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Algebraic geometry

Uniform bound for the effective Bogomolov conjecture

Borne uniforme pour la conjecture effective de Bogomolov

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ABSTRACT

We obtain a uniform bound for the effective Bogomolov conjecture, which depends only on the genus g of the curve. The bound grows as $O(g^{-3})$ as g tends to infinity.

RÉSUMÉ

On obtient une borne uniforme de la conjecture effective de Bogomolov, qui ne dépend que du genre g de la courbe. Cette borne croît comme $O(g^{-3})$ lorsque g tend vers l'infini.

1. Introduction

Fix an algebraically closed field k of characteristic zero and a smooth proper connected curve Y/k. Define K to be the field of rational functions on Y. Let C be a smooth proper geometrically connected curve of genus at least 2 over the function field K. Denote by $f: X \to Y$ the minimal regular model of the curve C over Y, where X is a smooth projective surface over k. Choose a divisor D of degree 1 on $\bar{C} = C \times_K \bar{K}$ and consider the embedding of C into its Jacobian $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ given on geometric points by $j_D(x) = [x] - D$. Define

$$a'(D) = \liminf_{x \in C(\bar{K})} \hat{h}(j_D(x)),$$

where $\hat{h}$ is the canonical Néron–Tate height on the Jacobian associated with the symmetric ample divisor $\Theta + [-1]^*\Theta$. As $C(\bar{K})$ may not be countable, the liminf is taken to mean the limit over the directed set of all cofinite subsets of $C(\bar{K})$ of the infimum of the heights of points in such a subset.

Many authors [3,4,6,9,14,15]... studied the following effective Bogomolov conjecture, which has been solved by Cinkir [3].

Conjecture 1.1. *If f is non-isotrivial, then there exists a positive number r_0 such that*

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$$\inf_{D\in Div^1(\bar{C})} a'(D) \geq r_0.$$

This conjecture concerns the finiteness of algebraic points of small height on a smooth complete curve over a global field.

By the Semistable Reduction Theorem, we may pass to a finite extension field $\tilde{K}$ over which $C_{\tilde{K}} = C \times_K \tilde{K}$ has semistable reduction. Let $\tilde{Y}/k$ be a smooth proper curve with field of rational functions $\tilde{K}$. To say that $C_{\tilde{K}}$ has semistable reduction means that there is a projective surface $\tilde{X}/k$ and a proper flat morphism $\tilde{f}: \tilde{X} \to \tilde{Y}$ so that

(1) $\tilde{f}$ has generic fiber isomorphic to $C_{\tilde{K}}$,
(2) the fibers of $\tilde{f}$ are connected and reduced with only nodal singularities,
(3) if Z is an irreducible component of a fiber and $Z \cong \mathbb{P}^1$, then Z meets the other components of the fiber in at least 2 points.

If we assume further that $\tilde{X}$ is a smooth surface (over k), then such a morphism $\tilde{f}: \tilde{X} \to \tilde{Y}$ is unique up to canonical isomorphism, and it may be characterized as the minimal regular model of $C_{\tilde{K}}$ over $\tilde{Y}$.

We divide the fiber singularities of $\tilde{f}$ into different types as follows. Choose a point $y \in \tilde{Y}(k)$ and a node $p \in \tilde{f}^{-1}(y)$. We say p is of type 0 if the normalization of $\tilde{f}^{-1}(y)$ at p is connected. Otherwise, the normalization at p has two connected components, and we say that p is of type i, where i is the minimum of the arithmetic genera of the two components. Let $\delta_i(\tilde{f}) = \delta_i(\tilde{X}/\tilde{Y})$ be the total number of nodes of type i in all fibers. By the uniqueness of the minimal regular model, the numbers $\delta_i(\tilde{f})$ are well-defined invariants of $C_{\tilde{K}}$.

In this note, we always assume that $\tilde{f}$ is not smooth. We rewrite Theorem 2.4 in [3] without semistable condition as follows, see also [6, Theorem 1.2].

