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Symmetry for extremal functions in subcritical Caffarelli–Kohn–Nirenberg inequalities

Symétrie des fonctions extrémales pour des inégalités de Caffarelli–Kohn–Nirenberg sous-critiques

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A R T I C L E I N F O

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A B S T R A C T

We use the formalism of the Rényi entropies to establish the symmetry range of extremal functions in a family of subcritical Caffarelli–Kohn–Nirenberg inequalities. By extremal functions we mean functions that realize the equality case in the inequalities, written with optimal constants. The method extends recent results on critical Caffarelli–Kohn–Nirenberg inequalities. Using heuristics given by a nonlinear diffusion equation, we give a variational proof of a symmetry result, by establishing a rigidity theorem: in the symmetry region, all positive critical points have radial symmetry and are therefore equal to the unique positive, radial critical point, up to scalings and multiplications. This result is sharp. The condition on the parameters is indeed complementary of the condition that determines the region in which symmetry breaking holds as a consequence of the linear instability of radial optimal functions. Compared to the critical case, the subcritical range requires new tools. The Fisher information has to be replaced by Rényi entropy powers, and since some invariances are lost, the estimates based on the Emden–Fowler transformation have to be modified.

R É S U M É

Nous utilisons le formalisme des entropies de Rényi pour établir le domaine de symétrie des fonctions extrémales dans une famille d'inégalités de Caffarelli–Kohn–Nirenberg sous-critiques. Par fonctions extrémales, il faut comprendre des fonctions qui réalisent le cas d'égalité dans les inégalités écrites avec des constantes optimales. La méthode étend des résultats récents sur les inégalités de Caffarelli–Kohn–Nirenberg critiques. En utilisant une heuristique donnée par une équation de diffusion non linéaire, nous donnons une preuve variationnelle d'un résultat de symétrie, grâce à un théorème de rigidité : dans la région de symétrie, tous les points critiques positifs sont à symétrie radiale et sont par conséquent égaux à l'unique point critique radial, positif, à une multiplication par une constante et à un changement d'échelle près. Ce résultat est optimal. La condition sur les paramètres est en effet complémentaire de celle qui définit la région dans laquelle il y a brisure de

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symétrie du fait de l'instabilité linéaire des fonctions radiales optimales. Comparé au cas critique, le domaine sous-critique nécessite de nouveaux outils. L'information de Fisher doit être remplacée par l'entropie de Rényi, et comme certaines invariances sont perdues, les estimations basées sur la transformation d'Emden–Fowler doivent être modifiées.

1. A family of subcritical Caffarelli–Kohn–Nirenberg interpolation inequalities

With the norms

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}, \quad \|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)},$$

let us define $\mathrm{L}^{q,\gamma}(\mathbb{R}^d)$ as the space of all measurable functions w such that $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)}$ is finite. Our functional framework is a space $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ of functions $w \in \mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)$ such that $\nabla w \in \mathrm{L}^{2,\beta}(\mathbb{R}^d)$, which is defined as the completion of the space $\mathscr{D}(\mathbb{R}^d \setminus \{0\})$ of the smooth functions on $\mathbb{R}^d$ with compact support in $\mathbb{R}^d \setminus \{0\}$, with respect to the norm given by $\|w\|^2 := (p_\star - p)\,\|w\|^2_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} + \|\nabla w\|^2_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}$.

Now consider the family of *Caffarelli–Kohn–Nirenberg interpolation inequalities* given by

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \quad \forall\, w \in \mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)\,. \tag{1}$$

Here the parameters β, γ and p are subject to the restrictions

$$d \ge 2\,, \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2} \tag{2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}\,.$$

These inequalities have been introduced, among others, by L. Caffarelli, R. Kohn and L. Nirenberg in [5]. We observe that $\vartheta = 1$ if $p = p_\star$, a case that has been dealt with in [14], and we shall focus on the sub-critical case $p < p_\star$. Throughout this paper, $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant in (1). We shall say that a function $w \in \mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ is an *extremal function* for (1) if equality holds in the inequality.

Symmetry in (1) means that the equality case is achieved by Aubin–Talenti-type functions

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d\,.$$

On the contrary, there is *symmetry breaking* if this is not the case, because the equality case is then achieved by a non-radial extremal function. It has been proved in [4] that *symmetry breaking* holds in (1) if

$$\gamma < 0 \quad \text{and} \quad \beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma\,, \tag{3}$$

where

$$\beta_{\rm FS}(\gamma) := d - 2 - \sqrt{(\gamma-d)^2 - 4\,(d-1)}\,.$$

For completeness, we will give a short proof of this result in Section 2. Our main result shows that, under Condition (2), *symmetry* holds in the complement of the set defined by (3), which means that (3) is the sharp condition for *symmetry breaking*. See Fig. 1.

Theorem 1.1. *Assume that* (2) *holds and that*

$$\beta \le \beta_{\rm FS}(\gamma) \quad \text{if} \quad \gamma < 0\,. \tag{4}$$

Then the extremal functions for (1) *are radially symmetric and, up to a scaling and a multiplication by a constant, equal to* $w_\star$.

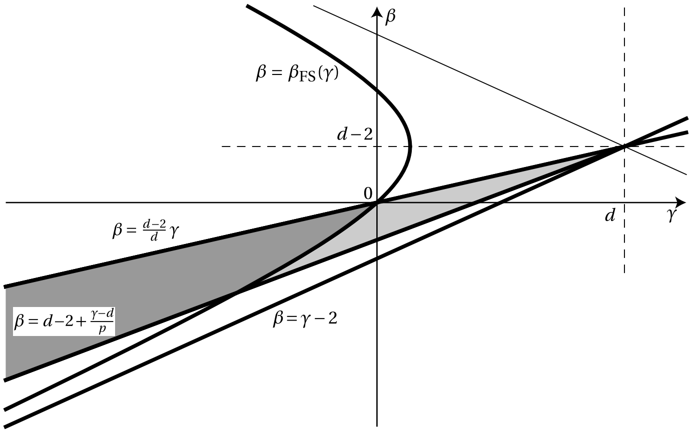

Fig. 1. In dimension $d=4$, with $p=1.2$, the grey area corresponds to the cone determined by $d-2+(\gamma-d)/p\le\beta<(d-2)\,\gamma/d$ and $\gamma\in(-\infty,d)$ in (2). The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola $(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$ or, equivalently $\beta=\beta_{\rm FS}(\gamma)$. Notice that the condition $p\le p_\star$ induces the restriction $\beta\ge d-2+(\gamma-d)/p$, so that the region of symmetry is bounded. The largest possible cone is achieved as $p\to1$ and is limited from below by the condition $\beta>\gamma-2$.

The above result is slightly stronger than just characterizing the range of (β,γ) for which equality in (1) is achieved by radial functions. Actually our method of proof allows us to analyze the symmetry properties not only of extremal functions of (1), but also of all positive solutions in $\mathrm H^p_{\beta,\gamma}(\mathbb R^d)$ of the corresponding Euler–Lagrange equations, that is, up to a multiplication by a constant and a dilation, of

$$-\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w\right)=|x|^{-\gamma}\left(w^{2p-1}-\,w^p\right)\quad\text{in}\quad\mathbb R^d\setminus\{0\}\,. \tag{5}$$

Theorem 1.2. *Assume that (2) and (4) hold. Then all positive solutions to (5) in $\mathrm H^p_{\beta,\gamma}(\mathbb R^d)$ are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star$.*

Up to a multiplication by a constant, we know that all non-trivial extremal functions for (1) are non-negative solutions to (5). Non-negative solutions to (5) are actually positive by the standard Strong Maximum principle. Theorem 1.1 is therefore a consequence of Theorem 1.2. In the particular case when $\beta=0$, the condition (2) amounts to $d\ge2$, $\gamma\in(0,2)$, $p\in\big(1,(d-\gamma)/(d-2)\big]$, and (1) can be written as

$$\|w\|_{\mathrm L^{2p,\gamma}(\mathbb R^d)}\le\mathsf C_{0,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm L^2(\mathbb R^d)}\,\|w\|^{1-\vartheta}_{\mathrm L^{p+1,\gamma}(\mathbb R^d)}\quad\forall\,w\in\mathrm H^p_{0,\gamma}(\mathbb R^d)\,.$$

In this case, we deduce from Theorem 1.1 that symmetry always holds. This is consistent with a previous result ($\beta=0$ and $\gamma>0$, close to 0) obtained in [17]. A few other cases were already known. The Caffarelli–Kohn–Nirenberg inequalities that were discussed in [14] correspond to the critical case $\theta=1$, $p=p_\star$ or, equivalently $\beta=d-2+(\gamma-d)/p$. Here by *critical* we simply mean that $\|w\|_{\mathrm L^{2p,\gamma}(\mathbb R^d)}$ scales like $\|\nabla w\|_{\mathrm L^{2,\beta}(\mathbb R^d)}$. The limit case $\beta=\gamma-2$ and $p=1$, which is an endpoint for (2), corresponds to Hardy-type inequalities: there is no extremal function, but optimality is achieved among radial functions: see [16]. The other endpoint is $\beta=(d-2)\,\gamma/d$, in which case $p_\star=d/(d-2)$. The results of Theorem 1.1 also hold in that case with $p=p_\star=d/(d-2)$, up to existence issues: according to [9], either $\gamma\ge0$, symmetry holds and there exists a symmetric extremal function, or $\gamma<0$, and then symmetry is broken, but there is no optimal function.

Inequality (1) can be rewritten as an interpolation inequality with same weights on both sides using a change of variables. Here we follow the computations in [4] (also see [14,15]). Written in spherical coordinates for a function

$$\widetilde w(r,\omega)=w(x)\,,\quad\text{with}\quad r=|x|\quad\text{and}\quad\omega=\frac{x}{|x|}\,,$$

inequality (1) becomes

$$\left(\int_0^\infty\!\!\int_{\mathbb S^{d-1}}|\widetilde w|^{2p}\,r^{d-\gamma-1}\,dr\,d\omega\right)^{\frac1{2p}}\le\mathsf C_{\beta,\gamma,p}\left(\int_0^\infty\!\!\int_{\mathbb S^{d-1}}|\nabla\widetilde w|^2\,r^{d-\beta-1}\,dr\,d\omega\right)^{\frac\vartheta2}\left(\int_0^\infty\!\!\int_{\mathbb S^{d-1}}|\widetilde w|^{p+1}\,r^{d-\gamma-1}\,dr\,d\omega\right)^{\frac{1-\vartheta}{p+1}},$$

where $|\nabla\widetilde w|^2=\left|\frac{\partial\widetilde w}{\partial r}\right|^2+\frac1{r^2}\,|\nabla_\omega\widetilde w|^2$ and $\nabla_\omega\widetilde w$ denotes the gradient of $\widetilde w$ with respect to the angular variable $\omega\in\mathbb S^{d-1}$. Next we consider the change of variables $r\mapsto s=r^\alpha$,

$$\widetilde w(r,\omega)=v(s,\omega)\quad\forall\,(r,\omega)\in\mathbb R^+\times\mathbb S^{d-1}, \tag{6}$$

where α and n are two parameters such that

$$n=\frac{d-\beta-2}{\alpha}+2=\frac{d-\gamma}{\alpha}\,.$$

Our inequality can therefore be rewritten as

$$\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|v|^{2p}\,s^{n-1}\,\mathrm{d}s\,\mathrm{d}\omega\right)^{\frac{1}{2p}}$$
$$\le \mathsf{K}_{\alpha,n,p}\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}\left(\alpha^2\left|\tfrac{\partial v}{\partial s}\right|^2+\tfrac{1}{s^2}\,|\nabla_\omega v|^2\right)s^{n-1}\,\mathrm{d}s\,\mathrm{d}\omega\right)^{\frac{\vartheta}{2}}\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|v|^{p+1}\,s^{n-1}\,\mathrm{d}s\,\mathrm{d}\omega\right)^{\frac{1-\vartheta}{p+1}},$$

with

$$\mathsf{C}_{\beta,\gamma,p}=\alpha^\zeta\,\mathsf{K}_{\alpha,n,p}\quad\text{and}\quad\zeta:=\frac{\vartheta}{2}+\frac{1-\vartheta}{p+1}-\frac{1}{2\,p}=\frac{(\beta+2-\gamma)\,(p-1)}{2\,p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}\,.$$

Using the notation

$$\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right),$$

with

$$\alpha=1+\frac{\beta-\gamma}{2}\quad\text{and}\quad n=2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

Inequality (1) is equivalent to a Gagliardo–Nirenberg type inequality corresponding to an artificial dimension n or, to be precise, to a Caffarelli–Kohn–Nirenberg inequality with weight $|x|^{n-d}$ in all terms. Notice that

$$p_\star=\frac{n}{n-2}\,.$$

Corollary 1.3. *Assume that α, n and p are such that*

$$d\ge 2\,,\quad \alpha>0\,,\quad n>d\quad\text{and}\quad p\in(1,p_\star]\,.$$

Then the inequality

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)\,,\tag{7}$$

holds with optimal constant $\mathsf{K}_{\alpha,n,p}=\alpha^{-\zeta}\,\mathsf{C}_{\beta,\gamma,p}$ as above and optimality is achieved among radial functions if and only if

$$\alpha\le\alpha_{\mathrm{FS}}\quad\text{with}\quad\alpha_{\mathrm{FS}}:=\sqrt{\frac{d-1}{n-1}}\,.\tag{8}$$

When symmetry holds, optimal functions are equal, up to a scaling and a multiplication by a constant, to

$$v_\star(x):=\left(1+|x|^2\right)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d\,.$$

We may notice that neither α_{FS} nor β_{FS} depend on p and that the curve $\alpha=\alpha_{\mathrm{FS}}$ determines the same threshold for the symmetry-breaking region as in the critical case $p=p_\star$. In the case $p=p_\star$, this curve was found by V. Felli and M. Schneider, who proved in [19] the linear instability of all radial critical points if $\alpha>\alpha_{\mathrm{FS}}$. When $p=p_\star$, symmetry holds under Condition (8) as was proved in [14]. Our goal is to extend this last result to the subcritical regime $p\in(1,p_\star)$.

