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Approximations of standard equivalence relations and Bernoulli percolation at p_u

Approximations de relations d'équivalence standard et percolation de Bernoulli à p_u

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A B S T R A C T

The goal of this note is to announce certain results in orbit equivalence theory, especially concerning the approximation of p.m.p. standard equivalence relations by increasing sequences of sub-relations, with application to the behavior of the Bernoulli percolation on Cayley graphs at the threshold p_u.

R É S U M É

Le but de cette note est d'annoncer certains résultats d'équivalence orbitale, concernant notamment la notion d'approximation de relations d'équivalence standard préservant la mesure de probabilité par suites croissantes de sous-relations, avec application au comportement en p_u de la percolation de Bernoulli sur les graphes de Cayley.

Version française abrégée

La notion de relation d'équivalence standard *hyperfinie* (*i.e.* réunion croissante de sous-relations standard finies) joue un rôle fondamental en théorie de l'équivalence orbitale. Plus généralement, on considère la notion d'*approximation* d'une relation d'équivalence mesurée standard $\mathcal{R}$, *i.e.* la possibilité d'écrire $\mathcal{R}$ comme une réunion croissante d'une suite de sous-relations d'équivalence standard $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \nearrow \mathcal{R}_n$. Une telle approximation est dite *triviale* s'il existe une partie borélienne A de mesure non nulle sur laquelle les restrictions coïncident à partir d'un certain rang : $\mathcal{R}_n \restriction A = \mathcal{R} \restriction A$. Nous établissons des conditions sous lesquelles les approximations de certaines relations d'équivalence *préservant la mesure de probabilité* (*p.m.p.*) sont nécessairement triviales.

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Théorème 0.1. *Soit G un groupe engendré par deux sous-groupes infinis de type fini H et K qui commutent. Considérons une action p.m.p. $G \overset{\alpha}{\curvearrowright} (X, \mu)$ sur l'espace borélien standard telle que H agit de manière fortement ergodique et K de manière ergodique. Alors, toute approximation de la relation d'équivalence engendrée $\mathcal{R}_\alpha$ est nécessairement triviale.*

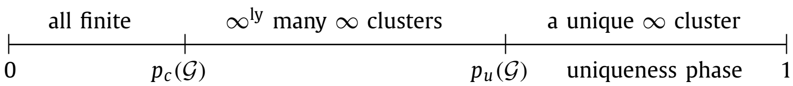
Puisque les actions par décalage de Bernoulli des groupes non moyennables sont automatiquement fortement ergodiques, ce résultat a des conséquences en théorie de la percolation de Bernoulli sur les graphes de Cayley. Pour des informations concernant les liens entre équivalence orbitale et percolation, on peut consulter [3]. En fait, le *couplage standard* permet de traduire l'étude relative aux variations du paramètre de rétention $p \in [0, 1]$ de la percolation en l'étude d'une famille croissante de relations d'équivalence standard p.m.p. $(\mathcal{R}_p)_{p \in [0,1]}$, telle que pour tout $q \in]0, 1]$, on a $\mathcal{R}_q = \bigcup_{p<q} \nearrow \mathcal{R}_p$. Le paramètre critique p_u (*cf.* [5]) est l'infimum des p pour lesquels on peut trouver une partie borélienne non négligeable A sur laquelle les restrictions $\mathcal{R}_1 \restriction A$ et $\mathcal{R}_p \restriction A$ coïncident (de tels p sont dits appartenir à la *phase d'unicité*). Pour les groupes dont les actions Bernoulli n'admettent pas d'approximation non triviale, le paramètre p_u lui-même n'appartient pas à la phase d'unicité. *C'est le cas des groupes qui apparaissent dans le théorème 0.1.* Des conditions d'exhaustion par des sous-groupes distingués (en un sens faible) nous permettent d'élargir encore la famille de nouveaux exemples.

Les notions de *dimension géométrique* et de *dimension approximative* d'une relation d'équivalence mesurée ont été introduites dans [2, section 5], où il est démontré qu'une non-annulation du d-ième nombre de Betti ℓ^2 fournit une minoration par d de ces deux notions de dimension. La première est analogue à la notion de dimension géométrique pour un groupe et la deuxième est le minimum des lim inf des dimensions géométriques le long des suites approximantes. Bien entendu, pour les relations non approximables, les deux notions de dimension coïncident. On peut alors exhiber des familles de groupes possédant des actions de dimensions approximatives variables.