Theorem 1.2 *([3], Theorem 2.4). Let C/K be a smooth proper geometrically connected curve of genus $g \geq 2$, and $d = [\tilde{K} : K]$. If $\tilde{f}$ is not smooth, then*

$$\inf_{D\in Div^1(\bar{C})} a'(D) \geq \frac{1}{2(2g+1)}\Big(\frac{(g-1)^2}{2g(7g+5)}\frac{\delta_0(\tilde{f})}{d} + \sum_{i\in(0,g/2]} \frac{2i(g-i)}{g}\frac{\delta_i(\tilde{f})}{d}\Big).$$

Then it is natural to consider the uniformity property (see [2]). The main purpose of this note is to give a uniform lower bound, as good as possible, which depends only on the genus of the curve.

Theorem 1.3. *Let C/K be a smooth proper geometrically connected curve of genus $g \geq 2$. If $\tilde{f}$ is not smooth, then*

$$\inf_{D\in Div^1(\bar{C})} a'(D) \geq \frac{(g-1)^2}{8g^3(4g+2)(7g+5)}.$$

Here we get a bound which grows as $O(g^{-3})$ as g tends to infinity.

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g, $\overline{\mathcal{M}}_g$ be its Deligne–Mumford compactification, and $\Delta_0, \Delta_1, \ldots, \Delta_{[g/2]}$ be the boundary divisors of $\overline{\mathcal{M}}_g$. Let $J: Y \to \overline{\mathcal{M}}_g$ be the induced moduli map of f [11], we call $\delta_i(f) = \deg(J^*\Delta_i)$ the *modular invariant* corresponding to Δ_i for each $i = 0, 1, \ldots, [g/2]$. Then [5]

$$\delta_i(f) = \frac{\delta_i(\tilde{f})}{d}, \quad i = 0, 1, \ldots, [g/2]. \tag{1.1}$$

We refer to [11] for modular invariants. More recently, the second author found that modular invariants are important to the study of holomorphic foliations and Poicaré problems, see [12]. Note that when $\tilde{f}$ is not smooth, there is at least one singular fiber and $\delta(f) \neq 0$. Thus there is at least one $0 \leq i \leq g/2$ with $\delta_i(f) \neq 0$. In order to prove Theorem 1.3, we give the lower bounds of modular invariants $\delta_i(f)$ as follows.

Theorem 1.4. *Let C/K be a smooth proper geometrically connected curve of genus $g \geq 2$, and $f: X \to Y$ be the minimal regular model of C. If $\delta_i(f) \neq 0$, then*

$$\delta_i(f) \geq \begin{cases} \frac{1}{4g^2}, & \text{if } i = 0, \\ \frac{1}{(4i+2)(4(g-i)+2)}, & \text{if } i \geq 1. \end{cases}$$

In order to prove the above theorems, we provide a simple formula (3.2) of $\delta_i(f)$, which can be applied to the study of pseudo-periodic maps and degeneration of Riemann surfaces [8]. Our method of this paper is new, and can be used to obtain more explicit results.

2. Preliminaries and notations

Let F be a singular fiber of $f : X \to Y$. Let $\bar{F}$ be a birational model of F such that $\bar{F}$ is normal crossing and $\bar{F}_{\mathrm{red}}$ has at worst ordinary double points as its singularities. If $\bar{F}$ has no (-1)-curves meeting other components in at most two points, then $\bar{F}$ is called *the minimal normal crossing model* of F (see [7, Section 2.1]). Denote by $\bar{f} : \bar{X} \to Y$ the obtained fibration by blowing up the singularities of the fibers of f such that each singular fiber of $\bar{f}$ is minimal normal crossing.

Let $\pi : \tilde{Y} \to Y$ be a base change of degree d. *The pullback fibration* $\tilde{f}$ of f under π is the relative minimal model of the desingularization of $\bar{X} \times_Y \tilde{Y} \to \tilde{Y}$. We can see the following diagram for this construction.

$$\begin{array}{ccccccccc}
\tilde{X} & \xleftarrow{\ \tilde{\rho}\ } & X_2 & \xrightarrow{\ \rho_2\ } & X_1 & \xrightarrow{\ \rho_1\ } & \bar{X} \times_Y \tilde{Y} & \xrightarrow{\ \Pi'\ } & \bar{X} \\
\tilde{f}\downarrow & & f_2\downarrow & & f_1\downarrow & & \downarrow & & \bar{f}\downarrow \\
\tilde{Y} & = & \tilde{Y} & = & \tilde{Y} & = & \tilde{Y} & \xrightarrow{\ \pi\ } & Y
\end{array}$$

Here ρ_1 is the normalization, ρ_2 is the minimal desingularization of X_1, and $\tilde{\rho} : X_2 \to \tilde{X}$ is the contraction of (-1)-curves in singular fibers. We call π a *stabilizing base change* if $\tilde{f}$ is semistable.