The change of variables $s=r^\alpha$ is an important intermediate step, because it allows one to recast the problem as a more standard interpolation inequality in which the *dimension* n is, however, not necessarily an integer. Actually n plays the role of a dimension in view of the scaling properties of the inequalities and, with respect to this *dimension*, they are critical if $p=p_\star$ and sub-critical otherwise. The critical case $p=p_\star$ has been studied in [14] using tools of entropy methods, a critical fast diffusion flow and, in particular, a reformulation in terms of a *generalized Fisher information*. In the subcritical range, we shall replace the entropy by a *Rényi entropy power* as in [21,18], and make use of the corresponding fast diffusion flow. As in [14], the flow is used only at the heuristic level in order to produce a well-adapted test function. The core of the method is based on the Bakry–Emery computation, also known as the *carré du champ method*, which is well adapted to optimal interpolation inequalities: see for instance [2] for a general exposition of the method and [12,13] for its use in the presence of nonlinear flows. Also see [6] for earlier considerations on the Bakry–Emery method applied to nonlinear flows and related functional inequalities in unbounded domains. However, in non-compact manifolds and in the presence of weights, integrations by parts have to be justified. In the critical case, one can rely on an additional invariance to use an Emden–Fowler transformation and rewrite the problem as an autonomous equation on a cylinder, which simplifies the estimates a lot. In the subcritical regime, estimates have to be adapted, since after the Emden–Fowler transformation, the problem in the cylinder is no longer autonomous.

This paper is organized as follows. We recall the computations that characterize the linear instability of radially symmetric minimizers in Section 2. In Section 3, we expose the strategy for proving symmetry in the subcritical regime when there are no weights. Section 4 is devoted to the Bakry–Emery computation applied to Rényi entropy powers, in the presence of weights. This provides a proof of our main results, if we admit that no boundary term appears in the integrations by parts in Section 4. To prove this last result, regularity and decay estimates of positive solutions to (5) are established in Section 5, which indeed show that no boundary term has to be taken into account (see Proposition 5.1).

2. Symmetry breaking

For completeness, we summarize known results on symmetry breaking for (1). Details can be found in [4]. With the notations of Corollary 1.3, let us define the functional

$$\mathscr{J}[v] := \vartheta \log\left(\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

obtained by taking the difference of the logarithm of the two terms in (7). Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where

$$\mu_\delta(x) := \frac{1}{(1+|x|^2)^\delta}\,.$$

Since $v_\star$ as defined in Corollary 1.3 is a critical point of $\mathscr{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathsf{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\;\mathscr{J}\left[v_\star+\varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathscr{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2p}{p-1}$ and

$$\mathscr{Q}[f] = \int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}\,.$$

The following *Hardy–Poincaré inequality* has been established in [4].

Proposition 2.1. *Let $d \ge 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \ge n$. Then*

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta \ge \Lambda \int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1} \tag{9}$$

holds for any $f \in \mathrm{L}^2(\mathbb{R}^d, |x|^{n-d}\,d\mu_{\delta+1})$, with $\mathsf{D}_\alpha f \in \mathrm{L}^2(\mathbb{R}^d, |x|^{n-d}\,d\mu_\delta)$, such that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$, with an optimal constant Λ given by

$$\Lambda = \begin{cases} 2\,\alpha^2\,(2\,\delta-n) & \text{if}\quad 0<\alpha^2 \le \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}\,,\\ 2\,\alpha^2\,\delta\,\eta & \text{if}\quad \alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}\,, \end{cases}$$

where η is the unique positive solution to

$$\eta\,(\eta+n-2) = \frac{d-1}{\alpha^2}\,.$$

Moreover, Λ is achieved by a non-trivial eigenfunction corresponding to the equality in (9). If $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}$, the eigenspace is generated by $\varphi_i(s,\omega) = s^\eta\,\omega_i$, with $i = 1, 2,\ldots d$ and the eigenfunctions are not radially symmetric, while in the other case the eigenspace is generated by the radially symmetric eigenfunction $\varphi_0(s,\omega) = s^2 - \frac{n}{2\,\delta-n}$.

As a consequence, $\mathscr{Q}$ is a nonnegative quadratic form if and only if $\frac{4\,p\,\alpha^2}{p-1} \le \Lambda$. Otherwise, $\mathscr{Q}$ takes negative values, and a careful analysis shows that symmetry breaking occurs in (1) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \quad\Longleftrightarrow\quad \eta < 1\,,$$

which means

$$\frac{d-1}{\alpha^2}=\eta\,(\eta+n-2)<n-1\,,$$

and this is equivalent to $\alpha>\alpha_{\rm FS}$.

3. The strategy for proving symmetry without weights

Before going into the details of the proof, we explain the strategy for the case of the Gagliardo–Nirenberg inequalities without weights. There are several ways to compute the optimizers, and the relevant papers are [11,7,8,6,2,18] (also see additional references therein). The inequality is of the form

$$\|w\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\le \mathsf{C}_{0,0,p}\ \|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta}\ \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\vartheta}\quad\text{with}\quad 1<p<\frac{d}{d-2} \tag{10}$$

and

$$\vartheta=\frac{d\,(p-1)}{p\,\big(d+2-p\,(d-2)\big)}\,.$$

It is known through the work in [11] that the optimizers of this inequality are, up to multiplications by a constant, scalings and translations, given by

$$w_\star(x)=\Big(1+|x|^2\Big)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d\,.$$

In our perspective, the idea is to use a version of the *carré du champ* or *Bakry–Emery method* introduced in [1]: by differentiating a relevant quantity along the flow, we recover the inequality in a form that turns out to be sharp. The version of the *carré du champ* we shall use is based on the *Rényi entropy powers* whose concavity as a function of t has been studied by M. Costa in [10] in the case of linear diffusions (see [21] and references therein for more recent papers). In [23], C. Villani observed that the *carré du champ* method gives a proof of the logarithmic Sobolev inequality in the Blachman–Stam form, also known as the Weissler form: see [3,24]. G. Savaré and G. Toscani observed in [21] that the concavity also holds in the nonlinear case, which has been used in [18] to give an alternative proof of the Gagliardo–Nirenberg inequalities, that we are now going to sketch.

The first step consists in reformulating the inequality in new variables. We set

$$u=w^{2p}\,,$$

which is equivalent to $w=u^{m-1/2}$, and consider the flow given by

$$\frac{\partial u}{\partial t}=\Delta u^m\,, \tag{11}$$

where m is related to p by

$$p=\frac{1}{2m-1}\,.$$

The inequalities $1<p<\frac{d}{d-2}$ imply that

$$1-\frac{1}{d}<m<1\,. \tag{12}$$

For some positive constant $\kappa>0$, one easily finds that the so-called Barenblatt–Pattle functions

$$u_\star(t,x)=\kappa^d\,t^{-\frac{d}{d\,m-d+2}}\,w_\star^{2p}\left(\kappa\,t^{-\frac{1}{d\,m-d+2}}\,x\right)=\Big(a+b\,|x|^2\Big)^{-\frac{1}{1-m}}$$

are self-similar solutions to (11), where $a=a(t)$ and $b=b(t)$ are explicit. Thus, we see that $w_\star=u_\star^{m-1/2}$ is an optimizer for (10) for all t and it makes sense to rewrite (10) in terms of the function u. Straightforward computations show that (10) can be brought into the form

$$\left(\int_{\mathbb{R}^d}u\,dx\right)^{(\sigma+1)\,m-1}\le C\,\mathcal{E}^{\sigma-1}\,\mathcal{I}\quad\text{where}\quad\sigma=\frac{2}{d\,(1-m)}-1 \tag{13}$$

for some constant C which does not depend on u, where

$$\mathscr{E} := \int_{\mathbb{R}^d} u^m \, dx$$

is a *generalized Ralston–Newman entropy*, also known in the literature as *Tsallis entropy*, and

$$\mathscr{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx$$

is the corresponding *generalized Fisher information*. Here we have introduced the *pressure variable*

$$\mathsf{P} = \frac{m}{1-m} \, u^{m-1} .$$

The *Rényi entropy power* is defined by

$$\mathscr{F} := \mathscr{E}^\sigma$$

as in [21,18]. With the above choice of σ, $\mathscr{F}$ is an affine function of t if $u = u_\star$. For an arbitrary solution to (11), we aim at proving that it is a concave function of t and that it is affine if and only if $u = u_\star$. For further references on related issues, see [11,22]. Note that one of the motivations for choosing the variable P is that it has a particular simple form for the self-similar solutions, namely

$$\mathsf{P}_\star = \frac{m}{1-m} \left(a + b \, |x|^2 \right) .$$

Differentiating $\mathscr{E}$ along the flow (11) yields

$$\mathscr{E}' = (1-m) \, \mathscr{I} ,$$

so that

$$\mathscr{F}' = \sigma \, (1-m) \, \mathscr{G} \quad \text{with} \quad \mathscr{G} := \mathscr{E}^{\sigma-1} \mathscr{I} .$$

More complicated is the derivative for the Fisher information:

$$\mathscr{I}' = -2 \int_{\mathbb{R}^d} u^m \left[\operatorname{Tr} \left(\left(\operatorname{Hess} \mathsf{P} - \tfrac{1}{d} \, \Delta \mathsf{P} \, \operatorname{Id} \right)^2 \right) + \left(m - 1 + \tfrac{1}{d} \right) (\Delta \mathsf{P})^2 \right] dx .$$

Here $\operatorname{Hess} \mathsf{P}$ and Id are respectively the Hessian of P and the $(d \times d)$ identity matrix. The computation can be found in [18]. Next we compute the second derivative of the *Rényi entropy power* $\mathscr{F}$ with respect to t:

$$\frac{(\mathscr{F})''}{\sigma \, \mathscr{E}^\sigma} = (\sigma - 1) \, \frac{\mathscr{E}'^{\,2}}{\mathscr{E}^2} + \frac{\mathscr{E}''}{\mathscr{E}} = (\sigma - 1) \, (1-m)^2 \, \frac{\mathscr{I}^2}{\mathscr{E}^2} + (1-m) \, \frac{\mathscr{I}'}{\mathscr{E}} =: (1-m) \, \mathscr{H} .$$

With $\sigma = \frac{2}{d} \, \frac{1}{1-m} - 1$, we obtain

$$\mathscr{H} = -2 \left\langle \operatorname{Tr} \left(\left(\operatorname{Hess} \mathsf{P} - \tfrac{1}{d} \, \Delta \mathsf{P} \, \operatorname{Id} \right)^2 \right) \right\rangle + (1-m) \, (1-\sigma) \left\langle (\Delta \mathsf{P} - \langle \Delta \mathsf{P} \rangle)^2 \right\rangle , \tag{14}$$

where we have used the notation

$$\langle A \rangle := \frac{\int_{\mathbb{R}^d} u^m \, A \, dx}{\int_{\mathbb{R}^d} u^m \, dx} .$$

Note that by (12), we have that $\sigma > 1$ and hence we find that $\mathscr{F}'' = (\mathscr{E}^\sigma)'' \le 0$, which also means that $\mathscr{G} = \mathscr{E}^{\sigma-1} \mathscr{I}$ is a non-increasing function. In fact it is strictly decreasing unless P is a polynomial function of order two in x and it is easy to see that the expression (14) vanishes precisely when P is of the form $a + b \, |x - x_0|^2$, where a, $b \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ are constants (but a and b may still depend on t).

Thus, while the left side of (13) stays constant along the flow, the right side decreases. In [18] it was shown that the right side decreases towards the value given by the self-similar solutions $u_\star$ and hence proves (10) in the sharp form. In our work we pursue a different tactic. The variational equation for the optimizers of (10) is given by

$$-\Delta w = a \, w^{2p-1} - b \, w^p .$$

A straightforward computation shows that this can be written in the form

$$2 \, m \, u^{m-2} \operatorname{div}\big(u \, \nabla \mathsf{P} \big) + |\nabla \mathsf{P}|^2 + c_1 \, u^{m-1} = c_2$$

for some constants c_1, c_2 whose precise values are explicit. This equation can also be interpreted as the variational equation for the sharp constant in (13). Hence, multiplying the above equation by Δu^m and integrating yields

$$\int_{\mathbb{R}^d} \left[2\,m\,u^{m-2}\,\mathrm{div}\big(u\,\nabla \mathsf{P}\big)+|\nabla \mathsf{P}|^2\right]\Delta u^m\,\mathrm{d}x+c_1\int_{\mathbb{R}^d} u^{m-1}\,\Delta u^m\,\mathrm{d}x=c_2\int_{\mathbb{R}^d}\Delta u^m\,\mathrm{d}x=0\,.$$

We recover the fact that, in the flow picture, $\mathscr{H}$ is, up to a positive factor, the derivative of $\mathscr{G}$ and hence vanishes. From the observations made above, we conclude that P must be a polynomial function of order two in x. In this fashion, one obtains more than just the optimizers, namely a classification of all positive solutions to the variational equation. The main technical problem with this method is the justification of the integrations by parts, which in the case at hand, without any weight, does not offer great difficulties: see, for instance, [6]. This strategy can also be used to treat the problem with weights, which will be explained next. Dealing with weights, however, requires some special care, as we shall see.

