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Number theory

Growth of class numbers of positive definite ternary unimodular Hermitian lattices over imaginary number fields

Croissance du nombre de classes de réseaux hermitiens unimodulaires sur un corps quadratique imaginaire

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ABSTRACT

We give a lower bound for class numbers of unimodular ternary Hermitian lattices over imaginary quadratic fields. This shows that class numbers of unimodular Hermitian lattices grow infinitely as the field discriminants grow.

RÉSUMÉ

Nous donnons une borne inférieure pour le nombre de classes de réseaux ternaires hermitiens unimodulaires sur corps quadratique imaginaire. Cela montre que le nombre de classes de réseaux unimodulaires hermitienne tend vers l'infini avec le discriminant du corps.

1. Introduction

As a variant of quadratic forms, we can define Hermitian lattices over rings of integers in imaginary quadratic fields. Also, thanks to many mathematicians, we can think about the local structure of Hermitian lattices [4,12,1]. Hasse–Minkowski's theorem for quadratic forms guarantees that local representations over $\mathbb{Z}_p$ for all prime spots p imply the global representation over $\mathbb{Q}$. But, in general, it does not imply the representation over $\mathbb{Z}$. Landherr's theorem is a similar result for Hermitian lattices [8]. But, it does not imply the representation over the ring $\mathcal{O}$ of integers, either. The measure of non-representability over $\mathcal{O}$ is presented by the number of non-isometric Hermitian lattices that are locally isometric to the given Hermitian

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lattice. This number is called the *class number* of that lattice. We give a lower bound for ternary unimodular Hermitian lattices involving discriminants of fields.

The exact formula for class numbers of binary or ternary unimodular Hermitian lattices was given by Hashimoto and Koseki [2, Main Theorems 5.1, 5.2]. Their formula was expressed by using the field class numbers, the numbers of prime divisors, Dirichlet characters, Bernoulli numbers, as well as field discriminants. So, it is hard to see the bounds for the class numbers by using Hashimoto–Koseki's formula.

Our formula for the lower bound involves only field discriminants, so that it is easy to calculate the formula, although the lower bound is not accurate. Besides, the inequality for the lower bound shows that the class number grows asymptotically according to the discriminant.

2. Preliminaries

Let $E=\mathbb{Q}(\sqrt{-m})$ for a square-free positive integer m and $\mathcal{O}=\mathcal{O}_E=\mathbb{Z}[\omega]$ be its ring of integers, where $\omega=\omega_m=\sqrt{-m}$ or $\frac{1+\sqrt{-m}}{2}$ if $m\equiv 1,2$ or $3 \pmod 4$, respectively.

The localization is defined according to the behaviors of primes. For a prime p we define $E_p:=E\otimes_{\mathbb{Q}}\mathbb{Q}_p$. Then the ring $\mathcal{O}_p$ of integers of E_p is defined as $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p$. If p is inert or ramifies in E, then $E_p=\mathbb{Q}_p(\sqrt{-m})$ and $\alpha\otimes\beta=\alpha\beta$ with $\alpha\in E$ and $\beta\in\mathbb{Q}_p$. If p splits in E, then $E_p=\mathbb{Q}_p\times\mathbb{Q}_p$ and $\alpha\otimes\beta=(\alpha\beta,\overline{\alpha}\beta)$, where $\bar{\cdot}$ is the canonical involution. Thus E_p allows the unique involution $\overline{\alpha\otimes\beta}=\overline{\alpha}\otimes\beta$ [12,1].

Definition 2.1. Let $F=E$ or E_p. A *Hermitian space* is a finite-dimensional vector space V over F equipped with a sesquilinear map $H: V\times V\to F$ satisfying the following conditions:

1. $H(\mathbf{x},\mathbf{y})=\overline{H(\mathbf{y},\mathbf{x})}$,
2. $H(a\mathbf{x},\mathbf{y})=aH(\mathbf{x},\mathbf{y})$,
3. $H(\mathbf{x}_1+\mathbf{x}_2,\mathbf{y})=H(\mathbf{x}_1,\mathbf{y})+H(\mathbf{x}_2,\mathbf{y})$.