Now we consider the above construction locally. Let F be a fiber of f over $p \in Y$. Assume that π is totally ramified over p, i.e. $\pi^{-1}(p)$ contains only one point $\tilde{p}$. In this case, π is defined locally by $z = w^d$. Denote by $\tilde{F}$ the fiber of $\tilde{f}$ over $\tilde{p} \in \tilde{Y}$. If $\tilde{F}$ is semistable, then $\tilde{F}$ is called the *d-th semistable model* of F.

In the following, we use C to denote an irreducible reduced component of $\bar{F}$.

Set $\Pi_2 = \Pi' \circ \rho_1 \circ \rho_2 : X_2 \to \bar{X}$, and $\Pi = \Pi_2 \circ \tilde{\rho}^{-1} : \tilde{X} \dashrightarrow \bar{X}$. Then Π is a well-defined rational morphism. For any irreducible smooth component $\tilde{C}$ of $\tilde{F}$, we can define the induced morphism $\Pi|_{\tilde{C}} : \tilde{C} \to C$ by the unique extension, where C is the image of $\Pi|_{\tilde{C}}$. Let $\Pi^{-1}(C)$ be the set of irreducible curves $\tilde{C}$ in $\tilde{F}$ with $\Pi(\tilde{C}) = C$, and $l(C) = |\Pi^{-1}(C)|$. Set $n_C = \deg(\Pi|_{\tilde{C}})$, which is independent of the choice of the irreducible component $\tilde{C}$ of $\Pi^{-1}(C)$. Then $n(C) = l(C) n_C$, where $n(C) = \mathrm{mult}_C(\bar{F})$ is the multiplicity of C in $\bar{F}$. It is easy to see that n_C is the period of the induced cyclic automorphism of $\tilde{C}$.

Let F be a singular fiber of f, and $\tilde{F}$ be its d-th semistable model. Let $\delta_i(\tilde{F})$ be the number of nodes of type i in $\tilde{F}$, then we define

$$\delta_i(F) := \frac{\delta_i(\tilde{F})}{d}, \quad i = 0, 1, \ldots, [g/2]. \tag{2.1}$$

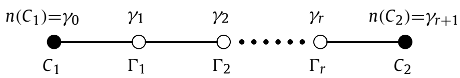
The definition is independent of the choice of the semistable model of F. Let $F_1, \ldots, F_s$ be all the singular fibers of f. If we choose a stabilizing base change totally ramified over $f(F_1), \ldots, f(F_s)$, then $\delta_i(\tilde{f}) = \delta_i(\tilde{F}_1) + \cdots + \delta_i(\tilde{F}_s)$. So we have that

$$\delta_i(f) = \delta_i(F_1) + \cdots + \delta_i(F_s), \quad i = 0, 1, \ldots, [g/2]. \tag{2.2}$$

If D is an irreducible singular component of $\tilde{F}$, then all the singularities of D are nodes for $\tilde{F}$ is semistable. After a base change of degree 2, the strict transform of D is smooth. Thus we may always assume that each irreducible component of $\tilde{F}$ is smooth, because $\delta_i(F)$ is independent of the choice of the semistable model $\tilde{F}$.

An irreducible component C of $\bar{F}$ is said to be *principal* if either C is not smooth rational, or C meets other components of $\bar{F}_{\mathrm{red}}$ at no less than three points.

Definition 2.1. Let F be a singular fiber of f, and $\bar{F}$ be its minimal normal crossing model. Let $\mathcal{C}$ be the following subgraph of the dual graph $G(\bar{F})$ of $\bar{F}$,