4. The Bakry–Emery computation and Rényi entropy powers in the weighted case

Let us adapt the above strategy to the case where there are weights in all integrals entering into the inequality, that is, let us deal with inequality (7) instead of inequality (10). In order to define a new, well-adapted fast diffusion flow, we introduce the diffusion operator $\mathscr{L}_\alpha := -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha$, which is given in spherical coordinates by

$$\mathscr{L}_\alpha u=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u\,,$$

where Δ_ω denotes the Laplace–Betrami operator acting on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ of the angular variables, and $'$ denotes here the derivative with respect to s. Consider the fast diffusion equation

$$\frac{\partial u}{\partial t}=\mathscr{L}_\alpha u^m \tag{15}$$

in the subcritical range $1-\frac{1}{n}<m=1-\frac{1}{\nu}<1$. The exponents m in (15) and p in (7) are related as in Section 3 by

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}$$

and ν is defined by

$$\nu:=\frac{1}{1-m}\,.$$

We consider the *Fisher information* defined as

$$\mathscr{I}[\mathsf{P}]:=\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,\mathrm{d}\mu\quad\text{with}\quad \mathsf{P}=\frac{m}{1-m}\,u^{m-1}\quad\text{and}\quad \mathrm{d}\mu=s^{n-1}\,\mathrm{d}s\,\mathrm{d}\omega=s^{n-d}\,\mathrm{d}x\,.$$

Here P is the *pressure variable*. Our goal is to prove that P takes the form $a+b\,s^2$, as in Section 3. It is useful to observe that (15) can be rewritten as

$$\frac{\partial u}{\partial t}=\mathsf{D}_\alpha^*\,(u\,\mathsf{D}_\alpha \mathsf{P})$$

and, in order to compute $\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}$, we will also use the fact that P solves

$$\frac{\partial \mathsf{P}}{\partial t}=(1-m)\,\mathsf{P}\,\mathscr{L}_\alpha \mathsf{P}-\,|\mathsf{D}_\alpha \mathsf{P}|^2\,. \tag{16}$$

4.1. First step: computation of $\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}$

Let us define

$$\mathscr{K}[\mathsf{P}]:=\mathscr{A}[\mathsf{P}]-(1-m)\,(\mathscr{L}_\alpha \mathsf{P})^2\quad\text{where}\quad\mathscr{A}[\mathsf{P}]:=\frac{1}{2}\,\mathscr{L}_\alpha\,|\mathsf{D}_\alpha \mathsf{P}|^2-\mathsf{D}_\alpha \mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha \mathsf{P}$$

and, on the boundary of the centered ball B_s of radius s, the boundary term

$$\mathsf{b}(s) := \int_{\partial B_s} \left(\tfrac{\partial}{\partial s} \left(\mathsf{P}^{\frac{m}{m-1}} \, |\mathsf{D}_\alpha \mathsf{P}|^2 \right) - 2\,(1-m)\, \mathsf{P}^{\frac{m}{m-1}}\, \mathsf{P}'\, \mathscr{L}_\alpha \mathsf{P} \right) d\varsigma$$

$$= s^{n-1} \left(\int_{\mathbb{S}^{d-1}} \left(\tfrac{\partial}{\partial s} \left(\mathsf{P}^{\frac{m}{m-1}} \, |\mathsf{D}_\alpha \mathsf{P}|^2 \right) - 2\,(1-m)\, \mathsf{P}^{\frac{m}{m-1}}\, \mathsf{P}'\, \mathscr{L}_\alpha \mathsf{P} \right) d\omega \right)(s)\,, \quad (17)$$

where by $d\varsigma = s^{n-1}\, d\omega$ we denote the standard Hausdorff measure on ∂B_s.

Lemma 4.1. *If u solves (15) and if*

$$\lim_{s\to 0_+} \mathsf{b}(s) = \lim_{S\to +\infty} \mathsf{b}(S) = 0\,, \quad (18)$$

then,

$$\frac{d}{dt} \mathscr{I}[\mathsf{P}] = -2 \int_{\mathbb{R}^d} \mathscr{K}[\mathsf{P}]\, u^m\, d\mu\,. \quad (19)$$

Proof. For $0 < s < S < +\infty$, let us consider the set $A_{(s,S)} := \left\{ x \in \mathbb{R}^d \,:\, s < |x| < S \right\}$, so that $\partial A_{(s,S)} = \partial B_s \cup \partial B_S$. Using (15) and (16), we can compute

$$\begin{aligned}
&\frac{d}{dt} \int_{A_{(s,S)}} u\, |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu \\
&= \int_{A_{(s,S)}} \frac{\partial u}{\partial t}\, |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu + 2 \int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \frac{\partial \mathsf{P}}{\partial t}\, d\mu \\
&= \int_{A_{(s,S)}} \mathscr{L}_\alpha (u^m)\, |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu + 2 \int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \Big((1-m)\, \mathsf{P}\, \mathscr{L}_\alpha \mathsf{P} - |\mathsf{D}_\alpha \mathsf{P}|^2 \Big)\, d\mu \\
&= \int_{A_{(s,S)}} u^m\, \mathscr{L}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu + 2\,(1-m) \int_{A_{(s,S)}} u\, \mathsf{P}\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \mathscr{L}_\alpha \mathsf{P}\, d\mu \\
&\quad + 2\,(1-m) \int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \mathsf{P}\, \mathscr{L}_\alpha \mathsf{P}\, d\mu - 2 \int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu \\
&\quad + \alpha^2 \int_{\partial B_S} \Big((u^m)'\, |\mathsf{D}_\alpha \mathsf{P}|^2 - u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2) \Big)\, d\varsigma - \alpha^2 \int_{\partial B_s} \Big((u^m)'\, |\mathsf{D}_\alpha \mathsf{P}|^2 - u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2) \Big)\, d\varsigma \\
&= -\int_{A_{(s,S)}} u^m\, \mathscr{L}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu + 2\,(1-m) \int_{A_{(s,S)}} u\, \mathsf{P}\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \mathscr{L}_\alpha \mathsf{P}\, d\mu + 2\,(1-m) \int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha \mathsf{P}\, \mathscr{L}_\alpha \mathsf{P}\, d\mu \\
&\quad + \alpha^2 \int_{\partial B_S} \Big((u^m)'\, |\mathsf{D}_\alpha \mathsf{P}|^2 + u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2) \Big)\, d\varsigma - \alpha^2 \int_{\partial B_s} \Big((u^m)'\, |\mathsf{D}_\alpha \mathsf{P}|^2 + u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2) \Big)\, d\varsigma\,,
\end{aligned}$$

where the last line is given by an integration by parts, upon exploiting the identity $u\, \mathsf{D}_\alpha \mathsf{P} = -\mathsf{D}_\alpha (u^m)$:

$$\begin{aligned}
\int_{A_{(s,S)}} u\, \mathsf{D}_\alpha \mathsf{P} \cdot \mathsf{D}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu &= -\int_{A_{(s,S)}} \mathsf{D}_\alpha (u^m) \cdot \mathsf{D}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu \\
&= \int_{A_{(s,S)}} u^m\, \mathscr{L}_\alpha |\mathsf{D}_\alpha \mathsf{P}|^2\, d\mu - \alpha^2 \int_{\partial B_S} u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2)\, d\varsigma + \alpha^2 \int_{\partial B_s} u^m\, \tfrac{\partial}{\partial s} (|\mathsf{D}_\alpha \mathsf{P}|^2)\, d\varsigma\,.
\end{aligned}$$

1) Using the definition of $\mathscr{A}[\mathsf{P}]$, we get that

$$-\int_{A_{(s,S)}} u^m\,\mathscr{L}_\alpha|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu=-2\int_{A_{(s,S)}} u^m\,\mathscr{A}[\mathsf{P}]\,d\mu-2\int_{A_{(s,S)}} u^m\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha\mathsf{P}\,d\mu\,. \tag{20}$$

2) Taking advantage again of $u\,\mathsf{D}_\alpha\mathsf{P}=-\,\mathsf{D}_\alpha(u^m)$, an integration by parts gives

$$\begin{aligned}\int_{A_{(s,S)}} u\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathsf{P}\,\mathscr{L}_\alpha\mathsf{P}\,d\mu&=-\int_{A_{(s,S)}}\mathsf{D}_\alpha(u^m)\cdot\mathsf{D}_\alpha\mathsf{P}\,\mathscr{L}_\alpha\mathsf{P}\,d\mu\\&=\int_{A_{(s,S)}} u^m\,(\mathscr{L}_\alpha\mathsf{P})^2\,d\mu+\int_{A_{(s,S)}} u^m\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha\mathsf{P}\,d\mu\\&\quad-\alpha^2\int_{\partial B_S} u^m\,\mathsf{P}'\,\mathscr{L}_\alpha\mathsf{P}\,d\varsigma+\alpha^2\int_{\partial B_s} u^m\,\mathsf{P}'\,\mathscr{L}_\alpha\mathsf{P}\,d\varsigma\end{aligned}$$

and, with $u\,\mathsf{P}=\frac{m}{1-m}\,u^m$, we find that

$$\begin{aligned}&2\,(1-m)\int_{A_{(s,S)}} u\,\mathsf{P}\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha\mathsf{P}\,d\mu+2\,(1-m)\int_{A_{(s,S)}} u\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathsf{P}\,\mathscr{L}_\alpha\mathsf{P}\,d\mu\\&=2\,(1-m)\int_{A_{(s,S)}} u^m\,(\mathscr{L}_\alpha\mathsf{P})^2\,d\mu+2\int_{A_{(s,S)}} u^m\,\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha\mathsf{P}\,d\mu\\&\quad-2\,(1-m)\,\alpha^2\int_{\partial B_S} u^m\,\mathsf{P}'\,\mathscr{L}_\alpha\mathsf{P}\,d\varsigma+2\,(1-m)\,\alpha^2\int_{\partial B_s} u^m\,\mathsf{P}'\,\mathscr{L}_\alpha\mathsf{P}\,d\varsigma\,.\end{aligned} \tag{21}$$

Summing (20) and (21), using (17) and passing to the limits as $s\to 0_+$, $S\to+\infty$, establishes (19). □

4.2. Second step: two remarkable identities

Let us define

$$\mathsf{k}[\mathsf{P}]:=\tfrac12\,\Delta_\omega\,|\nabla_\omega\mathsf{P}|^2-\nabla_\omega\mathsf{P}\cdot\nabla_\omega\Delta_\omega\,\mathsf{P}-\tfrac{1}{n-1}\,(\Delta_\omega\,\mathsf{P})^2-(n-2)\,\alpha^2\,|\nabla_\omega\mathsf{P}|^2$$

and

$$\mathscr{R}[\mathsf{P}]:=\mathscr{K}[\mathsf{P}]-\left(\frac1n-(1-m)\right)(\mathscr{L}_\alpha\mathsf{P})^2\,.$$

We observe that

$$\mathscr{R}[\mathsf{P}]=\frac12\,\mathscr{L}_\alpha\,|\mathsf{D}_\alpha\mathsf{P}|^2-\mathsf{D}_\alpha\mathsf{P}\cdot\mathsf{D}_\alpha\mathscr{L}_\alpha\mathsf{P}-\frac1n\,(\mathscr{L}_\alpha\mathsf{P})^2$$

is independent of m. We recall the result of [14, Lemma 5.1] and give its proof for completeness.

