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Ordinary differential equations/Partial differential equations

Almost automorphic evolution equations with compact almost automorphic solutions

Sur une classe d'équations d'évolution presque automorphes possédant des solutions compactes presque automorphes

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A R T I C L E I N F O

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A B S T R A C T

We prove that some almost automorphic evolution equations carry compact almost automorphic solutions. Moreover, we show that the almost automorphy of the coefficients is not necessary to obtain almost automorphic solutions. This improves the assumptions and the conclusion of a result of M. Zaki (Ann. Mat. Pura Appl. (4) 101 (1) (1974) 91–114), which gives the nature of solutions with relatively compact range for some almost automorphic evolution equations in Banach spaces. We note that many results in the literature can be improved in this direction.

R É S U M É

Nous montrons que certaines équations d'évolution presque automorphes possèdent des solutions compactes presque automorphes. De plus, nous montrons que la presque automorphie des coefficients n'est pas nécessaire pour obtenir des solutions presque automorphes. Cela améliore les hypothèses et la conclusion d'un résultat de M. Zaki (Ann. Mat. Pura Appl. (4) 101 (1) (1974) 91–114), qui donne la nature des solutions avec image relativement compacte pour certaines équations d'évolution presque automorphes dans les espaces de Banach. Nous notons que de nombreux résultats dans la littérature peuvent être améliorés dans cette direction.

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1. Introduction

In this work, we investigate the nature of solutions with relatively compact range for the following evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

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where A is the generator of a strongly asymptotically stable C_0-semigroup on a Banach space X and $f:\mathbb{R}\to X$ is an almost automorphic function in the sense of Stepanov.

Let us consider the following differential equation in $\mathbb{R}^n$:

$$x'(t) = G(t)x(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where the matrix $G(t)$ and the vector $f(t)$ are both continuous and ω-periodic for some $\omega > 0$. In [11], Massera proved that the existence of a bounded solution of Equation (1.2) on the positive real line is enough to get the existence of an ω-periodic solution. This result is known in the literature as the Massera theorem. Fixed-point theory plays an important role in this kind of results.

For almost periodic equations, the situation is more complicated, since one cannot use fixed-point arguments. Bohr and Neugebauer, see [8], extended Massera's theorem for Equation (1.2) to the almost periodic case when $G(t) = G$ is a constant matrix. In addition, they proved that a bounded solution of Equation (1.2) on $\mathbb{R}$ is automatically almost periodic. We note that this result does not hold for the periodic case.

In [5], Cooke proved that bounded solutions of the following differential equation

$$y^{(n)}(t) + A_1 y^{(n-1)}(t) + \ldots A_n y(t) = f(t),$$

are almost periodic when $f:\mathbb{R}\to H$ is almost periodic and A_i, $i = 1, \ldots, n$ are compact operators in a separable Hilbert space H. In [10], Haraux proved the same result for the following evolution equation

$$x'(t) + \widetilde{A}x(t) \ni f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

where $\widetilde{A}$ is a maximal monotone operator on $\mathbb{R}^2$. In [13,15], Zaidman proved that a bounded solution of Equation (1.1) is almost periodic when A is a self-adjoint operator in a Hilbert space. In another paper, Zaidman [16] proved this result for Equation (1.1) when A is a finite-rank operator. In [9], Goldstein extended the work of Zaidman by considering a more general finite dimensionality assumption when A is a closed linear operator in a Hilbert space.

Without some sort of finite dimensionality assumptions, one cannot predict in general that bounded solutions have a relatively compact range. In this case, it is more appropriate to look for almost periodic and almost automorphic solutions inside the set of solutions having a relatively compact range. Following this remark, Zaidman [14] showed that a bounded solution of Equation (1.1) with a relatively compact range is automatically almost periodic when f is also almost periodic and A generates a strongly asymptotically stable C_0-semigroup on a Banach space X. Under the same assumption on the operator A, Zaki [17] gave an analogous result for almost automorphic solutions. He proved that a bounded solution of Equation (1.1) with a relatively compact range is almost automorphic when f is also almost automorphic.