where C_1 and C_2 are two principal components of $\bar{F}$ (C_1, C_2 may be the same), $n(C_j) = \mathrm{mult}_{C_j}(\bar{F})$ $(j = 1, 2)$, $\gamma_k = \mathrm{mult}_{\Gamma_k}(\bar{F})$ $(1 \le k \le r)$, and $\Gamma_k \cong \mathbb{P}^1$ is not a principal component of $\bar{F}$. Then we call $\mathcal{C}$ a *principal chain* between C_1 and C_2, and we set $\mathcal{C} = \langle C_1, C_2 \rangle$ if there is no confusion. Set $d(\mathcal{C}) = \gcd(\gamma_0, \gamma_1, \ldots, \gamma_{r+1})$, and $\lambda_j = n(C_j)/d(\mathcal{C})$. Note that each $\tilde{C}_j \in \Pi^{-1}(C_j)$ connects with at least one of the $d(\mathcal{C})$ inverse images of $\mathcal{C}$ in $\tilde{F}$, so $l(C_j) \le d(\mathcal{C})$ and $\lambda_j \le n_{C_j}$. Let $0 \le \sigma_j < \lambda_j$ be integers satisfying

$$\sigma_1 \equiv \frac{\gamma_1}{d(\mathcal{C})} \pmod{\lambda_1}, \quad \sigma_2 \equiv \frac{\gamma_r}{d(\mathcal{C})} \pmod{\lambda_2}. \tag{2.3}$$

We denote by $PC(\bar{F})$ all the principal chains of $\bar{F}$. If all the nodes of $\Pi^{-1}(p)$ are of type i for any node p of $\mathcal{C}$, then we call $\mathcal{C}$ a *principal chain of type i*. Denote by $PC_i(\bar{F})$ all the principal chains of $\bar{F}$ of type i.

By Zariski's Lemma [1, III Lemma 8.2], we know that $\gamma_{i-1}+\gamma_{i+1}\equiv 0 \pmod{\gamma_i}$ $(i=1,\dots,r)$. Hence if $\sigma_1=0$, then $\gamma_0|\gamma_i$ $(i=1,2,\dots,r+1)$, and $\lambda_1=1$. Similarly, if $\sigma_2=0$, then $\lambda_2=1$. Let $\mathcal{C}=\langle C_1,C_2\rangle$ be a principal chain in Definition 2.1, we define

$$H(\mathcal{C}):=\sum_{i=0}^{r}\frac{\gcd(\gamma_i,\gamma_{i+1})^2}{\gamma_i\gamma_{i+1}}. \tag{2.4}$$

Lemma 2.2. *Let $\mathcal{C}=\langle C_1,C_2\rangle$ be a principal chain of $\bar{F}$, then*

$$H(\mathcal{C})=\frac{d(\mathcal{C})\mu_1}{n(C_1)}+\frac{d(\mathcal{C})\mu_2}{n(C_2)}+K(\mathcal{C})=\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}+K(\mathcal{C}), \tag{2.5}$$

where $K(\mathcal{C})\ge -1$ is an integer, and μ_1, μ_2 are integers with

$$0<\mu_1\le\lambda_1,\ \ \sigma_1\mu_1\equiv 1\ (mod\ \lambda_1),\quad 0<\mu_2\le\lambda_2,\ \ \sigma_2\mu_2\equiv 1\ (mod\ \lambda_2).$$

In particular, we have that

$$H(\mathcal{C})\ge\frac{1}{\operatorname{lcm}(\lambda_1,\lambda_2)}\ge\frac{1}{\lambda_1\lambda_2}\ge\frac{1}{n_{C_1}n_{C_2}}. \tag{2.6}$$

Proof. Without loss of generality, we may assume that $d(\mathcal{C})=1$. Hence $\gcd(\gamma_i,\gamma_{i+1})=1$ by Zariski's Lemma, and (2.5) is Lemma 5.2 (1) in [8]. Now we only need to prove the integer $K(\mathcal{C})\ge -1$. If $\mu_1=\lambda_1$ and $\mu_2=\lambda_2$, then $\lambda_1=\lambda_2=1$ (see NB below Lemma 5.2 (1) in [8]). So $H(\mathcal{C})$ is a positive integer for $H(\mathcal{C})>0$, and then $K(\mathcal{C})\ge -1$. For the remaining case, $0<\mu_1/\lambda_1+\mu_2/\lambda_2<2$, and thus $K(\mathcal{C})\ge -1$. Note that (2.6) is obtained from that $\operatorname{lcm}(\lambda_1,\lambda_2)H(\mathcal{C})$ is an integer bigger than 0 by (2.5). □

3. Proof of theorems

Firstly, we give a formula of $\delta_i(f)$.

