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A note on Fröberg's conjecture for forms of equal degrees

Une note sur la conjecture de Fröberg pour des formes de degrés égaux

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A R T I C L E   I N F O

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A B S T R A C T

In this note we study ideals generated by generic forms in polynomial rings over any algebraicly closed field of characteristic zero. We prove for many cases that the $(d+k)$-th graded component of an ideal generated by generic forms of degree d has the expected dimension (given by dimension count). And as a consequence of our result, we obtain that ideals generated by several generic forms of degrees d usually have the expected Hilbert series. The precise form of this expected Hilbert series, in general, is known as Fröberg's conjecture.

R É S U M É

Dans cette note, nous étudions les idéaux générés par des formes génériques dans des anneaux de polynômes sur un champ algébriquement clos de caractéristique nulle. Nous montrons que, dans de nombreux cas, la $(d+k)$-ième composante graduelle d'un idéal engendré par les formes génériques de degré d a la dimension attendue (donnée par certains calculs). Comme une conséquence de notre résultat, nous obtenons que les idéaux générés par plusieurs formes génériques de degré d ont habituellement la série de Hilbert prévue. Cette dernière affirmation est connue comme la conjecture de Fröberg.

1. Introduction and results

Let $S=\mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring in n variables, where $\mathbb{K}$ is an algebraicly closed field of characteristic zero. The number of variables n and the field $\mathbb{K}$ will be fixed throughout the whole paper. Denote by S_d the d-th graded component of S, i.e. the linear space of all homogeneous polynomials of degree d in n variables.

In [3] R. Fröberg formulated the following conjecture:

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Conjecture 1. *Let $f_1, \dots, f_z$ be generic forms of degrees $a_1, \dots, a_z$ respectively. Set $I = <f_1, \dots, f_z>$. The Hilbert series of S/I is given by:*

$$HS_{S/I}(t) = \left[\frac{\prod_{i=1}^{z}(1-t^{a_i})}{(1-t)^n} \right],$$

where $[..]$ *means that we truncate a real formal power series at its first negative term.*

He proved Conjecture 1 for 2 variables and noticed that the left-hand side is bigger or equal than the right-hand side in the lexicographic sense. Later in [1] D.J. Anick proved Conjecture 1 for 3 variables. The conjecture is trivial, when $z \leqslant n$, and according to R. Stanley's result in [6] it is true for $z = n+1$. For 4 variables, in [5] J. Migliore and R.M. Miró-Roig proved that any ideal generated by generic forms has weak Lefschetz property (strong L.P. is enough for a proof of the conjecture). In this note we present some related results in the case when all degrees $a_1, \dots, a_z$ are the same. As a consequence of our result we construct first nontrivial infinite series of examples in case $n > 3$, when Fröberg's conjecture holds.

Let $\mathcal{D}_d$ be any nonempty class of forms of degree d closed under the linear changes of coordinates. For example: $\mathcal{D}_d = S_d$ or $\mathcal{D}_d$ is the set of all d-th powers of linear forms.

We will work with the Hilbert function of an ideal; it is easy to convert it to Hilbert function of the quotient algebra, because the sum of dimensions of m-th graded components of S/I and of I is the dimension of S_m. For $\mathcal{D}_d$ and z, denote by $HF_{(\mathcal{D}_d,z)}(m)$ the dimension of the m-th graded component of an ideal generated by z generic forms from $\mathcal{D}_d$; and denote by $HS_{(\mathcal{D}_d,z)}(t) = \sum HF_{(\mathcal{D}_d,z)}(m)t^m$ the Hilbert series of this ideal. In [4] M. Hochster and D. Laksov found the values of $HF_{(S_d,z)}(d+1)$ for any d and z. Below we generalize their result for the $(d+k)$-th graded component, but we miss $2 \cdot \dim(S_k)$ possible values of z.

Theorem 1. *Let d and k be positive integers. Then*

- *for* $z \leqslant \frac{\dim(S_{d+k})}{\dim(S_k)} - \dim(S_k)$, $HF_{(\mathcal{D}_d,z)}(d+k) = z \cdot \dim(S_k)$;
- *for* $z \geqslant \frac{\dim(S_{d+k})}{\dim(S_k)} + \dim(S_k)$, $HF_{(\mathcal{D}_d,z)}(d+k) = \dim(S_{d+k})$.

Remark 1. The condition about zero characteristic is important, we use it inside technical Lemma 1. For example, if $\mathbb{K}$ is a field of characteristic 2, $n = 3$, $d = 2$ and $\mathcal{D}_2$ is the set of squares of linear forms, then $HF_{(\mathcal{D}_2,z)}(3) \leqslant \dim(S_3) - 1$, because the form $x_1x_2x_3$ does not belong to the 3rd graded component for any z.

