C. R. Acad. Sci. Paris, Ser. I 355 (2017) 20–23

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Group theory

Sylow 2-subgroups of solvable $\mathbb{Q}_1$-groups

2-Sous-groupes de Sylow des $\mathbb{Q}_1$-groupes résolubles

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A R T I C L E   I N F O

Article history:
Received 23 March 2016
Accepted after revision 8 November 2016
Available online 23 November 2016

Presented by the Editorial Board

A B S T R A C T

A finite group whose irreducible complex non-linear characters are rational is called a $\mathbb{Q}_1$-group. In this paper, we study the structure of a $\mathbb{Q}_1$-group through its Sylow 2-subgroups.

R É S U M É

Un groupe fini dont les caractères complexes non linéaires sont rationnels est appelé un $\mathbb{Q}_1$-groupe. Nous étudions dans cette Note la structure d'un $\mathbb{Q}_1$-groupe par le biais de ses 2-sous-groupes de Sylow.

1. Introduction and preliminary

A finite group G is said to be rational or $\mathbb{Q}$-group if every complex irreducible character of G is rational. It is well known that rational groups have even order and hence they have non-trivial Sylow 2-subgroups [8]. M. Isaacs and G. Navarro proved that if a Sylow 2-subgroup of a rational group has nilpotence class at most two, then it is rational [7]. However, there is no classification of such groups, but the structure of rational groups with quaternion or Abelian Sylow 2-subgroups was given in [8]. In [2], the authors proved that if G is a supersolvable rational group, then its Sylow 2-subgroup is also rational, and they studied solvable rational groups with an extraspacial Sylow 2-subgroup. In this paper, we will restrict our attention to finite groups G, whose non-linear complex irreducible characters are rational-valued. Such a group is called a $\mathbb{Q}_1$-group. We study the Sylow 2-subgroups of a $\mathbb{Q}_1$-group and determine the general structure of this type of solvable groups. This paper can be considered as an attempt to describe solvable non-Abelian $\mathbb{Q}_1$-groups. For some elementary properties of $\mathbb{Q}_1$-groups, we refer the reader to [1]. Similar to rational groups, $\mathbb{Q}_1$-groups have also even orders and hence they have non-trivial Sylow 2-subgroups [1].

Throughout the paper, we consider finite groups and we employ the following notation and terminology.

The semi-direct product of group K with group H is denoted by $K : H$. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order n. For a prime p and a non-negative integer n, the symbol $E(p^n)$ denotes the elementary Abelian p-group of order p^n,

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http://dx.doi.org/10.1016/j.crma.2016.11.001

p a prime number; Q_8 and D_8 are employed to denote the quaternion and dihedral group of order 8, respectively. Here we write $\pi(G)$ for $\pi(|G|)$, where $\pi(n)$ is the set of primes dividing the natural number n.

Let us mention some important consequences of rational groups and $\mathbb{Q}_1$-groups. Let G be a finite group. Let $\mathrm{nl}(G)$ denote the set of non-linear irreducible characters of G.

An element $x \in G$ is called rational if $\chi(x) \in \mathbb{Q}$ for every $\chi \in \mathrm{Irr}(G)$; otherwise it is called an irrational element. Also, $\chi \in \mathrm{Irr}(G)$ is called a rational character if $\chi(x) \in \mathbb{Q}$ for every $x \in G$.

Lemma 1.1. *([8, p. 11] and [5, p. 31]) A finite group G is a $\mathbb{Q}$-group if and only if for every $x \in G$ of order n, the elements x and x^m are conjugate in G, whenever $(m, n) = 1$. Equivalently, $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong \mathrm{Aut}(\langle x\rangle)$ for each $x \in G$.*

The detailed proofs of Theorems 1.2 and 1.4 are given in [1].