English version

1. Bernoulli bond percolation

Let $\mathcal{G} = (G, \mathrm{E})$ be a Cayley graph for a finitely generated group G. The *Bernoulli bond percolation* on $\mathcal{G}$, with retention parameter $p \in [0, 1]$, considers the i.i.d. assignment to each edge in E of the value 1 (open) with probability p and of the value 0 (closed) with probability $1 - p$. The number of infinite *clusters* (connected components of open edges), for the resulting probability measure $\mathbf{P}_p$ on $\{0, 1\}^{\mathrm{E}}$, is $\mathbf{P}_p$-a.s. either 0, 1 or ∞. There are two critical values, $0 < p_c(\mathcal{G}) \leq p_u(\mathcal{G}) \leq 1$, depending on the graph, which govern three regimes, as summarized in the following picture (see [5]):



While it is far from being entirely understood, there are some partial results concerning the situation at the threshold $p = p_u$, and our Theorem 1.1 contributes to this study.

For groups with infinitely many ends, $p_u = 1$ [8]; thus the percolation at $p = p_u$ belongs to the uniqueness phase. On the other hand, the percolation at the threshold $p = p_u$ does not belong to the uniqueness phase (and thus $p_u < 1$) for all Cayley graphs of infinite groups with Kazhdan's property (T) [9]. Y. Peres [11] proved that for a non-amenable direct product of infinite groups $G = H \times K$, and for any Cayley graph associated with a generating system $S = S_H \cup S_K$ with $S_H \subset H$ and $S_K \subset K$, the percolation at p_u does not belong to the uniqueness phase. We extend this result to a larger family of groups than direct products, and to any of their Cayley graphs.

Theorem 1.1 *(Nonuniqueness at p_u). Let G be a non-amenable group generated by two commuting infinite and finitely generated subgroups H and K. Then for every Cayley graph $\mathcal{G}$ of G, the percolation at $p_u(\mathcal{G})$ does not belong to the uniqueness phase.*

The same result holds when G admits an infinite normal subgroup H such that the pair (G, H) has the relative property (T). This has also been observed by C. Houdayer (personal communication). Using some weak forms of normality, we can extend the scope of our theorem, for instance when G is a nonamenable (generalized) Baumslag–Solitar group (see Theorem 2.6), or a nonamenable HNN-extension of $\mathbb{Z}^n$ relative to an isomorphism between two finite index subgroups.

Theorem 1.1 follows from a general result on approximations of standard probability measure preserving equivalence relations (Theorem 0.1). We refer to [3] and references therein for general information concerning connections between equivalence relations and percolation on graphs.

2. Approximations of standard equivalence relations

Let $\mathcal{R}$ be a standard probability measure preserving (*p.m.p.*) equivalence relation on the atomless probability standard Borel space (X, μ). See [1] for a general axiomatization of this notion.

Definition 2.1 *(Approximations).* An *approximation* $(\mathcal{R}_n)$ to $\mathcal{R}$ is an exhausting increasing sequence of standard sub-equivalence relations: $\bigcup_{n\in\mathbb{N}} \nearrow \mathcal{R}_n = \mathcal{R}$. An approximation is *trivial* if there is some n and a non-negligible Borel subset $A \subset X$ on which the restrictions coincide: $\mathcal{R}_n \restriction A = \mathcal{R} \restriction A$. We say that $\mathcal{R}$ is *non-approximable* if every approximation is trivial. An action $G \overset{\alpha}{\curvearrowright} (X,\mu)$ is *approximable* if its orbit equivalence relation $\mathcal{R}_G := \{(x, \alpha(g)(x)) : x \in X, g \in G\}$ is approximable.

For instance, all free p.m.p. actions of a non-finitely generated group are approximable. Finite standard equivalence relations are non-approximable.

Proposition 2.2 *(Approximable equivalence relations). The following are examples of approximable equivalence relations.*

(1) *Every aperiodic p.m.p. action of an (infinite) amenable group is approximable by a sequence of sub-equivalence relations with finite classes.*
(2) *Every ergodic non-strongly ergodic p.m.p. equivalence relation admits an approximation by $\mathcal{R}_n$ with diffuse ergodic decompositions.*
(3) *Any free product $\mathcal{R} = \mathcal{A} * \mathcal{B}$ of aperiodic p.m.p. equivalence relations is approximable.*

Item (1) follows from the Ornstein–Weiss theorem [10]. Item (2) relies heavily on results of Jones–Schmidt [7]. Recall that *strong ergodicity*, a reinforcement of ergodicity introduced by K. Schmidt, requires that: for every sequence (A_n) of Borel subsets of X such that $\lim_{n\to\infty} \mu(A_n \Delta g.A_n) = 0$ for each $g \in G$, we must have $\lim_{n\to\infty} \mu(A_n)(1-\mu(A_n)) = 0$. Item (3) will be developed in [4].

