_\Theta$ preserves a topological circle in the flag variety, on which it is topologically conjugated to the usual action of $\text{PSL}(2, \mathbb{Z})$ on the conformal boundary of the hyperbolic plane. This property is very similar to the one associated with Anosov representations of surface groups into $\text{PSL}(3, \mathbb{R})$. However, $\rho_\Theta$ cannot be Anosov since the Gromov boundary of $\text{PSL}(2, \mathbb{Z})$ is a Cantor set and not a circle. Thus the Schwartz maps $\psi$ and $\psi^*$ cease to be injective, contradicting a property of Anosov representations.

4. A new family of representations

In order to show that Schwartz representations are limits of Anosov representations, we define a new group of transformations of $CM$.

4.1. A new group of transformations of $CM$

Let $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$ be a convex marked box. Let us consider the unique affine chart in $P(V)$ such that $\Theta$ is seen as the “special square” where $p = (-1, 1), q = (1, 1), r = (1, -1)$ and $s = (-1, -1)$. Let $\lambda$ and $\mu$ be real numbers. Let $\Sigma_{(\lambda, \mu)} : CM \to CM$ be a new transformation of marked boxes such that the image of $\Theta$ is given by applying the matrix $\Sigma_{(\lambda, \mu)} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$ to this special square in $P(V)$. This new transformation has some interesting properties:
(1) it commutes with elements of \( \mathcal{H} \) (projective transformations), but it does not commute with elements of \( \mathcal{G} \setminus \mathcal{H} \) (dualities) acting on \( CM \).

(2) considering the particular case where \( \mu = 2 \lambda \) and let \( \sigma_\lambda := \sigma_{(\lambda, 2\lambda)} \), then the relation \( i\sigma_\lambda = \sigma_\lambda^{-1}i \) holds.

Let us define three more new transformations on \( CM \) as follows:

\[
i^\lambda \ := \ \sigma_\lambda i \quad \tau^1_\lambda \ := \ \sigma_\lambda \tau_1 \quad \tau^2_\lambda \ := \ \sigma_\lambda \tau_2.
\]

The semigroup \( \Theta^\lambda \) of \( S(CM) \), generated by \( i^\lambda, \tau^1_\lambda \) and \( \tau^2_\lambda \), is also an isomorphic group to the modular group \( (\text{PSL}(2, \mathbb{Z}) \cong \Theta \cong \Theta^\lambda) \) and, for \( \lambda = 0 \), of course \( \Theta^\lambda = \Theta \).

### 4.2. New representations

Given a convex marked box \( \Theta \) and a real number \( \lambda \), again let us consider the Farey lamination \( L_0 \) of \( \mathbb{H}^2 \) introduced in Remark 1; and the new group \( \Theta^\lambda \) of transformations of \( CM \). In order to circumvent the inconvenient of \( \Theta^\lambda \) not commuting with dualities acting on \( CM \), we restrict to the unique index 2 subgroup \( \text{PSL}(2, \mathbb{Z})_0 \) of \( \text{PSL}(2, \mathbb{Z}) \), isomorphic to \( \mathbb{Z}_3 \ast \mathbb{Z}_3 \). The main Theorem announced in this note is:

**Theorem 4.1.** Let \( \Theta \) be a convex marked box and let \( \lambda \in \mathbb{R} \). There is a representation \( \rho^\lambda_{\Theta} : \text{PSL}(2, \mathbb{Z})_0 \to \mathcal{H} \rtimes \mathcal{G} \) such that for every leaf \( e \) of \( L_0 \) and every \( \gamma \in \text{PSL}(2, \mathbb{Z})_0 \) we have:

\[
[\Theta](\gamma e) = \rho^\lambda_{\Theta}(\gamma)([\Theta](e)).
\]

Moreover, if \( \lambda \) is negative, then \( \rho^\lambda_{\Theta} \) is Anosov.

The key point of the our construction is: if \( \lambda \leq 0 \), then for any convex marked box \( \Theta \), we have \( \tau^1_\lambda(\Theta) \subseteq \Theta, \quad \tau^2_\lambda(\Theta) \subseteq \Theta, \quad \text{and} \quad \tau^1_\lambda(\Theta) \cup \tau^2_\lambda(\Theta) = \emptyset \) in \( \mathbb{P}(V) \). Furthermore, if \( \lambda \) is negative, then we have the same properties, but now for the closures of the interiors of the marked boxes. The Anosov character of the representations \( \rho^\lambda_{\Theta} \), for \( \lambda < 0 \), is a consequence of this stronger property.

**Remark 2.** When the marked box \( \Theta \) is symmetric, i.e. when \( t = (0, 1) \) and \( b = (0, -1) \) on the special affine chart, the Schwartz representation, restricted to the index 2 subgroup \( \text{PSL}(2, \mathbb{Z})_0 \), is the one arising by the inclusion \( \text{PSL}(2, \mathbb{Z})_0 \subset \text{PSL}(2, \mathbb{R}) \subset \text{PGL}(3, \mathbb{Z}) \) where the last inclusion is reducible, i.e. is such that \( \text{PSL}(2, \mathbb{R}) \) preserves a splitting of \( V \) as a sum of a line and a plane. The representation \( \rho^\lambda_{\Theta} \), for \( \lambda < 0 \), corresponds to the deformation of \( \text{PSL}(2, \mathbb{Z})_0 \) inside \( \text{PSL}(2, \mathbb{R}) \) consisting in opening up the cusp.

### 5. Conclusion

In summary, since the space of marked boxes up to projective transformations is 2-dimensional, we have defined a 3-dimensional family of representations \( \rho^\lambda_{\Theta} : \text{PSL}(2, \mathbb{Z})_0 \to \text{PGL}(3, \mathbb{R}) \) where \( \lambda \) is a real parameter. When \( \lambda \) vanishes, \( \rho^0_{\Theta} \) is the restriction of the Schwartz representation \( \rho_{\Theta} \) to \( \text{PSL}(2, \mathbb{Z})_0 \), and when \( \lambda \) is negative, \( \rho^\lambda_{\Theta} \) is Anosov. In particular, the Schwartz representations are limits of the Anosov representations in the space of all representations of \( \text{PSL}(2, \mathbb{Z})_0 \) into \( \text{PGL}(3, \mathbb{R}) \).

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