Lemma 4.2. *Let $d\in\mathbb{N}$, $n\in\mathbb{R}$ such that $n>d\ge 2$, and consider a function $\mathsf{P}\in C^3(\mathbb{R}^d\setminus\{0\})$. Then,*

$$\mathscr{R}[\mathsf{P}]=\alpha^4\left(1-\frac1n\right)\left[\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right]^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2+\frac{\mathsf{k}[\mathsf{P}]}{s^4}\,.$$

Proof. By definition of $\mathscr{R}[\mathsf{P}]$, we have

$$\begin{aligned}\mathscr{R}[\mathsf{P}]&=\frac{\alpha^2}{2}\left[\alpha^2\,\mathsf{P}'^2+\frac{|\nabla_\omega\mathsf{P}|^2}{s^2}\right]''+\frac{\alpha^2}{2}\,\frac{n-1}{s}\left[\alpha^2\,\mathsf{P}'^2+\frac{|\nabla_\omega\mathsf{P}|^2}{s^2}\right]'+\frac{1}{2\,s^2}\,\Delta_\omega\left[\alpha^2\,\mathsf{P}'^2+\frac{|\nabla_\omega\mathsf{P}|^2}{s^2}\right]\\&\quad-\alpha^2\,\mathsf{P}'\left(\alpha^2\,\mathsf{P}''+\alpha^2\,\frac{n-1}{s}\,\mathsf{P}'+\frac{\Delta_\omega\,\mathsf{P}}{s^2}\right)'-\frac{1}{s^2}\nabla_\omega\mathsf{P}\cdot\nabla_\omega\left(\alpha^2\,\mathsf{P}''+\alpha^2\,\frac{n-1}{s}\,\mathsf{P}'+\frac{\Delta_\omega\,\mathsf{P}}{s^2}\right)\\&\quad-\frac1n\left(\alpha^2\,\mathsf{P}''+\alpha^2\,\frac{n-1}{s}\,\mathsf{P}'+\frac{\Delta_\omega\,\mathsf{P}}{s^2}\right)^2,\end{aligned}$$

which can be expanded as

$$\mathscr{R}[\mathsf{P}] = \frac{\alpha^2}{2}\left[2\,\alpha^2\,\mathsf{P}''^{\,2} + 2\,\alpha^2\,\mathsf{P}'\,\mathsf{P}''' + 2\,\frac{|\nabla_\omega \mathsf{P}'|^2 + \nabla_\omega \mathsf{P}\cdot\nabla_\omega \mathsf{P}''}{s^2} - 8\,\frac{\nabla_\omega \mathsf{P}\cdot\nabla_\omega \mathsf{P}'}{s^3} + 6\,\frac{|\nabla_\omega \mathsf{P}|^2}{s^4}\right]$$
$$+\alpha^2\,\frac{n-1}{s}\left[\alpha^2\,\mathsf{P}'\,\mathsf{P}'' + \frac{\nabla_\omega \mathsf{P}\cdot\nabla_\omega \mathsf{P}'}{s^2} - \frac{|\nabla_\omega \mathsf{P}|^2}{s^3}\right] + \frac{1}{s^2}\left[\alpha^2\,\mathsf{P}'\,\Delta_\omega\,\mathsf{P}' + \alpha^2\,|\nabla_\omega \mathsf{P}'|^2 + \frac{\Delta_\omega\,|\nabla_\omega \mathsf{P}|^2}{2\,s^2}\right]$$
$$-\alpha^2\,\mathsf{P}'\left(\alpha^2\,\mathsf{P}''' + \alpha^2\,\frac{n-1}{s}\,\mathsf{P}'' - \alpha^2\,\frac{n-1}{s^2}\mathsf{P}' - 2\,\frac{\Delta_\omega\,\mathsf{P}}{s^3} + \frac{\Delta_\omega\,\mathsf{P}'}{s^2}\right)$$
$$-\frac{1}{s^2}\left(\alpha^2\,\nabla_\omega \mathsf{P}\cdot\nabla_\omega \mathsf{P}'' + \alpha^2\,\frac{n-1}{s}\nabla_\omega \mathsf{P}\cdot\nabla_\omega \mathsf{P}' + \frac{\nabla_\omega \mathsf{P}\cdot\nabla_\omega\Delta_\omega\,\mathsf{P}}{s^2}\right)$$
$$-\frac{1}{n}\left[\alpha^4\,\mathsf{P}''^{\,2} + \alpha^4\,\frac{(n-1)^2}{s^2}\,\mathsf{P}'^{\,2} + \frac{(\Delta_\omega\,\mathsf{P})^2}{s^4} + 2\,\alpha^4\,\frac{n-1}{s}\,\mathsf{P}'\,\mathsf{P}'' + 2\,\alpha^2\,\frac{\mathsf{P}''\,\Delta_\omega\,\mathsf{P}}{s^2} + 2\,\alpha^2\,\frac{n-1}{s^3}\mathsf{P}'\,\Delta_\omega\,\mathsf{P}\right].$$

Collecting terms proves the result. □

Now let us study the quantity $\mathsf{k}[\mathsf{P}]$ which appears in the statement of Lemma 4.2. The following computations are adapted from [12] and [14, Section 5]. For completeness, we give a simplified proof in the special case of the sphere $(\mathbb{S}^{d-1}, g)$ considered as a Riemannian manifold with standard metric g. We denote by $\mathrm{H}f$ the *Hessian* of f, which is seen as a $(d-1)\times(d-1)$ matrix, identify its trace with the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ and use the notation $\|\mathrm{A}\|^2 := \mathrm{A} : \mathrm{A}$ for the sum of the squares of the coefficients of the matrix A. It is convenient to define the *trace free Hessian*, the tensor $\mathrm{Z}f$ and its trace free counterpart respectively by

$$\mathrm{L}f := \mathrm{H}f - \frac{1}{d-1}\,(\Delta_\omega f)\,g\,,\quad \mathrm{Z}f := \frac{\nabla_\omega f\otimes\nabla_\omega f}{f}\quad\text{and}\quad \mathrm{M}f := \mathrm{Z}f - \frac{1}{d-1}\,\frac{|\nabla_\omega f|^2}{f}\,g$$

whenever $f \neq 0$. Elementary computations show that

$$\|\mathrm{L}f\|^2 = \|\mathrm{H}f\|^2 - \frac{1}{d-1}\,(\Delta_\omega f)^2\quad\text{and}\quad \|\mathrm{M}f\|^2 = \|\mathrm{Z}f\|^2 - \frac{1}{d-1}\,\frac{|\nabla_\omega f|^4}{f^2} = \frac{d-2}{d-1}\,\frac{|\nabla_\omega f|^4}{f^2}\,. \tag{22}$$

The Bochner–Lichnerowicz–Weitzenböck formula on $\mathbb{S}^{d-1}$ takes the simple form

$$\tfrac{1}{2}\,\Delta_\omega\,(|\nabla_\omega f|^2) = \|\mathrm{H}f\|^2 + \nabla_\omega(\Delta_\omega f)\cdot\nabla_\omega f + (d-2)\,|\nabla_\omega f|^2 \tag{23}$$

where the last term, *i.e.* $\mathrm{Ric}(\nabla_\omega f, \nabla_\omega f) = (d-2)\,|\nabla_\omega f|^2$, accounts for the Ricci curvature tensor contracted with $\nabla_\omega f\otimes\nabla_\omega f$.

We recall that $\alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$ and $\nu = 1/(1-m)$. Let us introduce the notations

$$\delta := \frac{1}{d-1} - \frac{1}{n-1}$$

and

$$\mathscr{B}[\mathsf{P}] := \int_{\mathbb{S}^{d-1}} \left(\tfrac{1}{2}\,\Delta_\omega(|\nabla_\omega \mathsf{P}|^2) - \nabla_\omega(\Delta_\omega \mathsf{P})\cdot\nabla_\omega \mathsf{P} - \tfrac{1}{n-1}\,(\Delta_\omega \mathsf{P})^2\right)\mathsf{P}^{1-\nu}\,d\omega\,,$$

so that

$$\int_{\mathbb{S}^{d-1}} \mathsf{k}[\mathsf{P}]\,\mathsf{P}^{1-\nu}\,d\omega = \mathscr{B}[\mathsf{P}] - (n-2)\,\alpha^2\int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{P}|^2\,\mathsf{P}^{1-\nu}\,d\omega\,.$$

Lemma 4.3. *Assume that $d \geq 2$ and $1/(1-m) = \nu > n > d$. There exists a positive constant $c(n,m,d)$ such that, for any positive function $\mathsf{P}\in C^3(\mathbb{S}^{d-1})$,*

$$\int_{\mathbb{S}^{d-1}} \mathsf{k}[\mathsf{P}]\,\mathsf{P}^{1-\nu}\,d\omega \geq (n-2)\left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right)\int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{P}|^2\,\mathsf{P}^{1-\nu}\,d\omega + c(n,m,d)\int_{\mathbb{S}^{d-1}} \frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\,\mathsf{P}^{1-\nu}\,d\omega\,.$$

Proof. If $d = 2$, we identify $\mathbb{S}^1$ with $[0, 2\pi) \ni \theta$ and denote by P_θ and $\mathsf{P}_{\theta\theta}$ the first and second derivatives of P with respect to θ. As in [14, Lemma 5.3], a direct computation shows that

$$\mathsf{k}[\mathsf{P}] = \frac{n-2}{n-1}\,|\mathsf{P}_{\theta\theta}|^2 - (n-2)\,\alpha^2\,|\mathsf{P}_\theta|^2 = (n-2)\left(\alpha_{\mathrm{FS}}^2\,|\mathsf{P}_{\theta\theta}|^2 - \alpha^2\,|\mathsf{P}_\theta|^2\right).$$

By the Poincaré inequality, we have

$$\int_{\mathbb{S}^1} \left|\frac{\partial}{\partial\theta}\left(\mathsf{P}^{\frac{1-\nu}{2}}\,\mathsf{P}_\theta\right)\right|^2 d\theta \ge \int_{\mathbb{S}^1} \left|\mathsf{P}^{\frac{1-\nu}{2}}\,\mathsf{P}_\theta\right|^2 d\theta\,.$$

On the other hand, an integration by parts shows that

$$\int_{\mathbb{S}^1} \mathsf{P}_{\theta\theta}\,\frac{|\mathsf{P}_\theta|^2}{\mathsf{P}}\,\mathsf{P}^{1-\nu}\,d\theta = \frac{1}{3}\int_{\mathbb{S}^1}\frac{\partial}{\partial\theta}\left(|\mathsf{P}_\theta|^2\,\mathsf{P}_\theta\right)\mathsf{P}^{-\nu}\,d\theta = \frac{\nu}{3}\int_{\mathbb{S}^1}\frac{|\mathsf{P}_\theta|^4}{\mathsf{P}^2}\,\mathsf{P}^{1-\nu}\,d\theta$$

and, as a consequence, by expanding the square, we obtain

$$\int_{\mathbb{S}^1} \left|\frac{\partial}{\partial\theta}\left(\mathsf{P}^{\frac{1-\nu}{2}}\,\mathsf{P}_\theta\right)\right|^2 d\theta = \int_{\mathbb{S}^1}\left|\mathsf{P}_{\theta\theta}+\frac{1-\nu}{2}\,\frac{|\mathsf{P}_\theta|^2}{\mathsf{P}}\right|^2\mathsf{P}^{1-\nu}\,d\theta = \int_{\mathbb{S}^1}|\mathsf{P}_{\theta\theta}|^2\,\mathsf{P}^{1-\nu}\,d\theta - \frac{(\nu-1)\,(\nu+3)}{12}\int_{\mathbb{S}^1}\frac{|\mathsf{P}_\theta|^4}{\mathsf{P}^2}\,\mathsf{P}^{1-\nu}\,d\theta\,.$$

The result follows with $c(n,m,2)=\frac{n-2}{n-1}\,\frac{1}{12}\,(\nu-1)\,(\nu+3)=\frac{n-2}{n-1}\,\frac{m\,(4-3\,m)}{12\,(1-m)^2}$ from

$$\int_{\mathbb{S}^1}|\mathsf{P}_{\theta\theta}|^2\,\mathsf{P}^{1-\nu}\,d\theta \ge \int_{\mathbb{S}^1}|\mathsf{P}_\theta|^2\,\mathsf{P}^{1-\nu}\,d\theta + \frac{(\nu-1)\,(\nu+3)}{12}\int_{\mathbb{S}^1}\frac{|\mathsf{P}_\theta|^4}{\mathsf{P}^2}\,\mathsf{P}^{1-\nu}\,d\theta\,.$$

Assume next that $d\ge 3$. We follow the method of [14, Lemma 5.2]. Applying (23) with $f=\mathsf{P}$ and multiplying by $\mathsf{P}^{1-\nu}$ yields, after an integration on $\mathbb{S}^{d-1}$, that $\mathscr{B}[\mathsf{P}]$ can also be written as

$$\mathscr{B}[\mathsf{P}] = \int_{\mathbb{S}^{d-1}}\left(\|\mathrm{H}\mathsf{P}\|^2+(d-2)\,|\nabla_\omega\mathsf{P}|^2-\tfrac{1}{n-1}\,(\Delta_\omega\mathsf{P})^2\right)\mathsf{P}^{1-\nu}\,d\omega\,.$$

We recall that $n>d\ge 3$ and set $\mathsf{P}=f^\beta$ with $\beta=\frac{2}{3-\nu}$. A straightforward computation shows that $\mathrm{H}f^\beta=\beta\,f^{\beta-1}\left(\mathrm{H}f+(\beta-1)\,\mathrm{Z}f\right)$ and hence

$$\mathscr{B}[\mathsf{P}] = \beta^2\int_{\mathbb{S}^{d-1}}\left(\|\mathrm{H}f+(\beta-1)\,\mathrm{Z}f\|^2+(d-2)\,|\nabla_\omega f|^2-\tfrac{1}{n-1}\left(\mathrm{Tr}\,(\mathrm{H}f+(\beta-1)\,\mathrm{Z}f)\right)^2\right)d\omega$$

$$= \beta^2\int_{\mathbb{S}^{d-1}}\left(\|\mathrm{L}f+(\beta-1)\,\mathrm{M}f\|^2+(d-2)\,|\nabla_\omega f|^2+\delta\left(\mathrm{Tr}\,(\mathrm{H}f+(\beta-1)\,\mathrm{Z}f)\right)^2\right)d\omega\,.$$