Proposition 2.3 *(Non-approximable equivalence relations). The following are examples of non-approximable equivalence relations.*

(1) *Every p.m.p. action of a Kazdhan property (T) group is non-approximable.*
(2) *Every p.m.p. action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, where $\mathbb{Z}^2$ acts ergodically, is non-approximable. More generally, this is the case for free actions of finitely generated relative property (T) pairs (G, H), where H is normal, infinite, and acts ergodically.*

We prove the following effective version of Theorem 0.1.

Theorem 2.4 *(Effective non-approximability). Let G be a countable group generated by two commuting subgroups H and K. Consider a p.m.p. action $G \curvearrowright (X,\mu)$ of G in which H acts strongly ergodically and K acts ergodically. Let $\mathcal{E}$ be any Borel sub-equivalence relation of $\mathcal{R}_G$. For each $g \in G$, set $A_g := \{x \in X : gx\,\mathcal{E}\,x\}$. Let S and T be generating sets for H and K respectively. Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{E}$ satisfies:*

(i) *$\mu(A_s) > 1-\delta$ for all $s \in S$, and*
(ii) *$\mu(A_t) > \epsilon$ for all $t \in T$,*

then there exists a Borel set $B \subseteq X$, with $\mu(B) > 1-\epsilon$, where the restrictions coincide: $\mathcal{E}\restriction B = \mathcal{R}_G \restriction B$.

Sketch of proof. Since the action of H is strongly ergodic, for every ϵ_0, we may find $\delta_0 > 0$ such that if $A \subseteq X$ is any Borel set satisfying $\sup_{s\in S} \mu(s^{-1}A \triangle A) < \delta_0$, then either $\mu(A) < \epsilon_0$ or $\mu(A) > 1-\epsilon_0$.

Given $\epsilon > 0$, we choose ϵ_0 such that $\epsilon_0 < \min\{\epsilon/8, 1/24\}$. Strong ergodicity for H delivers δ_0. We then choose δ satisfying the condition $\delta < \min\{\delta_0/2, 1-8\epsilon_0\}$.

By the commuting assumption, for every k in the group K, for every s in the generating set $S \subset H$, we have that $s^{-1}A_k \triangle A_k \subseteq X \setminus (A_s \cap k^{-1}A_s)$. Hence, by property (i), $\sup_{s\in S} \mu(s^{-1}A_k \triangle A_k) < 1 - \mu(A_s \cap k^{-1}A_s) < 2\delta < \delta_0$, so that for each $k \in K$

$$\text{either} \quad \mu(A_k) < \epsilon_0 \quad \text{or} \quad \mu(A_k) > 1-\epsilon_0. \tag{1}$$

Consider now the subset $K_0 := \{k \in K : \mu(A_k) > 1-\epsilon_0\}$ of K.

– Property (ii) along with (1) and $\epsilon_0 \leq \epsilon$, imply $T \subseteq K_0$.
– Since $\epsilon_0 < 1/3$, then K_0 is a subgroup of K. Indeed, clearly $K_0 = K_0^{-1}$, and if $k_0, k_1 \in K_0$ then $\mu(A_{k_0k_1}) \geq \mu(A_{k_1} \cap k_1^{-1}A_{k_0}) > 1-2\epsilon_0 > \epsilon_0$ hence $\mu(A_{k_0k_1}) > 1-\epsilon_0$ by (1), and thus $k_0k_1 \in K_0$.

It follows that $K_0 = K$. We have shown that $\mu(A_k) > 1-\epsilon_0$ for all $k \in K$.