In this work, we strengthen the result of Zaki in the sense that we obtain a stronger conclusion and using weaker assumptions. More specifically, under the same strong asymptotic stability in [17], we prove that a bounded solution of Equation (1.1) with a relatively compact range is even compact almost automorphic when f is only almost automorphic in the sense of Stepanov. We note that many results in the literature can be improved in this direction.

2. Almost automorphic functions

Let $(X, |.|)$ be a Banach space and $BC(\mathbb{R}, X)$ be the space of bounded continuous functions from $\mathbb{R}$ to X equipped with the supremum norm

$$|f|_\infty := \sup_{t\in\mathbb{R}} |f(t)|. \tag{2.1}$$

In [4], Bochner introduced the concept of almost automorphy, which is a generalization of the almost periodicity.

Definition 2.1. [4] A continuous function $f:\mathbb{R}\mapsto X$ is said to be almost automorphic if for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$, such that for each $t \in \mathbb{R}$

$$f(t + s'_n) \to \widetilde{f}(t)$$

and

$$\widetilde{f}(t - s'_n) \to f(t)$$

as $n\to\infty$. If the above limits hold uniformly in compact subsets of $\mathbb{R}$, then f is said to be compact almost automorphic.

Remark. If one of the convergences in Definition 2.1 holds uniformly on the whole real line, then we obtain the notion of almost periodicity.

Definition 2.2. [1,3,12] The Bochner transform f^b of a function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is the function $f^b : \mathbb{R} \to L^p([0,1], X)$ defined for each $t \in \mathbb{R}$ by

$$\left(f^b(t)\right)(s) = f(t+s) \quad \text{for } s \in [0,1].$$

Definition 2.3. [6] A function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is said to be Stepanov almost automorphic for some $p \geq 1$ (or S^p-almost automorphic) if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1], X)$ is almost automorphic.

The following Bochner-type characterization for the almost automorphy in the sense of Stepanov is essential for the rest of this work.

Proposition 2.4. [7] *A function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is S^p-almost automorphic if and only if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $g \in L^p_{\text{loc}}(\mathbb{R}, X)$, such that for each $t \in \mathbb{R}$*

$$\left(\int_t^{t+1} \left|f(s+s'_n) - g(s)\right|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0 \tag{2.2}$$

and

$$\left(\int_t^{t+1} \left|g(s-s'_n) - f(s)\right|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0, \tag{2.3}$$

as $n \to \infty$.

Remark. One can see from Proposition 2.4 that an almost automorphic function is automatically S^p-almost automorphic for all $p \geq 1$.

3. Main result

Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{3.1}$$

where $f : \mathbb{R} \to X$ is a given function and A is a linear operator on a Banach space X that generates a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 3.1. A mild solution of Equation (3.1) is a continuous function $x : \mathbb{R} \to X$ that satisfies, for each $t, \sigma \in \mathbb{R}$ with $t \geq \sigma$,

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)f(s)\,\mathrm{d}s.$$

The aim of this work is to improve the following result.

Theorem 3.2. [17] *If f is almost automorphic and the C_0-semigroup $(T(t))_{t\geq 0}$ is strongly asymptotically stable, i.e.* $\lim_{t\to\infty} T(t)x = 0$ *for each $x \in X$, then a mild solution of Equation (3.1) having a relatively compact range is almost automorphic.*

The following result shows that Theorem 3.2 holds under weaker assumptions on the forcing term f. In addition, the solution is shown to have more regularity than what is claimed in Theorem 3.2.