Lemma 3.1. *Let F be a singular fiber of f, and $\bar{F}$ be its minimal normal crossing model, then*

$$\delta_i(F)=\sum_{\mathcal{C}\in PC_i(\bar{F})}H(\mathcal{C}),\ \ i=0,1,\dots,[g/2]. \tag{3.1}$$

Proof. Suppose $\bar{F}=n(C_1)C_1+\cdots+n(C_t)C_t$. Let $\tilde{F}$ be a d-th semistable model of F with $n(C_i)|d$ $(i=1,2,\dots,t)$, and q be a node of $\tilde{F}$. By definition, q is a node of $\tilde{F}$ of type i if and only if $p=\Pi(q)$ is a node of some principal chain of $\bar{F}$ of type i. We may assume $p\in C_1\cap C_2$, then, by [10, Lemma 1.4], there are $d_p=\gcd(n(C_1),n(C_2))$ disjoint curves of type A_n, where n satisfies

$$\frac{\#\{q\in\tilde{F}: q \text{ is a node}, \Pi(q)=p\}}{d}=\frac{d_p(n+1)}{d}=\frac{\gcd(n(C_1),n(C_2))^2}{n(C_1)n(C_2)}.$$

Then the result is directly from the definition of $\delta_i(F)$. □

Theorem 3.2. *Let $F_1,\dots,F_s$ be all the singular fibers of f, and $PC_i(\bar{f})=PC_i(\bar{F}_1)\cup\cdots\cup PC_i(\bar{F}_s)$, for $0\le i\le[g/2]$. Then*

$$\delta_i(f)=\sum_{\mathcal{C}\in PC_i(\bar{f})}H(\mathcal{C})=\sum_{\mathcal{C}\in PC_i(\bar{f})}\left(\frac{d(\mathcal{C})\mu_{1,\mathcal{C}}}{n(C_{1,\mathcal{C}})}+\frac{d(\mathcal{C})\mu_{2,\mathcal{C}}}{n(C_{2,\mathcal{C}})}+K(\mathcal{C})\right). \tag{3.2}$$

Here we add $\mathcal{C}$ to each notation in order to differentiate, and the meaning of each notation is the same as that in Lemma 2.2.

Proof. It is directly from (2.2), Lemma 2.2 and Lemma 3.1. □

Now we prove two lemmas which play an important role in the proof of the main theorems.

Lemma 3.3. *Let F be a singular fiber of genus $g\ge 2$, C be a principal component of $\bar{F}$, and $\tilde{C}$ be an element of $\Pi^{-1}(C)$. Then*

(1) if either $g(\tilde{C})\ge 2$ or $g(\tilde{C})=1$ and $\Pi|_{\tilde{C}}:\tilde{C}\to C$ is ramified, then $n_C\le 4g(\tilde{C})+2$;
(2) if $g(\tilde{C})=1$ and $\Pi|_{\tilde{C}}:\tilde{C}\to C$ is un-ramified, then $n_C\le 2(g-1)$;

(3) assume $g(\tilde{C})=0$. If $\tilde{C}$ is not principal, then $n_C=2$. If $\tilde{C}$ is principal, then $n_C \le \min\{g+1-p_a(\tilde{F}-\tilde{C}), g+1\}$, where $p_a(\tilde{F}-\tilde{C})$ is the arithmetic genus of $\tilde{F}-\tilde{C}$.

In particular, $n_C \le 4g+2$.

Proof. (1). See Lemma B in [13].

(2). We denote also by C the component corresponding to C in F, then $n(C)=\mathrm{mult}_C(\bar{F})=\mathrm{mult}_C(F)$. Since $K_{X/Y}$ is nef, $n(C)CK_{X/Y} \le FK_{X/Y}=2g-2$. If $CK_{X/Y}=0$, then C is a (-2)-curve, which is impossible. Therefore, $CK_{X/Y} \ge 1$, and $n(C) \le 2g-2$.

(3). If $\tilde{C}$ is not principal, then $\tilde{C}(\tilde{F}-\tilde{C})=2$ for $\tilde{F}$ is semistable and $n_C \ne 1$ for C is a rational principal component of $\bar{F}$. Since $g(\tilde{C})=0$ and $\Pi|_{\tilde{C}}:\tilde{C}\to C$ is a cyclic covering of degree n_C, we know that there are exactly two branched points on C. Because C meets other components of $\bar{F}_{\mathrm{red}}$ at no less than three points, there is a third point on C, which is the intersection of C with some other component of $\bar{F}$, say D. Then $n(C)|n(D)=\mathrm{mult}_D(\bar{F})$. Thus $\tilde{C}$ meets $\Pi^{-1}(D)$ in at least n_C points. Hence $n_C \le \tilde{C}\cdot\Pi^{-1}(D) \le \tilde{C}(\tilde{F}-\tilde{C})=2$.