Theorem 2.7 of [6] then implies that $\mu(\{x \in X : \psi x\,\mathcal{E}\,x\}) > 1-4\epsilon_0$, for every element $\psi \in [\mathcal{R}_K]$ of the full group of the orbit equivalence relation $\mathcal{R}_K$ of K. Thus, by Lemma 2.14 of [6], there exists an $\mathcal{R}_K \cap \mathcal{E}$-invariant Borel set $B \subseteq X$ with $\mu(B) \geq 1-4\epsilon_0$ such that $\mathcal{R}_K \restriction B \subseteq \mathcal{E}\restriction B$. Indeed, $\mathcal{R}_K$ is *relatively non-approximable in* $\mathcal{R}_G$ (see below). We now claim that

$$\text{for each } g \in G, \text{ either } \mu(A_g) < 8\epsilon_0, \text{ or } g^{-1}B \cap B \subseteq A_g \text{ (thus in this case } \mu(A_g) > 1-8\epsilon_0). \tag{2}$$

If $\mu(A_g) > 8\epsilon_0$ for some $g \in G$. Then the set $A_g \cap g^{-1}B \cap B$ is a non-null subset of B, so it meets almost every $\mathcal{R}_K \restriction B$ equivalence class since $\mathcal{R}_K \restriction B$ is ergodic. For each $x \in g^{-1}B \cap B$ we can find some $k \in K$ such that $kx \in A_g \cap g^{-1}B \cap B$. Then $x, gx, kx, gkx \in B$ and $k, gkg^{-1} \in K$, so $x\,(\mathcal{R}_K \restriction B)\,kx\,(\mathcal{E} \restriction B)\,gkx = gkg^{-1}gx\,(\mathcal{R}_K \restriction B)\,gx$, whence $x \in A_g$.

Let $G_0 = \{g \in G : g^{-1}B \cap B \subseteq A_g\}$.

- Since $8\epsilon_0 < \epsilon$ and $1 > 1 - \delta > 8\epsilon_0$, then properties (i) and (ii) and Claim (2) imply that $S \cup T \subseteq G_0$.
- Since $\epsilon_0 < 1/24$ then G_0 is a subgroup of G: It is clear that $G_0^{-1} = G_0$ (since $A_{g^{-1}} = gA_g$). If $g_0, g_1 \in G_0$ then $\mu(A_{g_0}) \geq 1 - 8\epsilon_0$ and likewise $\mu(A_{g_1}) \geq 1 - 8\epsilon_0$, so that $\mu(A_{g_0g_1}) \geq \mu(A_{g_1} \cap g_1^{-1}A_{g_0}) \geq 1 - 16\epsilon_0 > 8\epsilon_0$ and hence $g_0g_1 \in G_0$ by Claim (2).

Therefore, $G_0 = G$. This shows that $\mathcal{R}_G \restriction B \subseteq \mathcal{E} \restriction B$. □

Consider a pair $\mathcal{S} \subset \mathcal{R}$ of p.m.p. standard equivalence relations. A standard sub-relation $\mathcal{S} \subset \mathcal{R}$ of p.m.p. standard equivalence relations is *relatively non-approximable* if for every approximation $(\mathcal{R}_n)$ of $\mathcal{R}$, there is some n and a non-negligible A with $\mathcal{S} \restriction A \subset \mathcal{R}_n \restriction A$. This notion is useful through several variants of the following proposition.

Proposition 2.5 *(Weak form of normality). If $\mathcal{R}$ contains a sub-equivalence relation $\mathcal{S}$ and if $\mathcal{R}$ is generated by a family $\phi_1, \phi_2, \cdots, \phi_p$ of isomorphisms of the space such that $\phi_i(\mathcal{S}) \cap \mathcal{S}$ is ergodic for each i, then every approximation $(\mathcal{R}_n)$ for which there is a non-negligible A with $\mathcal{S} \restriction A \subset \mathcal{R}_0 \restriction A$ has to be trivial.*

Consider such an approximation. We introduce the *Window Trick*:

Let $\mathcal{R}'_n := (\mathcal{R}_n \restriction A) \vee \mathcal{S}$ be the sub-relation of $\mathcal{R}$ generated by $\mathcal{R}_n \restriction A$ and $\mathcal{S}$. We claim that:

(a) $\mathcal{R}'_n \restriction A = \mathcal{R}_n \restriction A$, and
(b) $(\mathcal{R}'_n)$ is an approximation of $\mathcal{R}$.