We simply denote $H(\mathbf{v},\mathbf{v})$ by $H(\mathbf{v})$ and call it the *(Hermitian) norm* of $\mathbf{v}$.

Definition 2.2. Let $R=\mathcal{O}$ or $\mathcal{O}_p$. A *Hermitian R-lattice*, or briefly a lattice, is an R-module L in a Hermitian space (V,H) with $H(L,L)\subseteq R$. If $H(\mathbf{v})>0$ for any nonzero vector $\mathbf{v}\in L$, then we call L *positive definite*.

If L is free with a basis $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$, then we define

$$M_L:=\left[H(\mathbf{v}_i,\mathbf{v}_j)\right]_{n\times n},$$

and call it the *Gram matrix* of L. We often identify M_L with the lattice L. If M_L is diagonal, we simply write $L=\langle a_1,\dots,a_n\rangle$, where $a_i=H(\mathbf{v}_i)$ for $i=1,2,\dots,n$. The determinant of M_L is called the *discriminant* of L, denoted by dL. If dL is a unit, we call L *unimodular*. We define the *rank* of L by $\operatorname{rank} L:=\dim_F F\otimes L$. For unexplained terminology and for more information, see [9].

3. Lower bounds for class numbers of unimodular Hermitian lattices

If a positive definite Hermitian lattice represents every positive definite binary Hermitian lattice, we call it 2-universal. In [6], we classified all ternary and quaternary Hermitian lattices that are 2-universal.

$$\begin{array}{lll}
\mathbb{Q}(\sqrt{-1}): & \langle 1,1,1\rangle, & \langle 1,1\rangle\perp\begin{bmatrix}2&1\\1&2\end{bmatrix}\\
\mathbb{Q}(\sqrt{-2}): & \langle 1,1\rangle\perp\begin{bmatrix}2&-1+\omega_2\\-1+\overline{\omega}_2&2\end{bmatrix} & \\
\mathbb{Q}(\sqrt{-3}): & \langle 1,1,1\rangle, & \langle 1,1,2\rangle\\
\mathbb{Q}(\sqrt{-7}): & \langle 1,1,1\rangle & \\
\mathbb{Q}(\sqrt{-11}): & \langle 1,1\rangle\perp\begin{bmatrix}2&\omega_{11}\\\overline{\omega}_{11}&2\end{bmatrix}. &
\end{array}$$

We obtain a lower bound for class numbers of unimodular Hermitian lattices by investigating the ranks of 2-universal lattices. Denote the minimal rank of 2-universal Hermitian lattices over $\mathbb{Q}(\sqrt{-m})$ by $u_2(-m)$. We know that $u_2(-1)=3$, $u_2(-2)=4$, $u_2(-3)=3$, $u_2(-7)=3$, and $u_2(-11)=4$ from the above list.

Lemma 3.1. *Let L be a ternary unimodular lattice over an imaginary quadratic field E. Then L is locally 2-universal.*

Proof. Note that no ternary lattice is even unimodular. Let ℓ be an arbitrary positive definite binary Hermitian lattice over E and $p \in \mathbb{Q}$ be a prime.

If p is split in E, $\ell_p \to L_p$ by [1, 1.8]. If p is inert or a ramified nondyadic prime in E, $\ell_p \to L_p$ by [5, Theorem 4.4]. If $p = 2$ ramifies in E, $\ell_p \to L_p$ by [5, theorem 5.5]. Thus, L is locally 2-universal. $\square$