Theorem 1.2. *Let G be a non-Abelian $\mathbb{Q}_1$-group. Then the following are true:*

(1) *$|G|$ is even;*
(2) *each quotient of G is a $\mathbb{Q}_1$-group;*
(3) *if G is nilpotent, then it is a 2-group.*

Definition 1.3. Let G be a non-Abelian finite group. The vanishing-off subgroup of G is defined as follows:

$$V(G) = \langle g \in G \mid \exists \chi \in \mathrm{nl}(G) : \chi(g) \neq 0\rangle.$$

Notice that $V(G)$ is a characteristic subgroup of G. It is well known that $V(G)$ is the smallest subgroup $V \leq G$ such that every character in $\mathrm{nl}(G)$ vanishes on $G - V$.

Theorem 1.4. *Let G be a non-Abelian finite group. Then G is a $\mathbb{Q}_1$-group if and only if every element of $V(G)$ is a rational element of G.*

Our study is based on whether a Sylow 2-subgroup of a $\mathbb{Q}_1$-group G is included in $V(G)$ or not. The main result of this paper is as follows.

Main Theorem. *Suppose G is a non-Abelian solvable $\mathbb{Q}_1$-group with Sylow 2-subgroup P. Then one of the following occurs:*

(1) *if $P \subseteq V(G)$, then $G \cong V(G) : \mathbb{Z}_m$ or $G \cong V(G) : \mathrm{E}(p^n)$, where m is an odd integer and p is coprime to $|V(G)|$;*
(2) *if P is non-Abelian and $P \nsubseteq V(G)$, then $G \cong K : P$, where K is a $\{3, 5, 7\}$-group;*
(3) *if P is Abelian and $P \nsubseteq V(G)$ then $G \cong G' : (Z_m \times E(2^n))$, where the derived subgroup G' is a Hall subgroup of odd order and m, n are integers.*

2. Proof of the Main Theorem

We have divided the proof of the main results into a sequence of statements.

Lemma 2.1. *Let P be a Sylow p-subgroup of group G such that $P \nsubseteq V(G)$. Then G has a normal p-complement.*

Proof. Since $P \nsubseteq V(G)$, so there exists p-element g such that $g \notin V(G)$. Therefore, every non-linear irreducible character of G vanishes on g. By Remark 4.1 in [3] and a well-known theorem of Thompson [11], G has a normal p-complement. □

We point out the fact that similar to rational groups, we have the following:

Corollary 2.2. *Let G be a $\mathbb{Q}_1$-group with a non-Abelian Sylow 2-subgroup P. If P is a dihedral group or a generalized quaternion group, then P is isomorphic to D_8 or Q_8.*

Proof. By Lemma 2.1, if $P \nsubseteq V(G)$ then P is a $\mathbb{Q}_1$-group. Therefore, by [1, Remark 2.2], the results are achieved. And if $P \subseteq V(G)$, then every element of P is rational in G. Now, it follows, by a similar reasoning in proof of lemma 1 in [2], that $P \cong Q_8$ or by [8, p. 23], it may be concluded that $P \cong D_8$. □

But, when a $\mathbb{Q}_1$-group G has an Abelian Sylow 2-subgroup, the situation is different from that of rational groups. We will see it in Theorem 2.8.

Proposition 2.3. Suppose G is a $\mathbb{Q}_1$-group, $p \in \pi(G)$ and $p > 2$. If $p \mid |G : V(G)|$, then $p \nmid |V(G)|$.

Proof. Suppose, contrary to our claim, that p divides $|V(G)|$. So, there exists $x \in V(G)$ of order p^n. By Theorem 1.4, x is rational. Hence, by Lemma 1.1, we have

$$|N_G(\langle x \rangle)|/|C_G(\langle x \rangle)| = p^{n-1}(p-1).$$

On the other hand, by Lemma 2.1, G has a normal p-complement. Now, it follows from Theorem 5.26 of [6] that $N_G(\langle x \rangle)/C_G(\langle x \rangle)$ is a p-group, which is a contradiction. Therefore, $p \nmid |V(G)|$. □

By Proposition 2.3, if the Sylow 2-subgroups of a $\mathbb{Q}_1$-group G are included in $V(G)$, then $V(G)$ is a Hall subgroup of G. We need the following theorem.

Theorem 2.4. *([9]) Let G be a non-Abelian finite group. Then*

(1) *if G has a non-Abelian nilpotent quotient, then $G/V(G)$ is elementary Abelian.*
(2) *If G has a Frobenius quotient with an Abelian Frobenius complement, then $G/V(G)$ is cyclic.*

Therefore, the first part of Main Theorem is a consequence of Proposition 2.3 and Theorem 2.4.