Theorem 3.3. *If f is S^1-almost automorphic and the C_0-semigroup $(T(t))_{t\geq 0}$ is strongly asymptotically stable, i.e.* $\lim_{t\to\infty} T(t)x = 0$ *for each $x \in X$, then a mild solution of Equation (3.1) having a relatively compact range is compact almost automorphic.*

Lemma 3.4. [2, Theorem 4.1.2] *Let Y be a normed space and $(T_n)_n$ be a sequence of bounded linear operators on Y such that* $\sup_n |T_n| < \infty$. *If D is a dense subset of Y, and if for each $y \in D$*

$$T_n y \to T y \quad \textit{as } n \to \infty,$$

for some bounded linear operator T, then for every compact set K of Y

$$\sup_{y\in K} |T_n y - T y| \to 0 \quad \textit{as } n \to \infty.$$

The following lemma is needed in the proof of Theorem 3.3.

Lemma 3.5. *A mild solution of Equation (3.1) having a relatively compact range is uniformly continuous.*

Proof of Lemma 3.5. Let x be a solution of Equation (3.1) such that $K = \overline{\{x(t) : \ t \in \mathbb{R}\}}$ is compact. If x is not uniformly continuous, then there exist $\varepsilon > 0$ and two real sequences $(s_n)_n$ and $(h_n)_n$ such that $\lim_{n\to\infty} h_n = 0$ and

$$|x(s_n + h_n) - x(s_n)| > \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Assume without loss of generality that $h_n \geq 0$ for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} x(s_n + h_n) &= T(h_n)x(s_n) + \int_{s_n}^{s_n+h_n} T(s_n + h_n - s) f(s)\,\mathrm{d}s \\ &= T(h_n)x(s_n) + \int_0^{h_n} T(h_n - s) f(s + s_n)\,\mathrm{d}s. \end{aligned}$$

Let $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $|T(t)| \leq M_0 \mathrm{e}^{\omega_0 t}$ for all $t \geq 0$. Then

$$\begin{aligned} |x(s_n + h_n) - x(s_n)| &\leq |T(h_n)x(s_n) - x(s_n)| + \int_0^{h_n} |T(h_n - s)|\,|f(s + s_n)|\,\mathrm{d}s \\ &\leq \sup_{y\in K} |T(h_n)y - y| + M_0 \int_0^{h_n} \mathrm{e}^{\omega_0(h_n - s)}\,|f(s + s_n)|\,\mathrm{d}s \\ &\leq \sup_{y\in K} |T(h_n)y - y| + M_0 \mathrm{e}^{|\omega_0|h_n} \int_0^{h_n} |f(s + s_n)|\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

The semigroup $(T(t))_{t\geq 0}$ being strongly continuous, we have, for each $y \in K$, $T(h_n)y \to y$ as $n \to \infty$. This implies that $\sup_n |T(h_n)y| < \infty$ for each $y \in X$ and thus, by the Banach–Steinhaus Theorem, $\sup_n |T(h_n)| < \infty$. It follows from Lemma 3.4 that

$$\sup_{y\in K} |T(h_n)y - y| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

On the other hand, from the S^1-almost automorphy of f, there exist a subsequence $\left(s'_n\right)_n \subset (s_n)_n$ and a function $\widehat{f} \in L^1_{\text{loc}}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$

$$\int_t^{t+1} \left|f\left(s + s'_n\right) - \widehat{f}(s)\right| \mathrm{d}s \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Let $\left(h'_n\right)_n$ be the corresponding subsequence of $(h_n)_n$. We can assume that $0 \leq h_n \leq 1$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \int_0^{h'_n} \left|f\left(s + s'_n\right)\right| \mathrm{d}s &\leq \int_0^{h'_n} \left|f\left(s + s'_n\right) - \widehat{f}(s)\right| \mathrm{d}s + \int_0^{h'_n} \left|\widehat{f}(s)\right| \mathrm{d}s \\ &\leq \int_0^{1} \left|f\left(s + s'_n\right) - \widehat{f}(s)\right| \mathrm{d}s + \int_0^{h'_n} \left|\widehat{f}(s)\right| \mathrm{d}s. \end{aligned}$$