Now, the set $A_i^n := \{x \in X : x\mathcal{R}'_n\phi_i^{-1}(x)\}$ is $(\phi_i(\mathcal{S}) \cap \mathcal{S})$-invariant: if $x \in A_i^n$ and $(x, y) \in \phi_i(\mathcal{S}) \cap \mathcal{S}$, then $y \overset{\mathcal{S}}{\sim} x \overset{\mathcal{R}'_n}{\sim} \phi_i^{-1}(x) \overset{\mathcal{S}}{\sim} \phi_i^{-1}(y)$. So $y \in A_i^n$. Thus A_i^n has full measure as soon as it is non-negligible, and this happens for large enough n since $\mathcal{R}'_n$ is an approximation. Taking an n that is suitable for all i, we obtain $\mathcal{R}'_n = \mathcal{R}$. So that $\mathcal{R}'_n \restriction A = \mathcal{R}_n \restriction A = \mathcal{R} \restriction A$. □

Let $G = B(p, q) = \langle a, t | ta^pt^{-1} = a^q\rangle$ be a *Baumslag–Solitar group*. The kernel N of the modular map $G \to \mathbb{Q}^*$, $t \mapsto p/q$, $a \mapsto 1$ consists of the elements w of G that commute with a certain power a^{k_w} of a.

Theorem 2.6 *(Baumslag–Solitar groups). If the kernel N of the modular map acts strongly ergodically and all the (non-trivial) powers of a act ergodically, then the action of $B(p, q)$ is non-approximable.*

Indeed, one can find a finitely generated subgroup N_0 of N that already acts strongly ergodically. There is a common power a^k that commutes with N_0. Applying Theorem 0.1, we obtain that $G_0 = N_0 \cdot \langle a^k\rangle$ is non-approximable. Thus the sub-relation generated by G_0 is relatively non-approximable. Proposition 2.5 applied to the pair of relations generated by G_0 and $G_1 = N_0.\langle a\rangle$ with $\phi_1 = a$, first; and then, the same proposition applied to the pair generated by $G_1 < B(p, q)$ with $\phi_1 = t$, proves the result. □

We also obtain similar results for (most) inner amenable groups and various related families of groups.

3. Approximate and geometric dimensions

Besides consequences in Bernoulli bond percolation, Theorem 2.4 allows us to obtain some information about the *approximate dimension*.

A standard p.m.p. equivalence relation $\mathcal{R}$, when considered as a measured groupoid, may act on bundles (fields) of simplicial complexes $x \mapsto \Sigma_x$ over X. The action is *proper* if its restriction to the 0-skeleton $x \mapsto \Sigma_x^{(0)}$ of the sub-bundle is smooth. The *dimension* of such a bundle is the maximum dimension of a fiber Σ_x, and the bundle is said to be *contractible* if (almost) every fiber is contractible. The *geometric dimension* of $\mathcal{R}$ is the minimum of the dimensions of the $\mathcal{R}$-bundles which are proper and contractible. The *approximate dimension* of $\mathcal{R}$ is the minimum of the dimensions d such that $\mathcal{R}$ admits an approximation $(\mathcal{R}_n)$ by sub-relations of dimension d. These notions were introduced in [2, section 5].

For instance, smooth equivalence relations have geometric dimension $= 0$. Aperiodic treeable equivalence relations are exactly those with geometric dimension $= 1$. Their approximate dimension is $= 0$ if and only if they are hyperfinite and is $= 1$ otherwise. One can show the general inequalities: approx-dim $\leq$ geom-dim $\leq$ approx-dim $+ 1$. It is unknown whether there are groups admitting free p.m.p. actions with different geometric dimensions. As for approximate dimension, various situations may occur. For instance, we obtain:

Proposition 3.1. *Let $G_d := \mathbf{F}_2 \times \mathbf{F}_2 \cdots \times \mathbf{F}_2 \times \mathbb{Z}$ be the direct product of d copies of the free group $\mathbf{F}_2$ and one copy of $\mathbb{Z}$. All its free p.m.p. actions have geometric dimension $= d+1$. It admits both free p.m.p. actions with approximate dimension $= d$ and $= d+1$.*

As already mentioned, free products of equivalence relations are always approximable. This is no longer the case for free actions of amalgamated free products over an infinite central subgroup $G = G_1 *_C G_2$ when the common subgroup has indices greater than 3 in the factors (apply Theorem 0.1 to, say, the Bernoulli shift action with $H = G$ and $K = C$). This allows us to produce examples of group actions that are amalgamated free products of treeable over amenable, but which are not *approxi-treeable* (approximable by treeable): take for instance G_1 and G_2 abelian.

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