From Lemma 3.1, we assert that if a positive definite binary Hermitian lattice is given over $\mathbb{Q}(\sqrt{-m})$, then there exists a ternary unimodular lattice that represents it globally. This follows from the corresponding result for quadratic lattices [3]. Thus, there exist 2-universal Hermitian lattices for all $\mathbb{Q}(\sqrt{-m})$, because a unimodular lattice L is locally 2-universal and thus we can make a 2-universal Hermitian lattice by summing up orthogonally all non-isometric lattices in the genus. We obtain an upper bound for minimal ranks as follows:

$$u_2(-m) \le 3h(L).$$

For example, from [11] we have that over $\mathbb{Q}(\sqrt{-19})$

$$\operatorname{gen} I_3 = \operatorname{cls}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cup \operatorname{cls}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & \omega_{19} \\ 0 & \overline{\omega}_{19} & 3 \end{bmatrix} \cup \operatorname{cls}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1+\omega_{19} \\ 1 & 1+\overline{\omega}_{19} & 3 \end{bmatrix}.$$

Thus the following lattice of rank 8 is trivially 2-universal over $\mathbb{Q}(\sqrt{-19})$.

$$\langle 1,1,1\rangle \perp \begin{bmatrix} 2 & \omega_{19} \\ \overline{\omega}_{19} & 3 \end{bmatrix} \perp \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1+\omega_{19} \\ 1 & 1+\overline{\omega}_{19} & 3 \end{bmatrix}.$$

Now consider binary sublattices $\ell(k) := \begin{bmatrix} k & \omega \\ \overline{\omega} & c_k \end{bmatrix}$ with $2 \le k \le c_k$ and $k(c_k - 1) \le \omega\overline{\omega} < kc_k$. Then the largest index of k's is given as $n := \left\lfloor \frac{\sqrt{4\omega\overline{\omega}+1}+1}{2} \right\rfloor$ from $n \le c_n \le \frac{\omega\overline{\omega}}{n} + 1$.

Let $\{\mathbf{v}_k, \mathbf{w}_k\}$ be the basis of $\ell(k)$ with $\mathbf{v}_k \cdot \mathbf{v}_k = k$, $\mathbf{w}_k \cdot \mathbf{w}_k = c_k$, and $\mathbf{v}_k \cdot \mathbf{w}_k = \omega$. Also let $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$. We show that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent.

Suppose that those vectors are not linearly independent. Then

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + \omega(b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n) = 0 \tag{1}$$

with $a_i, b_i \in \mathbb{Z}$. Note that

$$|\mathbf{v}_i \cdot \mathbf{v}_j|^2 \le (\mathbf{v}_i \cdot \mathbf{v}_i)(\mathbf{v}_j \cdot \mathbf{v}_j) = ij < \omega\overline{\omega}$$

unless $i = j = n$. Thus $\mathbf{v}_i \cdot \mathbf{v}_j \in \mathbb{Z}$ for $1 \le i, j \le n$. Multiplying (1) by $\mathbf{v}_j$ and comparing both sides, we conclude that $(b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n) \cdot \mathbf{v}_j = 0$. Since the norm of $b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$ vanishes and the concerned 2-universal Hermitian lattice is positive definite, $b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = 0$. Thus we can write

$$a_{k_1}\mathbf{v}_{k_1} + \cdots + a_{k_N}\mathbf{v}_{k_N} = 0$$

with nonzero $a_{k_i} \in \mathbb{Z}$ and $k_1 < k_2 < \cdots < k_N \le n$. But we obtain a contradiction by multiplying both sides by $\mathbf{w}_{k_N}$, since $\mathbf{v}_{k_i} \cdot \mathbf{w}_{k_N} \in \mathbb{Z}$ for $i < N$ and $\mathbf{v}_{k_N} \cdot \mathbf{w}_{k_N} = \omega$.

The 2-universal lattice contains $\langle 1,1\rangle$ and $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent, so that the rank of 2-universal lattice is bigger than n. That is,

$$u_2(-m) \ge \text{the number of } c_n\text{'s} + 2 = \left\lfloor \frac{\sqrt{4\omega\overline{\omega}+1}+1}{2} \right\rfloor + 1.$$

We have that the equality holds for $u_2(-2) = u_2(-7) = 3$.