Since $\left|\widehat{f}(.)1_{[0,h_n']}(.)\right| \le \left|\widehat{f}(.)1_{[0,1]}(.)\right| \in L^1(\mathbb{R},\mathbb{R})$ for all $n \in \mathbb{N}$ and $\widehat{f}(.)1_{[0,h_n']}(.)$ converges almost everywhere to 0, then it follows from Lebesgue's dominated convergence theorem that $\int_0^{h_n'} \left|\widehat{f}(s)\right| \mathrm{d}s \to 0$ as $n \to \infty$. On the other hand, using (3.5) we have $\int_0^1 \left|f\left(s+s_n'\right) - \widehat{f}(s)\right| \mathrm{d}s \to 0$ as $n \to \infty$ and thus

$$\int_0^{h_n'} \left|f\left(s+s_n'\right)\right| \mathrm{d}s \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Therefore, we conclude from (3.3), (3.4) and (3.6) that

$$\left|x(s_n'+h_n') - x(s_n')\right| \to 0 \quad \text{as } n \to \infty,$$

which contradicts (3.2). We conclude that $x(\cdot)$ must be uniformly continuous. $\square$

Proof of Theorem 3.3. Let $K = \overline{\{x(t) : t \in \mathbb{R}\}}$ and $(t_n)_n$ be a sequence of real numbers. From the almost automorphy of f, there exist a subsequence $\left(t_n'\right)_n \subset (t_n)_n$ and a measurable function $g : \mathbb{R} \to X$ such that, for each $t \in \mathbb{R}$

$$\int_t^{t+1} \left|f(s+t_n') - g(s)\right| \mathrm{d}s \to 0 \tag{3.7}$$

$$\int_t^{t+1} \left|g(s-t_n') - f(s)\right| \mathrm{d}s \to 0.$$

If we denote $x_n(t) = x(t+t_n')$, then for each $n \in \mathbb{N}$, $x_n \in C(\mathbb{R}, X)$ and satisfies $x_n(t) \in K$ for each $t \in \mathbb{R}$. Therefore, for each $t \in \mathbb{R}$, the set $\{x_n(t) : n \in \mathbb{N}\}$ is a relatively compact subset of X. Since x is uniformly continuous (Lemma 3.5), the sequence $(x_n)_n$ is equicontinuous on $\mathbb{R}$. In view of Arzelà–Ascoli's theorem, we can assert that $\{x_n : n \in \mathbb{N}\}$ is a relatively compact subset of $C(\mathbb{R}, X)$ endowed with the topology of compact convergence. Thus from the sequence $(t_n')_n$, one can extract another subsequence $(t_n'')_n \subset (t_n')_n \subset (t_n)_n$ such that

$$x(t+t_n'') \to y(t) \tag{3.8}$$

uniformly on compact subsets of $\mathbb{R}$, where $y \in C(\mathbb{R}, X)$. Since x is a mild solution of Equation (3.1), we obtain for each $t, s \in \mathbb{R}$ with $t \ge s$ and $n \in \mathbb{N}$

$$x(t+t_n'') = x(s+t_n'') + \int_s^t T(t-\tau) f\left(\tau + t_n''\right) \mathrm{d}\tau. \tag{3.9}$$

By letting $n \to \infty$ in (3.9) using (3.7) and (3.8) we deduce that

$$y(t) = y(s) + \int_s^t T(t-\tau) g(\tau) \mathrm{d}\tau.$$

Now by applying the same argument to the function y using this time the returning sequence $\left(-t_n''\right)_n$, we have another subsequence $(t_n''')_n \subset (t_n'')_n \subset (t_n')_n \subset (t_n)_n$ such that

$$y(t - t_n''') \to z(t) \tag{3.10}$$

uniformly on compact subsets of $\mathbb{R}$, where z is a solution of Equation (3.1).