Using (22), we deduce from

$$\int_{\mathbb{S}^{d-1}}\Delta_\omega f\,\frac{|\nabla_\omega f|^2}{f}\,d\omega = \int_{\mathbb{S}^{d-1}}\frac{|\nabla_\omega f|^4}{f^2}\,d\omega - 2\int_{\mathbb{S}^{d-1}}\mathrm{H}f:\mathrm{Z}f\,d\omega$$

$$= \frac{d-1}{d-2}\int_{\mathbb{S}^{d-1}}\|\mathrm{M}f\|^2\,d\omega - 2\int_{\mathbb{S}^{d-1}}\mathrm{L}f:\mathrm{Z}f\,d\omega - \frac{2}{d-1}\int_{\mathbb{S}^{d-1}}\Delta_\omega f\,\frac{|\nabla_\omega f|^2}{f}\,d\omega$$

that

$$\int_{\mathbb{S}^{d-1}}\Delta_\omega f\,\frac{|\nabla_\omega f|^2}{f}\,d\omega = \frac{d-1}{d+1}\left[\int_{\mathbb{S}^{d-1}}\frac{d-1}{d-2}\,\|\mathrm{M}f\|^2\,d\omega - 2\int_{\mathbb{S}^{d-1}}\mathrm{L}f:\mathrm{Z}f\,d\omega\right]$$

$$= \frac{d-1}{d+1}\left[\int_{\mathbb{S}^{d-1}}\frac{d-1}{d-2}\,\|\mathrm{M}f\|^2\,d\omega - 2\int_{\mathbb{S}^{d-1}}\mathrm{L}f:\mathrm{M}f\,d\omega\right]$$

on the one hand, and from (23) integrated on $\mathbb{S}^{d-1}$ that

$$\int_{\mathbb{S}^{d-1}}(\Delta_\omega f)^2\,d\omega = \frac{d-1}{d-2}\int_{\mathbb{S}^{d-1}}\|\mathrm{L}f\|^2\,d\omega + (d-1)\int_{\mathbb{S}^{d-1}}|\nabla_\omega f|^2\,d\omega$$

on the other hand. Hence we find that

$$\int_{\mathbb{S}^{d-1}} \big(\mathrm{Tr}\,(\mathrm{H} f+(\beta-1)\,\mathrm{Z} f)\big)^2\,d\omega=\int_{\mathbb{S}^{d-1}} \left((\Delta_\omega f)^2+2\,(\beta-1)\,\Delta_\omega f\,\frac{|\nabla_\omega f|^2}{f}+(\beta-1)^2\,\frac{|\nabla_\omega f|^4}{f^2}\right)d\omega$$

$$=\frac{d-1}{d-2}\int_{\mathbb{S}^{d-1}} \|\mathrm{L} f\|^2\,d\omega+(d-1)\int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2\,d\omega$$

$$+2\,(\beta-1)\,\frac{d-1}{d+1}\left[\int_{\mathbb{S}^{d-1}} \frac{d-1}{d-2}\,\|\mathrm{M} f\|^2\,d\omega-2\int_{\mathbb{S}^{d-1}} \mathrm{L} f:\mathrm{M} f\,d\omega\right]$$

$$+(\beta-1)^2\,\frac{d-1}{d-2}\int_{\mathbb{S}^{d-1}} \|\mathrm{M} f\|^2\,d\omega\,.$$

Altogether, we obtain

$$\mathscr{B}[\mathsf{P}]=\beta^2\int_{\mathbb{S}^{d-1}} \left(\mathsf{a}\,\|\mathrm{L} f\|^2+2\,\mathsf{b}\,\mathrm{L} f:\mathrm{M} f+\mathsf{c}\,\|\mathrm{M} f\|^2\right)d\omega+\beta^2\left(d-2+\delta\,(d-1)\right)\int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2\,d\omega,$$

where

$$\mathsf{a}=1+\delta\,\frac{d-1}{d-2}\,,\quad \mathsf{b}=(\beta-1)\left(1-2\,\delta\,\frac{d-1}{d+1}\right)\quad\text{and}\quad \mathsf{c}=(\beta-1)^2\left(1+\delta\,\frac{d-1}{d-2}\right)+2\,(\beta-1)\,\frac{\delta\,(d-1)^2}{(d+1)\,(d-2)}\,.$$

A tedious but elementary computation shows that

$$\mathscr{B}[\mathsf{P}]=\mathsf{a}\,\beta^2\int_{\mathbb{S}^{d-1}} \left\|\mathrm{L} f+\tfrac{\mathsf{b}}{\mathsf{a}}\,\mathrm{M} f\right\|^2\,d\omega+\left(\mathsf{c}-\tfrac{\mathsf{b}^2}{\mathsf{a}}\right)\beta^2\int_{\mathbb{S}^{d-1}} \|\mathrm{M}\,f\|^2\,d\omega+\beta^2\,(n-2)\,\alpha_{\rm FS}^2\int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2\,d\omega$$

can be written in terms of P as

$$\mathscr{B}[\mathsf{P}]=\int_{\mathbb{S}^{d-1}} \mathrm{Q}[\mathsf{P}]\,\mathsf{P}^{1-\nu}\,d\omega+(n-2)\,\alpha_{\rm FS}^2\int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{P}|^2\,\mathsf{P}^{1-\nu}\,d\omega,$$

where

$$\mathrm{Q}[\mathsf{P}]:=\alpha_{\rm FS}^2\,\frac{n-2}{d-2}\left\|\mathrm{L}\mathsf{P}+\tfrac{3\,(\nu-1)\,(n-d)}{(d+1)\,(n-2)\,(\nu-3)}\,\mathrm{M}\mathsf{P}\right\|^2+\tfrac{(d-1)\,(\nu-1)\,(n-d)\,[((4\,d-5)\,n+d-8)\,\nu+9\,(n-d))]}{(d-2)\,(d+1)^2\,(\nu-3)^2\,(n-2)\,(n-1)}\,\|\mathrm{M}\mathsf{P}\|^2$$

is positive definite. This concludes the proof in the case $d\ge 3$ with $c(n,m,d)=\frac{m\,(n-d)\,[4\,(d+1)\,(n-2)-9\,m\,(n-d)]}{(d+1)^2\,(3\,m-2)^2\,(n-2)\,(n-1)}$. □

Let us recall that

$$\mathscr{K}[\mathsf{P}]=\mathscr{R}[\mathsf{P}]+\left(\frac{1}{n}-(1-m)\right)(\mathscr{L}_\alpha \mathsf{P})^2\,.$$

We can collect the two results of Lemmas 4.2 and 4.3 as follows.

Corollary 4.4. *Let $d\in\mathbb{N}$, $n\in\mathbb{R}$ be such that $n>d\ge 2$, and consider a positive function $\mathsf{P}\in C^3(\mathbb{R}^d\setminus\{0\})$. If u is related to P by $\mathsf{P}=\frac{m}{1-m}\,u^{m-1}$ for some $m\in(1-\frac{1}{n},1)$, then there exists a positive constant $c(n,m,d)$ such that*

$$\int_{\mathbb{R}^d} \mathscr{R}[\mathsf{P}]\,u^m\,d\mu\ge\alpha^4\left(1-\frac{1}{n}\right)\int_{\mathbb{R}^d}\left[\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega \mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right]^2 u^m\,d\mu+2\,\alpha^2\int_{\mathbb{R}^d}\frac{1}{s^2}\left|\nabla_\omega \mathsf{P}'-\frac{\nabla_\omega \mathsf{P}}{s}\right|^2 u^m\,d\mu$$

$$+(n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)\int_{\mathbb{R}^d}\frac{1}{s^4}\,|\nabla_\omega \mathsf{P}|^2\,u^m\,d\mu+c(n,m,d)\int_{\mathbb{R}^d}\frac{1}{s^4}\,\frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\,u^m\,d\mu\,.$$

4.3. Third step: concavity of the Rényi entropy powers and consequences

We keep investigating the properties of the flow defined by (15). Let us define the *entropy* as

$$\mathcal{E} := \int_{\mathbb{R}^d} u^m \, d\mu$$

and observe that

$$\mathcal{E}' = (1-m)\,\mathcal{I}$$

if u solves (15), after integrating by parts. The fact that boundary terms do not contribute, *i.e.*

$$\lim_{s\to 0_+} \int_{\partial B_s} u^m \, \mathsf{P}' \, d\varsigma = \lim_{S\to +\infty} \int_{\partial B_S} u^m \, \mathsf{P}' \, d\varsigma = 0 \tag{24}$$

will be justified in Section 5: see Proposition 5.1. Note that we use $'$ both for derivation w.r.t. t and w.r.t. s, at least when this does not create any ambiguity. As in Section 3, we introduce the *Rényi entropy power*

$$\mathcal{F} := \mathcal{E}^\sigma$$

for some exponent σ to be chosen later, and find that $\mathcal{F}' = \sigma\,(1-m)\,\mathcal{G}$ where $\mathcal{G} := \mathcal{E}^{\sigma-1}\,\mathcal{I}$. With $\mathcal{H} := \mathcal{E}^{-\sigma}\,\mathcal{G}'$, by using Lemma 4.1, we also find that $\mathcal{E}^{-\sigma}\,\mathcal{F}'' = \sigma\,(1-m)\,\mathcal{H}$ where

$$\begin{aligned}
\mathcal{E}^2\,\mathcal{H} = \mathcal{E}^{2-\sigma}\,\mathcal{G}' &= \frac{1}{\sigma\,(1-m)}\,\mathcal{E}^{2-\sigma}\,\mathcal{F}'' = (1-m)\,(\sigma-1)\left(\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu\right)^2 - 2\int_{\mathbb{R}^d} u^m\,d\mu \int_{\mathbb{R}^d} \mathcal{K}[\mathsf{P}]\,u^m\,d\mu \\
&= (1-m)\,(\sigma-1)\left(\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu\right)^2 - 2\left(\frac{1}{n}-(1-m)\right)\int_{\mathbb{R}^d} u^m\,d\mu \int_{\mathbb{R}^d} (\mathcal{L}_\alpha \mathsf{P})^2\,u^m\,d\mu \\
&\quad - 2\int_{\mathbb{R}^d} u^m\,d\mu \int_{\mathbb{R}^d} \mathcal{R}[\mathsf{P}]\,u^m\,d\mu
\end{aligned}$$

if $\lim_{s\to 0_+} \mathsf{b}(s) = \lim_{S\to+\infty} \mathsf{b}(S) = 0$. Using $u\,\mathsf{D}_\alpha \mathsf{P} = -\,\mathsf{D}_\alpha(u^m)$, we know that

$$\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu = -\int_{\mathbb{R}^d} \mathsf{D}_\alpha(u^m)\cdot \mathsf{D}_\alpha \mathsf{P}\,d\mu = \int_{\mathbb{R}^d} u^m\,\mathcal{L}_\alpha \mathsf{P}\,d\mu$$

and so, with the choice

$$\sigma = \frac{2}{n}\,\frac{1}{1-m} - 1\,,$$

we may argue as in Section 3 and get that

$$\mathcal{E}^2\,\mathcal{H} + (1-m)\,(\sigma-1)\,\mathcal{E}\int_{\mathbb{R}^d} u^m \left|\mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu + 2\,\mathcal{E}\int_{\mathbb{R}^d} \mathcal{R}[\mathsf{P}]\,u^m\,d\mu = 0$$

if $\lim_{s\to 0_+} \mathsf{b}(s) = \lim_{S\to+\infty} \mathsf{b}(S) = 0$. So, if $\alpha \le \alpha_{\rm FS}$ and P is of class C^3, by Corollary 4.4, as a function of t, $\mathcal{F}$ is concave, that is, $\mathcal{G} = \mathcal{E}^{\sigma-1}\,\mathcal{I}$ is non-increasing in t. Formally, $\mathcal{G}$ converges towards a minimum, for which necessarily $\mathcal{L}_\alpha \mathsf{P}$ is a constant and $\mathcal{R}[\mathsf{P}] = 0$, which proves that $\mathsf{P}(x) = \mathsf{a} + \mathsf{b}\,|x|^2$ for some real constants a and b, according to Corollary 4.4. Since $\frac{2\,(1-\vartheta)}{\vartheta\,(p+1)} = \sigma - 1$, the minimization of $\mathcal{G}$ under the mass constraint $\int_{\mathbb{R}^d} u\,d\mu = \int_{\mathbb{R}^d} v^{2p}\,d\mu$ is equivalent to the *Caffarelli–Kohn–Nirenberg interpolation inequalities* (1), since for some constant κ,

$$\mathcal{G} = \mathcal{E}^{\sigma-1}\,\mathcal{I} = \kappa\left(\int_{\mathbb{R}^d} v^{p+1}\,d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} |\mathsf{D}_\alpha v|^2\,d\mu \quad \text{with} \quad v = u^{m-1/2}\,.$$

We emphasize that (15) preserves mass, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} v^{2p}\,\mathrm{d}\mu = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u\,\mathrm{d}\mu = \int_{\mathbb{R}^d} \mathscr{L}_\alpha u^m\,\mathrm{d}\mu = 0$ because, as we shall see in Proposition 5.1, no boundary terms appear when integrating by parts if v is an extremal function associated with (7). In particular, for mass conservation we need

$$\lim_{s\to 0_+}\int_{\partial B_s} u\,\mathsf{P}'\,\mathrm{d}\varsigma = \lim_{S\to+\infty}\int_{\partial B_S} u\,\mathsf{P}'\,\mathrm{d}\varsigma = 0\,. \tag{25}$$

The above remarks on the monotonicity of $\mathscr{G}$ and the symmetry properties of its minimizers can in fact be extended to the analysis of the symmetry properties of all critical points of $\mathscr{G}$. This is actually the contents of Theorem 1.2.