Now let us think about finiteness theorems for 2-universal lattices of higher ranks. For any infinite set S of positive definite integral quadratic forms of bounded rank, there exists a finite subset S_0 of S such that any positive definite integral quadratic form that represents all elements of S_0 represents all elements of S [7]. This theorem holds for Hermitian lattices. Denote the finite set ensuring 2-universality over $E = \mathbb{Q}(\sqrt{-m})$ by S_{-m}. We have constructed binary lattices $\begin{bmatrix} k & \omega \\ \overline{\omega} & c_k \end{bmatrix}$ that should be represented. Besides, $\langle 1,1\rangle$ should be also represented independently. Thus the cardinality of S_{-m} has a lower bound:

$$\#S_{-m} \ge \text{the number of } c_n\text{'s} + 1 = \left\lfloor \frac{\sqrt{4\omega\overline{\omega}+1}+1}{2} \right\rfloor.$$

If we denote the discriminant of $E = \mathbb{Q}(\sqrt{-m})$ by d_E, $d_E = -4m = -4\omega\overline{\omega}$ if $m \not\equiv 3 \pmod 4$ and $d_E = -m = 1 - 4\omega\overline{\omega}$ if $m \equiv 3 \pmod 4$. Then the above results can be rephrased as following:

Theorem 3.2. *$u_2(d_E)$ and $\#S_{d_E}$ are $\Omega(\sqrt{|d_E|})$.*

In the above theorem, $f(n) = \Omega(g(n))$ means $\liminf_{n\to\infty} | f(n)/g(n) | > 0$. Roughly speaking, the two quantities increase as $|d_E|$ increases.

Corollary 3.3. *Let E be an imaginary quadratic field and d_E its discriminant. If L is a ternary unimodular Hermitian lattice over E, then $h(L) = \Omega(\sqrt{|d_E|})$.*

Proof. It is clear from $h(L) \geq u_2(d_E)/3$. $\square$

Remark 1. By Hashimoto–Koseki's result [2], we see that

$$h(L) = \frac{1}{6} \cdot \frac{1}{2^t} h(E)^2 |d_E| + O\left(\frac{1}{2^t} h(E)^2\right)$$

where t is the number of distinct prime divisors of d_E and $h(E)$ is the class number of E. It is known that $t \leq 1.3841 \frac{\log d_E}{\log\log d_E}$ [10], so that one can verify that $h(L)$ increases asymptotically as $|d_E|$ increases, although it is not easy to write the lower bound explicitly.

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References

[1] L.J. Gerstein, Integral decomposition of Hermitian forms, Amer. J. Math. 92 (1970) 398–418.
[2] K. Hashimoto, H. Koseki, Class numbers of positive definite binary and ternary unimodular Hermitian forms, Tohoku Math. J. (2) 41 (2) (1989) 171–216.
[3] J.S. Hsia, J.P. Prieto-Cox, Representations of positive definite Hermitian forms with approximation and primitive properties, J. Number Theory 47 (2) (1994) 175–189.
[4] N. Jacobson, A note on Hermitian forms, Bull. Amer. Math. Soc. 46 (1940) 264–268.
[5] A.A. Johnson, Integral representation of Hermitian forms over local fields, J. Reine Angew. Math. 229 (1968) 57–80.
[6] M.-H. Kim, P.-S. Park, 2-Universal Hermitian lattices over imaginary quadratic fields, Ramanujan J. 22 (2) (2010) 139–151.
[7] B.M. Kim, M.-H. Kim, B.-K. Oh, A finiteness theorem for representability of quadratic forms by forms, J. Reine Angew. Math. 581 (2005) 23–30.
[8] W. Landherr, Äquivalenz Hermitscher Formen über einen beliebigen algebraischen Zahlkörper, Abh. Math. Semin. Univ. Hamb. 11 (1935) 245–248.
[9] O.T. O'Meara, Introduction to Quadratic Forms, Springer-Verlag, New York, 1973.
[10] G. Robin, Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n, Acta Arith. 42 (4) (1983) 367–389.
[11] A. Schiemann, Classification of Hermitian forms with the neighbour method, J. Symb. Comput. 26 (4) (1998) 487–508.
[12] G. Shimura, Arithmetic of unitary groups, Ann. Math. (2) 79 (1964) 369–409.