If $\tilde{C}$ is principal, set $\tilde{C}\cap(\tilde{F}-\tilde{C})=\{q_1,\ldots,q_m\}$, then $m=\tilde{C}(\tilde{F}-\tilde{C})\ge 3$. Since

$$p_a(\tilde{F})=p_a(\tilde{F}-\tilde{C})+p_a(\tilde{C})+\tilde{C}(\tilde{F}-\tilde{C})-1=p_a(\tilde{F}-\tilde{C})+m-1,$$

we know that $m=g+1-p_a(\tilde{F}-\tilde{C})$. Let $\sigma\in\mathrm{Aut}(\tilde{C})$ be the induced cyclic automorphism of $\tilde{C}$, then $\sigma(q_i)\in\{q_1,\ldots,q_m\}$ for any $1\le i\le m$. Denote by τ the restriction action of σ on $\{q_1,\ldots,q_m\}$, then τ is a permutation. Let $\tau=\tau_1\cdot\tau_2\cdots\tau_k$ be the decomposition of τ into disjoint cycles, and let $m_0=\max\{|\tau_1|,\ldots,|\tau_k|\}$ where $|\tau_i|$ is the length of the cycle τ_i. Because these m_0 points induce either m_0-1 cycles or m_0 curves E_k's with $p_a(E_k)\ge 1$ in $\tilde{F}$ by semi-stability of $\tilde{F}$, we have $m_0\le g+1$. If $m_0\le 2$, then τ^{m_0} fixes all the m points. If $m_0\ge 3$, then τ^{m_0} fixes $m_0\ge 3$ points. In all cases, τ^{m_0} fixes no less than 3 points. So σ^{m_0} is the identity map and $n_C=m_0\le\min\{m,g+1\}$. □

Lemma 3.4. *Let $\mathcal{C}=\langle C_1,C_2\rangle\in PC(\bar{F})$, and $\tilde{\mathcal{C}}=\langle\tilde{C}_1,\tilde{C}_2\rangle$ be a principal chain of $\tilde{F}$ with $\Pi(\tilde{\mathcal{C}})=\mathcal{C}$. If $g(\tilde{C}_1)=1$ and $\tilde{C}_1\to C_1$ is un-ramified, then*

$$H(\mathcal{C})\ge\frac{1}{4g-2}.$$

Proof. Since $\tilde{C}_1\to C_1$ is un-ramified, we have that $n(C_1)|\mathrm{mult}_D(\bar{F})=n(D)$ for any irreducible component D of $\bar{F}$ intersecting with C_1. Moreover, $n(C_1)=d(\mathcal{C})$ and $\lambda_1=1$ by Zariski's Lemma. We know that $n_{C_2}\le 4(g-1)+2=4g-2$ by Lemma 3.3, and then by Lemma 2.2

$$H(\mathcal{C})=\frac{d(\mathcal{C})\mu_1}{n(C_1)}+\frac{d(\mathcal{C})\mu_2}{n(C_2)}+K(\mathcal{C})\ge\frac{1}{n_{C_2}}\ge\frac{1}{4g-2}. \quad \square$$

Proof of Theorem 1.4. (1) If $\delta_0(F)\ne 0$, then there is a principal chain $\mathcal{C}=\langle C_1,C_2\rangle\in PC_0(\bar{F})$. If $C_1=C_2$, then by Lemma 2.2, 3.1 and 3.3, we know that

$$\delta_0(F)\ge H(\mathcal{C})\ge\frac{1}{\mathrm{lcm}(\lambda_1,\lambda_2)}=\frac{1}{\lambda_1}\ge\frac{1}{n_{C_1}}\ge\frac{1}{4g+2}.$$

If $g(\tilde{C}_1)=g(\tilde{C}_2)=0$, then $n_{C_j}\le g+1$ $(j=1,2)$ by Lemma 3.3 (3), and $\delta_0(F)\ge 1/(g+1)^2$ by (2.6).