Proof of Theorem 1.2. Let w be a positive solution to equation (5). As pointed out above, by choosing

$$w(x) = u^{m-1/2}(r^\alpha,\omega)\,,$$

we know that u is a critical point of $\mathscr{G}$ under a mass constraint on $\int_{\mathbb{R}^d} u\,\mathrm{d}x$, so that we can write the corresponding Euler–Lagrange equation as $\mathrm{d}\mathscr{G}[u] = C$, for some constant C. That is, $\int_{\mathbb{R}^d} \mathrm{d}\mathscr{G}[u]\cdot\mathscr{L}_\alpha u^m\,\mathrm{d}\mu = C\int_{\mathbb{R}^d}\mathscr{L}_\alpha u^m\,\mathrm{d}\mu = 0$ thanks to (25). Using $\mathscr{L}_\alpha u^m$ as a test function amounts to apply the flow of (15) to $\mathscr{G}$ with initial datum u and compute the derivative with respect to t at $t=0$. This means

$$0 = \int_{\mathbb{R}^d} \mathrm{d}\mathscr{G}[u]\cdot\mathscr{L}_\alpha u^m\,\mathrm{d}\mu = \mathscr{E}^\sigma\,\mathscr{H}$$

$$= -\,(1-m)\,(\sigma-1)\,\mathscr{E}^{\sigma-1}\int_{\mathbb{R}^d} u^m\left|\mathscr{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,\mathrm{d}\mu}{\int_{\mathbb{R}^d} u^m\,\mathrm{d}\mu}\right|^2\mathrm{d}\mu - 2\,\mathscr{E}^{\sigma-1}\int_{\mathbb{R}^d}\mathscr{R}[\mathsf{P}]\,u^m\,\mathrm{d}\mu$$

if $\lim_{s\to 0_+}\mathsf{b}(s) = \lim_{S\to+\infty}\mathsf{b}(S) = 0$ and (24) holds. Here we have used Lemma 4.1. We emphasize that this proof is purely variational and does not rely on the properties of the solutions to (15), although using the flow was very useful to explain our strategy. All we need is that no boundary term appears in the integrations by parts. Hence, in order to obtain a complete proof, we have to prove that (18), (24) and (25) hold with b defined by (17), whenever u is a critical point of $\mathscr{G}$ under mass constraint. This will be done in Proposition 5.1. Using Corollary 4.4, we know that $\mathscr{R}[\mathsf{P}] = 0$, $\nabla_\omega\mathsf{P} = 0$ a.e. in $\mathbb{R}^d$ and $\mathscr{L}_\alpha\mathsf{P} = \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,\mathrm{d}\mu}{\int_{\mathbb{R}^d} u^m\,\mathrm{d}\mu}$ a.e. in $\mathbb{R}^d$, with $\mathsf{P} = \frac{m}{1-m}\,u^{m-1}$. We conclude as in [14, Corollary 5.5] that P is an affine function of s^2. □

5. Regularity and decay estimates

In this last section, we prove the regularity and decay estimates on w (or on P or u) that are necessary to establish the absence of boundary terms in the integrations by parts of Section 4.

Proposition 5.1. *Under Condition (2), if w is a positive solution in $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ of (5), then (18), (24) and (25) hold with b as defined by (17), $u = v^{2p}$ and v given by (6).*

To prove this result, we split the proof in several steps: we will first establish a uniform bound and a decay rate for w inspired by [17] in Lemmas 5.2, 5.3, and then follow the methodology of [14, Appendix] in the subsequent Lemma 5.4.

Lemma 5.2. *Let β, γ and p satisfy the relations (2). Any positive solution w of (5) such that*

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} + \|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)} + \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} < +\infty\,. \tag{26}$$

is uniformly bounded and tends to 0 at infinity, uniformly in $|x|$.

Proof. The strategy of the first part of the proof is similar to the one in [17, Lemma 3.1], which was restricted to the case $\beta = 0$.

Let us set $\delta_0 := 2\,(p_\star - p)$. For any $A > 0$, we multiply (5) by $(w\wedge A)^{1+\delta_0}$ and integrate by parts (or, equivalently, plug it in the weak formulation of (5)): we point out that the latter is indeed an admissible test function since $w\in\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$. In that way, by letting $A\to+\infty$, we obtain the identity

$$\frac{4\,(1+\delta_0)}{(2+\delta_0)^2}\int_{\mathbb{R}^d}\left|\nabla w^{1+\delta_0/2}\right|^2|x|^{-\beta}\,\mathrm{d}x + \int_{\mathbb{R}^d} w^{p+1+\delta_0}\,|x|^{-\gamma}\,\mathrm{d}x = \int_{\mathbb{R}^d} w^{2p+\delta_0}\,|x|^{-\gamma}\,\mathrm{d}x\,.$$

By applying (1) with $p = p_\star$ (so that $\vartheta = 1$) to the function $w = w^{1+\delta_0/2}$, we deduce that

$$\|w\|^{2+\delta_0}_{\mathrm{L}^{2p+\delta_1,\gamma}(\mathbb{R}^d)} \le \frac{(2+\delta_0)^2}{4\,(1+\delta_0)}\,\mathsf{C}^2_{\beta,\gamma,p_\star}\,\|w\|^{2p+\delta_0}_{\mathrm{L}^{2p+\delta_0,\gamma}(\mathbb{R}^d)}$$

with $2\,p + \delta_1 = p_\star\,(2+\delta_0)$. Let us define the sequence $\{\delta_n\}$ by the induction relation $\delta_{n+1} := p_\star\,(2+\delta_n) - 2\,p$ for any $n \in \mathbb{N}$, that is,

$$\delta_n = 2\,\tfrac{p_\star - p}{p_\star - 1}\left(p_\star^{n+1} - 1\right) \quad \forall\, n \in \mathbb{N}\,,$$

and take $q_n = 2\,p + \delta_n$. If we repeat the above estimates with δ_0 replaced by δ_n and δ_1 replaced by δ_{n+1}, we get

$$\|w\|^{2+\delta_n}_{\mathrm{L}^{q_{n+1},\gamma}(\mathbb{R}^d)} \le \frac{(2+\delta_n)^2}{4\,(1+\delta_n)}\,\mathsf{C}^2_{\beta,\gamma,p_\star}\,\|w\|^{q_n}_{\mathrm{L}^{q_n,\gamma}(\mathbb{R}^d)}\,.$$

By iterating this estimate, we obtain the estimate

$$\|w\|_{\mathrm{L}^{q_n,\gamma}(\mathbb{R}^d)} \le C_n\,\|w\|^{\zeta_n}_{\mathrm{L}^{2p_\star,\gamma}(\mathbb{R}^d)} \quad \text{with} \quad \zeta_n := \frac{(p_\star - 1)\,p_\star^n}{p - 1 + (p_\star - p)\,p_\star^n}\,,$$

where the sequence $\{C_n\}$ is defined by $C_0 = 1$ and

$$C_{n+1}^{2+\delta_n} = \frac{(2+\delta_n)^2}{4\,(1+\delta_n)}\,\mathsf{C}^2_{\beta,\gamma,p_\star}\,C_n^{q_n} \quad \forall\, n \in \mathbb{N}\,.$$

The sequence $\{C_n\}$ converges to a finite limit C_∞. Letting $n \to \infty$ we obtain the uniform bound

$$\|w\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_\infty\,\|w\|^{\zeta_\infty}_{\mathrm{L}^{2p_\star,\gamma}(\mathbb{R}^d)} \le C_\infty\left(\mathsf{C}_{\beta,\gamma,p_\star}\,\|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\right)^{\zeta_\infty} \le C_\infty\left(\mathsf{C}_{\beta,\gamma,p_\star}\,\|w\|^p_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)}\right)^{\zeta_\infty}\,,$$

where $\zeta_\infty := \frac{p_\star - 1}{p_\star - p} = \lim_{n\to\infty} \zeta_n$.

In order to prove that $\lim_{|x|\to+\infty} w(x) = 0$, we can suitably adapt the above strategy. We shall do it as follows: we truncate the solution so that the truncated function is supported outside of a ball of radius R_0 and apply the iteration scheme. Up to an enlargement of the ball, that is, outside of a ball of radius $R_\infty = a\,R_0$ for some fixed numerical constant $a > 1$, we get that $\|w\|_{\mathrm{L}^\infty(B^c_{R_\infty})}$ is bounded by the energy localized in $B^c_{R_0}$. The conclusion will hold by letting $R_0 \to +\infty$. Let us give some details.

Let $\xi \in C^\infty(\mathbb{R}^+)$ be a cut-off function such that $0 \le \xi \le 1$, $\xi \equiv 0$ in $[0,1)$ and $\xi \equiv 1$ in $(2,+\infty)$. Given $R_0 \ge 1$, consider the sequence of radii defined by

$$R_{n+1} = \left(1 + \frac{1}{n^2}\right) R_n \quad \forall\, n \in \mathbb{N}\,.$$

By taking logarithms, it is immediate to deduce that $\{R_n\}$ is monotone increasing and that there exists $a > 1$ such that

$$R_\infty := \lim_{n\to\infty} R_n = a\,R_0\,.$$

Let us then define the sequence of radial cut-off functions $\{\xi_n\}$ by

$$\xi_n(x) := \xi^2\left(\frac{|x| - R_n}{R_{n+1} - R_n} + 1\right) \quad \forall\, x \in \mathbb{R}^d\,.$$

Direct computations show that there exists some constant $c > 0$, which is independent of n and R_0, such that

$$|\nabla \xi_n(x)| \le c\,\frac{n^2}{R_n}\,\chi_{B_{R_{n+1}}\setminus B_{R_n}}\,, \quad \left|\nabla \xi_n^{1/2}(x)\right| \le c\,\frac{n^2}{R_n}\,\chi_{B_{R_{n+1}}\setminus B_{R_n}}\,, \quad |\Delta \xi_n(x)| \le c\,\frac{n^4}{R_n^2}\,\chi_{B_{R_{n+1}}\setminus B_{R_n}} \quad \forall\, x \in \mathbb{R}^d\,. \tag{27}$$

From here on we denote by c, c', *etc.* positive constants that are all independent of n and R_0. We now introduce the analogue of the sequence $\{\delta_n\}$ above, which we relabel $\{\sigma_n\}$ to avoid confusion. Namely, we set $\sigma_0 := 2\,p - 2$ and $\sigma_{n+1} = p_\star\,(2+\sigma_n) - 2$, so that $\sigma_n = 2\,(p\,p_\star^n - 1)$. If we multiply (5) by $\xi_n\,w^{1+\sigma_n}$ and integrate by parts, we obtain:

$$\int_{\mathbb{R}^d} \nabla\!\left(\xi_n\,w^{1+\sigma_n}\right)\cdot\nabla w\,|x|^{-\beta}\,\mathrm{d}x + \int_{\mathbb{R}^d} \xi_n\,w^{p+1+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x = \int_{\mathbb{R}^d} \xi_n\,w^{2p+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x\,,$$

whence

$$\frac{4\,(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d} \xi_n\left|\nabla w^{1+\sigma_n/2}\right|^2 |x|^{-\beta}\,\mathrm{d}x + \frac{1}{2+\sigma_n}\int_{\mathbb{R}^d} \nabla\xi_n\cdot\nabla w^{2+\sigma_n}\,|x|^{-\beta}\,\mathrm{d}x \le \int_{B^c_{R_n}} w^{2p+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x\,.$$

By integrating by parts the second term in the l.h.s. and combining this estimate with

$$\int_{\mathbb{R}^d} \left|\nabla\left(\xi_n^{1/2}\, w^{1+\sigma_n/2}\right)\right|^2 |x|^{-\beta}\,\mathrm{d}x \le 2\int_{\mathbb{R}^d} \xi_n \left|\nabla w^{1+\sigma_n/2}\right|^2 |x|^{-\beta}\,\mathrm{d}x + 2\int_{\mathbb{R}^d} \left|\nabla\xi_n^{1/2}\right|^2 w^{2+\sigma_n}\,|x|^{-\beta}\,\mathrm{d}x\,,$$

we end up with

$$\frac{2\,(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d} \left|\nabla\left(\xi_n^{1/2}\, w^{1+\sigma_n/2}\right)\right|^2 |x|^{-\beta}\,\mathrm{d}x - \frac{4\,(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d} \left|\nabla\xi_n^{1/2}\right|^2 w^{2+\sigma_n}\,|x|^{-\beta}\,\mathrm{d}x$$
$$-\frac{1}{2+\sigma_n}\int_{\mathbb{R}^d}\left(|x|^{-\beta}\Delta\xi_n - \beta\,|x|^{-\beta-2}x\cdot\nabla\xi_n\right) w^{2+\sigma_n}\,\mathrm{d}x \le \int_{B_{R_n}^c} w^{2p+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x\,.$$

Thanks to (27), we can deduce that

$$\int_{\mathbb{R}^d} \left|\nabla\left(\xi_n^{1/2}\, w^{1+\sigma_n/2}\right)\right|^2 |x|^{-\beta}\,\mathrm{d}x \le \int_{B_{R_{n+1}}\setminus B_{R_n}} \left(\frac{2\,c^2+c}{R_n^2}\,n^4 + \frac{\beta\,c}{R_n}\,n^2\,|x|^{-1}\right) w^{2+\sigma_n}\,|x|^{-\beta}\,\mathrm{d}x$$
$$+\frac{(2+\sigma_n)^2}{2\,(1+\sigma_n)}\int_{B_{R_n}^c} w^{2p+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x\,.$$

In particular,

$$\int_{\mathbb{R}^d} \left|\nabla\left(\xi_n^{1/2}\, w^{1+\sigma_n/2}\right)\right|^2 |x|^{-\beta}\,\mathrm{d}x \le c'n^4\int_{B_{R_n}^c} w^{2+\sigma_n}\,|x|^{-\beta-2}\,\mathrm{d}x + \frac{(2+\sigma_n)^2}{2\,(1+\sigma_n)}\,\|w\|_\infty^{2p-2}\int_{B_{R_n}^c} w^{2+\sigma_n}\,|x|^{-\gamma}\,\mathrm{d}x\,.$$