Since x is also another solution of Equation (3.1) we have for each $t, \sigma \in \mathbb{R}$ with $t \ge \sigma$

$$|x(t) - z(t)| \le |T(t-\sigma)(x(\sigma) - z(\sigma))|. \tag{3.11}$$

We note that

$$\overline{\{z(t) : t \in \mathbb{R}\}} \subset \overline{\{y(t) : t \in \mathbb{R}\}} \subset \overline{\{x(t) : t \in \mathbb{R}\}} = K.$$

Thus there exists a compact set $\widetilde{K}$ such that $\{x(t)-z(t) : t \in \mathbb{R}\} \subset \widetilde{K}$. We deduce from (3.11) that

$$|x(t)-z(t)| \leq \sup_{y\in\widetilde{K}} |T(t-\sigma)y| \quad \text{for } t \geq \sigma. \tag{3.12}$$

By letting $\sigma \to -\infty$ in (3.12) using the strong asymptotic stability of the C_0-semigroup $(T(t))_{t\geq 0}$ together with Lemma 3.4 and the Banach–Steinhaus Theorem, we deduce that $x = z$. Thus the compact almost automorphy of x follows from (3.8) and (3.10). $\square$

Remark 3.6. An alternative proof for Theorem 3.3 can be given as follows: using Lemma 3.5, one can see that the almost automorphic solution in Theorem 3.2 is in fact uniformly continuous and thus it is compact almost automorphic according to the following lemma.

Lemma 3.7. *A function f is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

Proof of Lemma 3.7. Let $f : \mathbb{R} \to X$ be an almost automorphic function that is uniformly continuous. Let $(s_n)_n$ be a sequence of real numbers. Then there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$ such that, for each $t \in \mathbb{R}$

$$f(t+s'_n) \to \widetilde{f}(t) \tag{3.13}$$

and

$$\widetilde{f}(t-s'_n) \to f(t) \tag{3.14}$$

as $n \to \infty$. Consider the sequence of functions defined for each $n \in \mathbb{N}$ by

$$f_n(t) = f(t+s'_n) \quad \text{for } t \in \mathbb{R}.$$

Since f is uniformly continuous, then the family $(f_n)_n$ is equicontinuous. It follows that the convergence (3.13) holds uniformly in compact subsets of $\mathbb{R}$.

On the other hand, from (3.13), one can see that $\widetilde{f}$ is also uniformly continuous. Using the same argument, the convergence (3.14) also holds uniformly in compact subsets of $\mathbb{R}$.

Now, if f is compact almost automorphic, then it is almost automorphic. To show that f is uniformly continuous, take two sequences $(t_n)_n$ and $(s_n)_n$ such that $|t_n - s_n| \to 0$ as $n \to \infty$ and show that $\alpha_n = |f(t_n) - f(s_n)| \to 0$ as $n \to \infty$. Let $(\alpha'_n)_n = \left(\left|f(t'_n) - f(s'_n)\right|\right)_n$ be a subsequence of $(\alpha_n)_n$. Since f is compact almost automorphic, there exist a subsequence $(s''_n)_n \subset (s'_n)_n$ and a function $g : \mathbb{R} \to X$ such that $f(t+s''_n) \to g(t)$ uniformly on compact subsets of $\mathbb{R}$. In addition, the function g is continuous. Let $a, b \in \mathbb{R}$ be such that $a \leq t_n - s_n \leq b$ for all $n \in \mathbb{N}$. Then, from the compact almost automorphy of f and the continuity of g, we have

$$\begin{aligned} \alpha''_n = \left|f(t''_n) - f(s''_n)\right| &\leq \left|f(t''_n - s''_n + s''_n) - g(t''_n - s''_n)\right| + \left|g(t''_n - s''_n) - g(0)\right| + \left|g(0) - f(s''_n)\right| \\ &\leq \sup_{a\leq t\leq b} \left|f(t+s''_n) - g(t)\right| + \left|g(t''_n - s''_n) - g(0)\right| + \left|g(0) - f(s''_n)\right| \to 0 \end{aligned}$$

as $n \to \infty$. Thus we showed that every subsequence of $(\alpha_n)_n$ has a subsequence that converges to 0. We conclude that the whole sequence $(\alpha_n)_n$ converges to 0. Therefore f is uniformly continuous. $\square$

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