In order to prove the next lemma, we recall the definition of a p-rational group. Let $\mathbb{Q}_n$ be a field that is obtained by adjoining a primitive nth root of unity to $\mathbb{Q}$. Also, suppose that p is a prime divisor of $|G|$ and $\mathbb{Q}_n$ is the smallest complex field that contains all character values of G. Then, we say that G is p-rational, if p does not divide n.

Lemma 2.5. *Let P be a Sylow p-subgroup of the $\mathbb{Q}_1$-group G such that $P \subseteq V(G)$. Then G is p-rational.*

Proof. Let $x \in G$ and $\chi \in \mathrm{Irr}(G)$. If χ is non-linear, then, by hypothesis, $\chi(x)$ is rational. Now, suppose that χ is linear and x_p and $x_{p'}$ are p-part and p'-part of x, respectively. Then $\chi(x) = \chi(x_p)\chi(x_{p'})$. Since $P \subseteq V(G)$, so $\chi(x_p)$ is rational. Therefore $\chi(x)$ is a p'-root of unity. We deduce that G is p-rational. □

Corollary 2.6. *Suppose that G is a solvable $\mathbb{Q}_1$-group and P is a Sylow 2-subgroup of G. If P has nilpotence class two, then*

(a) *If $P \nsubseteq V(G)$ then P is a $\mathbb{Q}_1$-group;*
(b) *If $P \subseteq V(G)$ then P is a $\mathbb{Q}$-group.*

Proof. Assume that P is a Sylow 2-subgroup of G. If $P \nsubseteq V(G)$, then it follows from Lemma 2.1 that G has a normal 2-complement. Thus, by part two of Theorem 1.2, P is a $\mathbb{Q}_1$-group. If $P \subseteq V(G)$, then by Lemma 2.5, G is 2-rational. Now, we conclude from the main result of [7] that P is a rational group. □

Let G be a $\mathbb{Q}_1$-group. If G has a Frobenius quotient, then every prime number can be a divisor of $|G|$, because, for every odd prime p, a Frobenius group of order $p(p-1)$ with cyclic complement of order $p-1$ is a $\mathbb{Q}_1$-group [10]. The case where G has a non-Abelian nilpotent quotient is discussed in the next theorem. Indeed, we show that, in this case, the prime divisors of $|G|$ are in $\{2, 3, 5, 7\}$.

Theorem 2.7. *Suppose G is a solvable $\mathbb{Q}_1$-group with a non-Abelian nilpotent quotient. Then, $\pi(G) \subseteq \{2, 3, 5, 7\}$.*

Proof. By [5, Lemma 12.3], every finite solvable group G has a non-Abelian nilpotent or Frobenius quotient, because if K is a maximal normal subgroup of G such that G/K is non-Abelian, then $(G/K)'$ is the unique minimal normal subgroup of G/K. Thus, we can assume that G/K is a nilpotent $\mathbb{Q}_1$-group.

Hence, by Theorem 1.2(3), G/K is a 2-group. On the other hand, by [9, Lemma 3.3], we have $K \subseteq V(G)$. Hence, we conclude from Theorem 2.4 that $G/V(G)$ must be an elementary Abelian 2-group.

Now, assume that $x \in G - V(G)$ and for some $\lambda \in \mathrm{lin}(G)$, $\lambda(x)$ be irrational. Then $x^2 \in V(G)$. Thus $\lambda(x)^2 = \lambda(x^2) = \pm 1$. Consequently, $\lambda(x) = \pm \mathrm{i}$. Since for every $\chi \in \mathrm{nl}(G)$, we have $\chi(x) = 0$ and every element of $V(G)$ is rational in G, so $|\mathbb{Q}(G) : \mathbb{Q}| = 2$, where $\mathbb{Q}(G)$ is the smallest complex field containing all character values of G. Hence, it follows from [4, Theorem 2.3(c)] that $\pi(G) \subseteq \{2, 3, 5, 7\}$. □

Now, suppose that G is a non-Abelian $\mathbb{Q}_1$-group with non-Abelian Sylow 2-subgroup P. If $P \nsubseteq V(G)$, then it follows from Lemma 2.1 that G has a normal 2-complement, say K. Therefore, G has a non-Abelian nilpotent quotient. Consequently, by Theorem 2.7, we have $\pi(K) \subseteq \{3, 5, 7\}$. This completes the proof of the second part of the Main Theorem.