Since (2) implies that $\beta+2>\gamma$, by exploiting the explicit expression of σ_n and applying (1) with $p=p_\star$ (and $\vartheta=1$) to the function $\xi_n^{1/2}\,w^{1+\sigma_n/2}$, we can rewrite our estimate as

$$\|w\|^{2+\sigma_n}_{\mathrm{L}^{2+\sigma_{n+1},\gamma}(B_{R_{n+1}}^c)} \le c''\,p_\star^n\,\|w\|^{2+\sigma_n}_{\mathrm{L}^{2+\sigma_n,\gamma}(B_{R_n}^c)}\,.$$

After iterating the scheme and letting $n\to\infty$, we end up with

$$\|w\|_{\mathrm{L}^\infty(B_{R_\infty}^c)} \le c'''\,\|w\|_{\mathrm{L}^{2p,\gamma}(B_{R_0}^c)}\,.$$

Since w is bounded in $\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)$, in order to prove the claim, it is enough to let $R_0\to+\infty$. □

Lemma 5.3. *Let β, γ and p satisfy the relations (2). Any positive solution w of (5) satisfying (26) is such that $w\in C^\infty(\mathbb{R}^d\setminus\{0\})$ and there exist two positive constants, C_1 and C_2 with $C_1<C_2$, such that for $|x|$ large enough,*

$$C_1\,|x|^{(\gamma-2-\beta)/(p-1)} \le w(x) \le C_2\,|x|^{(\gamma-2-\beta)/(p-1)}\,.$$

Proof. By Lemma 5.2 and elliptic bootstrapping methods we know that $w\in C^\infty(\mathbb{R}^d\setminus\{0\})$. Let us now consider the function $h(x):=C\,|x|^{(\gamma-2-\beta)/(p-1)}$, which satisfies the differential inequality

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla h\right) + (1-\varepsilon)\,|x|^{-\gamma}\,h^p \ge 0 \quad \forall\,x\in\mathbb{R}^d\setminus\{0\}$$

for any $\varepsilon\in(0,1)$ and C such that $C^{p-1} > \frac{2-\gamma+\beta}{1-\varepsilon}\,\frac{d-\gamma-p\,(d-2-\beta)}{(p-1)^2}$. On the other hand, by Lemma 5.2, w^{2p-1} is negligible compared to w^p as $|x|\to\infty$ and, as a consequence, for any $\varepsilon>0$ small enough, there is an $R_\varepsilon>0$ such that

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla w\right) + (1-\varepsilon)\,|x|^{-\gamma}\,w^p \le 0 \quad \text{if} \quad |x|\ge R_\varepsilon\,.$$

With $q:=(1-\varepsilon)\,|x|^{-\gamma}\,\frac{h^p-w^p}{h-w}\ge 0$, it follows that

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla(h-w)\right) + q\,(h-w) \ge 0 \quad \text{if} \quad |x|\ge R_\varepsilon\,.$$

Hence, for C large enough, we know that $h(x)\ge w(x)$ for any $x\in\mathbb{R}^d$ such that $|x|=R_\varepsilon$, and we also have that $\lim_{|x|\to+\infty}\left(h(x)-w(x)\right)=0$. Using the Maximum Principle, we conclude that $0\le w(x)\le h(x)$ for any $x\in\mathbb{R}^d$ such that $|x|\ge R_\varepsilon$. The lower bound uses a similar comparison argument. Indeed, since

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla w\right)+|x|^{-\gamma}\,w^p\geq 0\quad\forall\,x\in\mathbb{R}^d\setminus\{0\}$$

and

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla h\right)+|x|^{-\gamma}\,h^p\leq 0\quad\forall\,x\in\mathbb{R}^d\setminus\{0\},$$

if we choose C such that $C^{p-1}\leq(2-\gamma+\beta)\,\frac{d-\gamma-p\,(d-2-\beta)}{(p-1)^2}$, we easily see that

$$w(x)\geq\left(\min_{|x|=1}w(x)\wedge C\right)|x|^{(\gamma-2-\beta)/(p-1)}\quad\forall x\in\mathbb{R}^d\setminus B_1\,.$$

This concludes the proof. □

Our next goal is to obtain growth and decay estimates, respectively, on the functions P and u as they appear in the proof of Theorem 1.2 in Section 4, in order to prove Proposition 5.1. We also need to estimate their derivatives near the origin and at infinity. Let us start by reminding the change of variables (6), which in particular, by Lemma 5.3, implies that for some positive constants C_1 and C_2,

$$C_1\,s^{2/(1-p)}\leq v(s,\omega)\leq C_2\,s^{2/(1-p)}\quad\text{as } s\to+\infty\,.$$

Then we perform the Emden–Fowler transformation

$$v(s,\omega)=s^a\,\varphi(z,\omega)\quad\text{with}\quad z=-\log s\,,\quad a=\frac{2-n}{2}\,,\tag{28}$$

and see that φ satisfies the equation

$$-\alpha^2\,\varphi''-\Delta_\omega\,\varphi+a^2\alpha^2\varphi=\mathrm{e}^{((n-2)\,p-n)\,z}\,\varphi^{2p-1}-\mathrm{e}^{((n-2)\,p-n-2)\,z/2}\,\varphi^p=:h\quad\text{in }\mathscr{C}:=\mathbb{R}\times\mathbb{S}^{d-1}\ni(z,\omega)\,.\tag{29}$$

From here on we shall denote by $'$ the derivative with respect either to z or to s, depending on the argument. By definition of φ and using Lemma 5.3, we obtain that

$$\varphi(z,\omega)\sim\mathrm{e}^{\left(\frac{2-n}{2}+\frac{2}{p-1}\right)z}\quad\text{as } z\to-\infty\,,$$

where we say that $f(z,\omega)\sim g(z,\omega)$ as $z\to+\infty$ (resp. $z\to-\infty$) if the ratio f/g is bounded from above and from below by positive constants, independently of ω, and for z (resp. $-z$) large enough. Concerning $z\to+\infty$, we first note that Lemma 5.2 and (28) show that $\varphi(z,\omega)\leq O(\mathrm{e}^{a\,z})$. The lower bound can be established by a comparison argument as in [14, Proposition A.1], after noticing that $|h(z,\omega)|\leq O(\mathrm{e}^{(a-2)z})$. Hence we obtain that

$$\varphi(z,\omega)\sim\mathrm{e}^{a\,z}=\mathrm{e}^{\frac{2-n}{2}\,z}\quad\text{as } z\to+\infty\,.$$

Moreover, uniformly in ω, we have that

$$|h(z,\omega)|\leq O\left(\mathrm{e}^{-\frac{n+2}{2}\,z}\right)\quad\text{as } z\to+\infty\,,\quad|h(z,\omega)|\sim\mathrm{e}^{\left(-\frac{n+2}{2}+\frac{2\,p}{p-1}\right)z}\quad\text{as } z\to-\infty\,,$$

which in particular implies

$$|h(z,\omega)|=o\big(\varphi(z,\omega)\big)\quad\text{as } z\to+\infty\quad\text{and}\quad|h(z,\omega)|\sim\varphi(z,\omega)\quad\text{as } z\to-\infty\,.$$

Finally, using [20, Theorem 8.32, p. 210] on local $C^{1,\delta}$ estimates, as $|z|\to+\infty$ we see that all first derivatives of φ converge to 0 at least with the same rate as φ. Next, [20, Theorem 8.10, p. 186] provides local $\mathrm{W}^{k+2,2}$ estimates which, together with [20, Corollary 7.11, Theorem 8.10, and Corollary 8.11], up to choosing k large enough, prove that

$$|\varphi'(z,\omega)|,\;|\varphi''(z,\omega)|,\;|\nabla_\omega\varphi(z,\omega)|,\;|\nabla_\omega\varphi'(z,\omega)|,\;|\nabla_\omega\varphi''(z,\omega)|,\;|\Delta_\omega\,\varphi(z,\omega)|\leq O\left(\varphi(z,\omega)\right),\tag{30}$$

uniformly in ω. Here we denote by ∇_ω the differentiation with respect to ω. As a consequence, we have, uniformly in ω, and for $\ell\in\{0,1,2\}$, $t\in\{0,1\}$,

$$|\partial_z^\ell\nabla_\omega^t h(z,\omega)|\leq O\left(\mathrm{e}^{-\frac{n+2}{2}\,z}\right)\quad\text{as } z\to+\infty\,,\quad|\partial_z^\ell\nabla_\omega^t h(z,\omega)|\leq O\left(\mathrm{e}^{\left(-\frac{n+2}{2}+\frac{2\,p}{p-1}\right)z}\right)\quad\text{as } z\to-\infty\,,\tag{31}$$

$$|\Delta_\omega h(z,\omega)|\leq O\left(\mathrm{e}^{-\frac{n+2}{2}\,z}\right)\quad\text{as } z\to+\infty\,,\quad|\Delta_\omega h(z,\omega)|\leq O\left(\mathrm{e}^{\left(-\frac{n+2}{2}+\frac{2\,p}{p-1}\right)z}\right)\quad\text{as } z\to-\infty\,.\tag{32}$$

Lemma 5.4. *Let β, γ and p satisfy the relations (2) and assume $\alpha\leq\alpha_{\rm FS}$. For any positive solution w of (5) satisfying (26), the* pressure function $\mathsf{P}=\frac{m}{1-m}\,u^{m-1}$ *is such that P'', P'/s, P/s^2, $\nabla_\omega\mathsf{P}'/s$, $\nabla_\omega\mathsf{P}/s^2$ and $\mathscr{L}_\alpha\mathsf{P}$ are of class C^∞ and bounded as $s\to+\infty$. On the other hand, as $s\to0_+$ we have*

(i) $\int_{\mathbb{S}^{d-1}} |\mathsf{P}'(s,\omega)|^2\,d\omega \le O(1)$,
(ii) $\int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{P}(s,\omega)|^2\,d\omega \le O(s^2)$,
(iii) $\int_{\mathbb{S}^{d-1}} |\mathsf{P}''(s,\omega)|^2\,d\omega \le O(1/s^2)$,
(iv) $\int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \mathsf{P}'(s,\omega) - \frac{1}{s}\nabla_\omega \mathsf{P}(s,\omega)\right|^2 d\omega \le O(1)$,
(v) $\int_{\mathbb{S}^{d-1}} \left|\frac{1}{s^2}\Delta_\omega \mathsf{P}(s,\omega)\right|^2 d\omega \le O(1/s^2)$.

Proof. By using the change of variables (28), we see that

$$\mathsf{P}(s,\omega) = \tfrac{p+1}{p-1}\,e^{-\frac{1}{2}(n-2)(p-1)z}\,\varphi^{1-p}(z,\omega)\,, \quad z = -\log s\,.$$

From (30) we easily deduce that uniformly in ω, P'', P'/s, P/s^2, $\nabla_\omega \mathsf{P}'/s$, $\nabla_\omega \mathsf{P}/s^2$ and $\mathscr{L}_\alpha \mathsf{P}$ are of class C^∞ and bounded as $s \to +\infty$. Moreover, as $s \to 0_+$, we obtain that

$$\left|\mathsf{P}'(s,\omega)\right| \le O\left(\frac{1}{s}\left(\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a\right)\right) \quad \text{and} \quad \left|\frac{1}{s}\nabla_\omega \mathsf{P}(s,\omega)\right| \le O\left(\frac{1}{s}\left(\frac{\nabla_\omega\varphi(z,\omega)}{\varphi(z,\omega)}\right)\right)$$

are of order at most $1/s$ uniformly in ω. Similarly we obtain that

$$|\mathsf{P}''(s,\omega)| \le O\left(\frac{1}{s^2}\left(\frac{\varphi''(z,\omega)}{\varphi(z,\omega)} - p\,\frac{|\varphi'(z,\omega)|^2}{|\varphi(z,\omega)|^2} + \big(1-2a(1-p)\big)\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} + a^2(1-p) - a\right)\right),$$
$$\left|\frac{\nabla_\omega \mathsf{P}'(s,\omega)}{s} - \frac{a(1-p)}{s^2}\nabla_\omega \mathsf{P}(s,\omega)\right| \le O\left(\frac{1}{s^2}\left(\frac{\nabla_\omega\varphi'(z,\omega)}{\varphi(z,\omega)} - \frac{p\,\varphi'(z,\omega)\,\nabla_\omega\varphi(z,\omega)}{|\varphi(z,\omega)|^2}\right)\right),$$
$$\frac{1}{s^2}|\Delta_\omega \mathsf{P}(s,\omega)| \le O\left(\frac{1}{s^2}\left(\frac{\Delta_\omega\varphi(z,\omega)}{\varphi(z,\omega)} - p\,\frac{|\nabla_\omega\varphi(z,\omega)|^2}{|\varphi(z,\omega)|^2}\right)\right),$$

are at most of order $1/s^2$ uniformly in ω. This shows that $|\mathsf{b}(s)| \le O(s^{n-4})$ as $s \to 0_+$ and concludes the proof if $4 \le d < n$. When $d = 2$ or 3 and $n \le 4$, more detailed estimates are needed. Properties (i)–(v) amount to prove that

(i) $\int_{\mathbb{S}^{d-1}} \left|\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a\right|^2 d\omega \le O(e^{-2z})$,
(ii) $\int_{\mathbb{S}^{d-1}} \left|\frac{\nabla_\omega\varphi(z,\omega)}{\varphi(z,\omega)}\right|^2 d\omega \le O(e^{-2z})$,
(iii) $\int_{\mathbb{S}^{d-1}} \left|\frac{\varphi''(z,\omega)}{\varphi(z,\omega)} - p\,\frac{|\varphi'(z,\omega)|^2}{|\varphi(z,\omega)|^2} + \big(1-2a(1-p)\big)\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} + a^2(1-p) - a\right|^2 d\omega \le O(e^{-2z})$,
(iv) $\int_{\mathbb{S}^{d-1}} \left|\frac{\nabla_\omega\varphi'(z,\omega)}{\varphi(z,\omega)} - \frac{p\,\varphi'(z,\omega)\,\nabla_\omega\varphi(z,\omega)}{|\varphi(z,\omega)|^2}\right|^2 d\omega \le O(e^{-2z})$,
(v) $\int_{\mathbb{S}^{d-1}} \left|\frac{\Delta_\omega\varphi(z,\omega)}{\varphi(z,\omega)} - p\,\frac{|\nabla_\omega\varphi(z,\omega)|^2}{|\varphi(z,\omega)|^2}\right|^2 d\omega \le O(e^{-2z})$,

as $z \to +\infty$.