Now we may assume that C_1 and C_2 are distinct and $g(\tilde{C}_2)\ge 1$. Then $g\ge g(\tilde{C}_1)+g(\tilde{C}_2)+1$ by [13, Lemma A]. So the rest are the following three cases.

(i) If $g(\tilde{C}_1)=0$, let $g(\tilde{C}_2)=a$, then $1\le a\le g-1$ and $p_a(\tilde{F}-\tilde{C}_1)\ge a$. Thus $n_{C_1}\le g+1-a$, $n_{C_2}\le 4a+2$. So

$$\delta_0(F)\ge H(\mathcal{C})\ge\frac{1}{n_{C_1}n_{C_2}}\ge\frac{1}{(g+1-a)(4a+2)}\ge\frac{1}{4g^2}.$$

(ii) If $g(\tilde{C}_j)=1$, and $\tilde{C}_j\to C_j$ is un-ramified for $j=1$ or $j=2$. Then by Lemma 3.4, we have $\delta_0(F)\ge H(\mathcal{C})\ge 1/(4g-2)$.

(iii) In the remaining cases, we may assume that $n_{C_1}\le 4\alpha+2$ where $\alpha=g(\tilde{C}_1)\ge 1$, and $n_{C_2}\le 4(g-\alpha-1)+2$. Thus

$$\delta_0(F)\ge H(\mathcal{C})\ge\frac{1}{n_{C_1}n_{C_2}}\ge\frac{1}{(4\alpha+2)(4(g-\alpha)-2)}\ge\frac{1}{4g^2}. \tag{3.3}$$

(2) If $i\ge 1$ and $\delta_i(F)\ne 0$, then there is a principal chain $\mathcal{C}=\langle C_1,C_2\rangle\in PC_i(\bar{F})$. Denote by $\tilde{\mathcal{C}}=\langle\tilde{C}_1,\tilde{C}_2\rangle$ a principal chain of $\tilde{F}$ with $\Pi(\tilde{\mathcal{C}})=\mathcal{C}$. Let $\tilde{p}$ be the intersection point of $\tilde{C}_1$ with other components of $\tilde{\mathcal{C}}$, and let $Bl_{\tilde{p}}(\tilde{F})$ be the blow-up of $\tilde{F}$ at $\tilde{p}$. Let $\tilde{F}_1$ and $\tilde{F}_2$ be the two connected components of $Bl_{\tilde{p}}(\tilde{F})$, with $\tilde{C}_1\subseteq\tilde{F}_1$, $p_a(\tilde{F}_1)=i$ and $p_a(\tilde{F}_2)=g-i$.

By Lemma 3.4, we may assume that if $g(C_j) = 1$ $(j = 1, 2)$, then $\tilde{C}_j \to C_j$ is ramified in the following. If $g(\tilde{C}_1) = 0$, then $n_{C_1} \le g + 1 - p_a(\tilde{F}_2) = i + 1$. If $g(\tilde{C}_1) \ge 1$, then $n_{C_1} \le 4p_a(\tilde{F}_1) + 2 = 4i + 2$. Hence $n_{C_1} \le 4i + 2$ always holds true. Similarly, $n_{C_2} \le 4(g - i) + 2$. Thus the result is directly from (2.6). □

Proof of Theorem 1.3. For $g \ge 2$, the result is from the following two cases by Theorem 1.2 and 1.4:

Case 1: $\delta_0(f) > 0$. Then

$$\inf_{D \in \mathrm{Div}^1(\bar{C})} a'(D) \ge \frac{1}{4g+2} \frac{(g-1)^2}{2g(7g+5)} \delta_0(f) \ge \frac{(g-1)^2}{8g^3(4g+2)(7g+5)}.$$

Case 2: $\delta_0(f) = 0$. Since $\delta(f) > 0$, we have $\delta_i(f) > 0$ for some $i > 0$. Then

$$\inf_{D \in \mathrm{Div}^1(\bar{C})} a'(D) \ge \frac{1}{4g+2} \frac{2i(g-i)}{g} \delta_i(f) \ge \frac{2i(g-i)}{(4g+2)g((4i+2)(4(g-i)+2)}$$
$$\ge \frac{(g-1)^2}{8g^3(4g+2)(7g+5)}. \quad \square$$

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