Now, we prove the third part of the Main Theorem, which we restate here.

Theorem 2.8. *Let G be a $\mathbb{Q}_1$-group with a Sylow 2-subgroup P. If P is Abelian and $P \nsubseteq V(G)$ then $G \cong G' : (\mathbb{Z}_m \times E(2^n))$, where G' is Hall derivation subgroup of odd order and m, n are integers.*

Proof. First, we show that $P \cap V(G)$ is an elementary Abelian 2-group. Take $x \in P \cap V(G)$ to be of order 2^k. We have $|N_G(\langle x\rangle)/C_G(\langle x\rangle)| = 2^{k-1}$. Since P is Abelian, $P \subseteq C_G(\langle x\rangle)$. Consequently, $k = 1$. Therefore $P \cap V(G) \cong E(2^n)$ for some integer n.

Since $P \nsubseteq V(G)$, there exists a 2-element g such that $g \notin V(G)$. Therefore, for every $\chi \in \mathrm{nl}(G)$, $\chi(g) = 0$. Hence, by [9, Lemma 2.1], $C_G(\langle g\rangle) = |G : G'|$. The order of G' is odd, because $P \subseteq C_G(\langle g\rangle)$. On the other hand, by [9, Lemma 3.2], $G' \leqslant V(G)$. We know that every rational element of odd order belongs to G'. Thus G' is a normal 2-complement for $V(G)$. By Proposition 2.3, G' is also a Hall subgroup of G.

By Theorem 2.4, there are two cases for G. We claim G has not a non-Abelian nilpotent quotient. If it is true then for some subgroup K, G/K is a non-Abelian nilpotent $\mathbb{Q}_1$-group, so by Theorem 1.2, G/K is a non-Abelian 2-group, which violates the fact that P is an Abelian subgroup. Hence, $G/V(G)$ is cyclic, therefore, it is isomorphic to $\mathbb{Z}_m$ for some integer m. The statements of the theorem should now be clear. □

Acknowledgements

The authors wishes to express their thanks to the referee for her/his useful comments, specially for drawing the author's attention to Theorem 2.8.

References

[1] M.R. Darafsheh, A. Iranmanesh, S.A. Moosavi, Groups whose non-linear irreducible characters are rational valued, Arch. Math. (Basel) 94 (5) (2010) 411–418.
[2] M.R. Darafsheh, H. Sharifi, Sylow 2-subgroups of solvable $\mathbb{Q}$-groups, Extr. Math. 22 (1) (2007) 83–91.
[3] S. Dolfi, E. Pacifici, L. Sanus, P. Spiga, On the orders of zeros of irreducible characters, J. Algebra 321 (2009) 345–352.
[4] E. Farias e Soares, Big primes and character values for solvable groups, J. Algebra 100 (1986) 305–324.
[5] I.M. Isaacs, Character Theory of Finite Groups, Academic Press, 1976.
[6] I.M. Isaacs, Finite Group Theory, American Mathematical Society, 2008.
[7] M. Isaacs, G. Navarro, Sylow 2-subgroups of rational groups, Math. Z. 272 (3–4) (2012) 937–945.
[8] D. Kletzing, Structure and Representations of $\mathbb{Q}$-Group, Lecture Notes in Mathematics, vol. 1084, Springer-Verlag, 1984.
[9] M.L. Lewis, The vanishing-off subgroup, J. Algebra 321 (2009) 1313–1325.
[10] M. Norooz-Abadian, H. Sharifi, Frobenius $\mathbb{Q}_1$-groups, Arch. Math. (Basel) 105 (2015) 509–517.
[11] J.G. Thompson, Normal p-complements and irreducible characters, J. Algebra 14 (1970) 129–134.