Step 1: Proof of (ii) and (iv). If w is a positive solution to (5), then φ is a positive solution to (29). With $\ell \in \{0,1,2\}$, applying the operator $\nabla_\omega \partial_z^\ell$ to the equation (29) we obtain:

$$-\alpha^2(\nabla_\omega\partial_z^\ell\varphi)'' - \nabla_\omega\Delta_\omega\partial_z^\ell\varphi + a^2\alpha^2\nabla_\omega\partial_z^\ell\varphi = \nabla_\omega\partial_z^\ell h(z,\omega) \quad \text{in } \mathscr{C}\,.$$

Define

$$\chi_\ell(z) := \frac{1}{2}\int_{\mathbb{S}^{d-1}} |\nabla_\omega\partial_z^\ell\varphi|^2\,d\omega\,,$$

which by (30) converges to 0 as $z \to \pm\infty$. Assume first that χ_ℓ is a positive function. After multiplying the above equation by $\nabla_\omega\partial_z^\ell\varphi$, integrating over $\mathbb{S}^{d-1}$, integrating by parts and using

$$\chi_\ell' = \int_{\mathbb{S}^{d-1}} \nabla_\omega\partial_z^\ell\varphi\,\nabla_\omega\partial_z^\ell\varphi'\,d\omega$$

and

$$\chi_\ell'' = \int_{\mathbb{S}^{d-1}} \nabla_\omega\partial_z^\ell\varphi\,\nabla_\omega\partial_z^\ell\varphi''\,d\omega + \int_{\mathbb{S}^{d-1}} |\nabla_\omega\partial_z^\ell\varphi'|^2\,d\omega\,,$$

we see that χ_ℓ satisfies

$$-\chi_\ell'' + \int_{\mathbb{S}^{d-1}} |\nabla_\omega \partial_z^\ell \varphi'|^2 \, d\omega + \frac{1}{\alpha^2} \left(\int_{\mathbb{S}^{d-1}} |\Delta_\omega \partial_z^\ell \varphi|^2 \, d\omega - \lambda_1 \int_{\mathbb{S}^{d-1}} |\nabla_\omega \partial_z^\ell \varphi|^2 \, d\omega \right) + 2 \left(a^2 + \frac{\lambda_1}{\alpha^2} \right) \chi_\ell = \frac{h_\ell}{\alpha^2} \,,$$

with $h_\ell := \int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell h \, \nabla_\omega \partial_z^\ell \varphi \, d\omega$. Then, using $\int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell \varphi \, d\omega = 0$, by the Poincaré inequality we deduce

$$\int_{\mathbb{S}^{d-1}} |\Delta_\omega \partial_z^\ell \varphi|^2 \, d\omega \geq \lambda_1 \int_{\mathbb{S}^{d-1}} |\nabla_\omega \partial_z^\ell \varphi|^2 \, d\omega$$

as *e.g.* in [12, Lemma 7], where $\lambda_1 := d-1$. A Cauchy–Schwarz inequality implies that

$$-\chi_\ell'' + \frac{|\chi_\ell'|^2}{2\,\chi_\ell} + 2 \left(a^2 + \frac{\lambda_1}{\alpha^2} \right) \chi_\ell \leq \frac{|h_\ell|}{\alpha^2} \,.$$

The function $\zeta_\ell := \sqrt{\chi_\ell}$ satisfies

$$-\zeta_\ell'' + \left(a^2 + \frac{\lambda_1}{\alpha^2} \right) \zeta_\ell \leq \frac{|h_\ell|}{2\,\alpha^2\,\zeta_\ell} \,.$$

By the Cauchy–Schwarz inequality and (31) we infer that $|h_\ell/\zeta_\ell| = O\big(e^{(a-2)\,z}\big)$ for $z \to +\infty$, and $|h_\ell/\zeta_\ell| = O\big(e^{(a+2/(p-1))\,z}\big)$ for $z \to -\infty$. By a simple comparison argument based on the Maximum Principle, and using the convergence of χ_ℓ to 0 at $\pm\infty$, we infer that

$$\zeta_\ell(z) \leq -\frac{e^{-\nu\,z}}{2\,\nu\,\alpha^2} \int_{-\infty}^{z} e^{\nu\,t} \, \frac{|h_\ell(t)|}{\zeta_\ell(z)} \, dt - \frac{e^{\nu\,z}}{2\,\nu\,\alpha^2} \int_{z}^{\infty} e^{-\nu\,t} \, \frac{|h_\ell(t)|}{\zeta_\ell(z)} \, dt$$

if $\nu := \sqrt{a^2 + \lambda_1/\alpha^2}$. This is enough to deduce that $\zeta_\ell(z) \leq O\big(e^{(a-1)z}\big)$ as $z \to +\infty$ after observing that the condition

$$-\nu = -\sqrt{a^2 + \lambda_1/\alpha^2} \leq a - 1$$

is equivalent to the inequality $\alpha \leq \alpha_{FS}$. Hence we have shown that if χ_ℓ is a positive function, then for $\alpha \leq \alpha_{FS}$,

$$\chi_\ell(z) \leq O\big(e^{2\,(a-1)\,z}\big) \quad \text{as } z \to +\infty \,. \tag{33}$$

In the case where χ_ℓ is equal to 0 at some points of $\mathbb{R}$, it is enough to do the above comparison argument on maximal positivity intervals of χ_ℓ to deduce the same asymptotic estimate. Finally we observe that $\varphi(z,\omega) \sim e^{a\,z}$ as $z \to +\infty$, which ends the proof of (ii) considering the above estimate for χ_ℓ when $\ell = 0$. Moreover, the same estimate for $\ell = 1$ together with (ii) and (30) proves (iv).

Step 2: Proof of (v). By applying the operator Δ_ω to (29), we obtain

$$-\alpha^2\,(\Delta_\omega\,\varphi)'' - \Delta_\omega^2\,\varphi + a^2\,\alpha^2\,\Delta_\omega\,\varphi = \Delta_\omega\,h \quad \text{in } \mathscr{C} \,.$$

We proceed as in Step 1. With similar notations, by defining

$$\chi_3(z) := \frac{1}{2} \int_{\mathbb{S}^{d-1}} |\Delta_\omega\,\varphi|^2 \, d\omega \,,$$

after multiplying the equation by $\Delta_\omega\,\varphi$ and using the fact that

$$-\int_{\mathbb{S}^{d-1}} \Delta_\omega\,\varphi\,\Delta_\omega^2\,\varphi \, d\omega = \int_{\mathbb{S}^{d-1}} |\nabla_\omega \Delta_\omega\,\varphi|^2 \, d\omega \geq \lambda_1 \int_{\mathbb{S}^{d-1}} |\Delta_\omega\,\varphi|^2 \, d\omega \,,$$

we obtain

$$-\chi_3'' + \frac{|\chi_3'|^2}{2\,\chi_\ell} + 2 \left(a^2 + \frac{\lambda_1}{\alpha^2} \right) \chi_3 \leq \frac{|h_3|}{\alpha^2}$$

with $h_3 := \int_{\mathbb{S}^{d-1}} \Delta_\omega\,h\,\Delta_\omega\,\varphi \, d\omega$. Again using the same arguments as above, together with (32), we deduce that

$$\chi_3(z) \leq O\big(e^{2\,(a-1)\,z}\big) \quad \text{as } z \to +\infty \,.$$

This ends the proof of (v), using (ii), (30) and noticing again that $\varphi(z,\omega) \sim e^{a\,z}$ as $z \to +\infty$.

Step 3: Proof of (i) and (iii). Let us consider a positive solution φ to (29) and define on $\mathbb{R}$ the function

$$\varphi_0(z) := \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \varphi(z,\omega)\,d\omega\,.$$

By integrating (29) on $\mathbb{S}^{d-1}$, we know that φ_0 solves

$$-\varphi_0'' + a^2\,\varphi_0 = \frac{1}{\alpha^2\,|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} h(z,\omega)\,d\omega =: \frac{h_0(z)}{\alpha^2} \quad \forall\, z \in \mathbb{R}\,,$$

with

$$|h_0(z)| \le O\left(e^{-\frac{n+2}{2}\,z}\right) \quad \text{as } z \to +\infty\,, \quad |h_0(z)| \sim e^{\left(-\frac{n+2}{2} + \frac{2\,p}{p-1}\right)z} \quad \text{as } z \to -\infty\,.$$

From the integral representation

$$\varphi_0(z) = -\frac{e^{a\,z}}{2\,a\,\alpha^2} \int_{-\infty}^{z} e^{-at}\,h_0(t)\,dt - \frac{e^{-a\,z}}{2\,a\,\alpha^2} \int_{z}^{\infty} e^{at}\,h_0(t)\,dt\,,$$

we deduce that as $z \to +\infty$, $\varphi_0(z) \sim e^{a\,z}$ and

$$\frac{\varphi_0'(z) - a\,\varphi_0(z)}{\varphi(z,\omega)} \sim e^{-2a\,z} \int_{z}^{\infty} e^{at}\,h_0(t)\,dt = O\,(e^{-2\,z})\,.$$

If we define the function $\psi(z,\omega) := e^{-a\,z}\left(\varphi(z,\omega) - \varphi_0(z)\right)$, we may observe that it is bounded for z positive and, moreover,

$$\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a = O\,(e^{-2\,z}) + \frac{\psi'(z,\omega)}{e^{-a\,z}\,\varphi(z,\omega)} \quad \text{as } z \to +\infty\,.$$

We recall that $e^{-a\,z}\,\varphi(z,\omega)$ is bounded away from 0 by a positive constant as $z \to +\infty$. Hence we know that

$$\left|\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a\right| \le O\left(|\psi'(z,\omega)|\right) + O\,(e^{-2\,z})\,. \tag{34}$$

By the Poincaré inequality and estimate (33) with $\ell = 0$, we have

$$\int_{\mathbb{S}^{d-1}} |\psi|^2\,d\omega = e^{-2az} \int_{\mathbb{S}^{d-1}} |\varphi - \varphi_0|^2\,d\omega \le \frac{e^{-2az}}{\lambda_1} \int_{\mathbb{S}^{d-1}} |\nabla_\omega \varphi|^2\,d\omega \le O\,(e^{-2z})\,.$$

Moreover, by the estimate (33) with $\ell = 1$, we also obtain

$$e^{-2az} \int_{\mathbb{S}^{d-1}} |\varphi' - \varphi_0'|^2\,d\omega \le \frac{e^{-2az}}{\lambda_1} \int_{\mathbb{S}^{d-1}} |\nabla_\omega \varphi'|^2\,d\omega \le O\,(e^{-2z})\,.$$

Hence, since $\psi' = -a\,\psi + e^{-az}\,(\varphi' - \varphi_0')$, the above estimates imply that

$$\int_{\mathbb{S}^{d-1}} |\psi|^2\,d\omega + \int_{\mathbb{S}^{d-1}} |\psi'|^2\,d\omega \le O\,(e^{-2z})\,,$$

which together with (34) ends the proof of (i).

To prove (iii), we first check that

$$\frac{\varphi''}{\varphi} - p\,\frac{|\varphi'|^2}{|\varphi|^2} + \left(1 - 2\,a\,(1-p)\right)\frac{\varphi'}{\varphi} + a^2\,(1-p) - a = O\,(|\psi'| + |\psi'|^2 + |\psi''|) + O\,(e^{-2\,z})\,,$$

and so it remains to prove that $\int_{\mathbb{S}^{d-1}} |\psi''|^2\,d\omega$ is of order $O\,(e^{-2\,z})$. Since

$$\psi'' = a^2\,\psi - 2\,a\,e^{-az}\,(\varphi' - \varphi_0') + e^{-az}\,(\varphi'' - \varphi_0'')\,,$$

using the above estimates, we have only to estimate the term with the second derivatives. This can be done as above by the Poincaré inequality,

$$e^{-2az}\int_{\mathbb{S}^{d-1}}|\varphi''-\varphi_0''|^2\,d\omega\le\frac{e^{-2az}}{\lambda_1}\int_{\mathbb{S}^{d-1}}|\nabla_\omega\varphi''|^2\,d\omega\le O\,(e^{-2z})\,,$$

based on the estimate (33) with $\ell=2$. This ends the proof of (iii). □

Proof of Proposition 5.1. It is straightforward to verify that the boundedness of P'', P'/s, P/s^2, $\nabla_\omega\mathsf{P}'/s$, $\nabla_\omega\mathsf{P}/s^2$, $\mathscr{L}_\alpha\mathsf{P}$ as $s\to+\infty$ and the integral estimates (i)–(v) as $s\to0^+$ from Lemma 5.4 are enough in order to establish (18), (24) and (25). □

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