In [2], M. Aubry got the result of the first type; his result covers only a thin set of cases (d is larger than some complicated function of k and z). In [5], J. Migliore and R.M. Miró-Roig also wrote a similar result as a consequence of Anick's work; however, their result holds only for small z (the upper bound for z depends only on d, k; it does not depend on number of variables).

As a consequence of Theorem 1, we get the following statement.

Proposition 2. *Let d and z be positive integers. If there exists r such that*

$$\frac{\dim(S_{d+r+1})}{\dim(S_{r+1})} + \dim(S_{r+1}) \leqslant z \leqslant \frac{\dim(S_{d+r})}{\dim(S_r)} - \dim(S_r),$$

then the Hilbert series of the ideal generated by z generic forms from $\mathcal{D}_d$ is given by

$$HS_{(\mathcal{D}_d,z)} = \sum_{k=0}^{\infty} \min(z \cdot \dim(S_k), \dim(S_{d+k}))t^{d+k} = \frac{1}{(1-t)^n} - \left[\frac{(1-t^d)^z}{(1-t)^n} \right].$$

(*Recall that $\mathcal{D}_d$ is any nonempty subset of S_d closed under linear changes of coordinates.*)

Of course, all interesting cases correspond to $z \leqslant \dim(S_d)$; otherwise $HF_{(\mathcal{D}_d,z)}(m) = \dim(S_m)$ for $m \geqslant d$. Denote by $p_d = \frac{\#\{z \leqslant \dim(S_d) \text{ satisfying Proposition 2}\}}{\dim(S_d)}$ the "probability" that a given $z \leqslant \dim(S_d)$ is covered by Proposition 2.

Example 1. For $n = 5$ and $d = 10$, $\dim(S_d) = 1001$;

$$\dim(S_1) = 5 \text{ and } \frac{\dim(S_{d+1})}{\dim(S_1)} = 273;$$

$$\dim(S_2) = 15 \text{ and } \frac{\dim(S_{d+2})}{\dim(S_2)} = \frac{364}{3} = 121\frac{1}{3};$$

$$\dim(S_3) = 35 \text{ and } \frac{\dim(S_{d+3})}{\dim(S_3)} = 68.$$

Then the Hilbert series is given by Fröberg's conjecture at least if the number of generators z belongs to one of the following intervals:

- $z \geqslant 278$;
- $268 \geqslant z \geqslant 137$;
- $106 \geqslant z \geqslant 103$.

In other words, the Hilbert series is the standard one, except possibly for $141 = 9 + 30 + 102$ cases. Thus

$$p_{10} = 1 - \frac{141}{1001} = 0,859..$$

For larger d: $p_{15} = 0,927..$; $p_{25} = 0,968..$; $p_{40} = 0,986..$

Proposition 3. *For any fixed number of variables, the probability p_d tends to 1 as $d \to +\infty$.*

Proposition 3 means that Proposition 2 gives the criterion which covers a huge number of nontrivial cases for large d. As a consequence, we get that Fröberg's conjecture is true for many previously unknown cases for large d when the degrees of all forms are the same.

2. Proofs

For a proof of Theorem 1, we need the following technical lemma, which is true for fields of characteristic zero.

Lemma 1. *Let H be a nonempty linear subspace of S_d closed under the linear change of coordinates $x_1, \dots, x_n$, then H coincides with S_d.*

Proof. Our proof consists of two parts. (i) We choose a form $f(x_1, \dots, x_n) \in H$ which has all monomials of degree d with nonzero coefficients (choose any form and make generic change of coordinates).

(ii) Note that if $g \in S_d$, then polynomial $g(2x_1, x_2 \dots, x_n) - 2^t g(x_1, \dots, x_n)$ has a monomial $\beta = x_1^{\beta_1} \cdots, x_n^{\beta_n}$ with a nonzero coefficient if and only if $\beta \in \mathrm{supp}(g)$ and $\beta_1 \neq t$. Since $\mathrm{supp}(f)$ has all monomials of degree d, all monomials of degree d belong to a linear span of forms $f(2^{\alpha_1}x_1, \dots, 2^{\alpha_n}x_n)$, $\alpha_i \in \mathbb{N}$. □

Proof of Theorem 1. Fix d, k and $\mathcal{D}_d$. For a given z, define a_z as the dimension of the $(d+k)$-th graded component of the intersection of the two ideals generated by z forms from $\mathcal{D}_d$ and by one extra form from $\mathcal{D}_d$, which are generic. In other words, if $g_1, \dots, g_z, g$ are generic forms, then

$$a_z := \dim\left(<g_1, \dots, g_z>_{d+k} \cap <g>_{d+k}\right).$$

Lemma 2. *If $a_{z+1} = a_z \neq 0$, then $a_z = \dim(S_k)$ and $HF_{(\mathcal{D}_d, z)}(d+k) = \dim(S_{d+k})$.*

Proof. Consider generic forms $g_1, \dots, g_z, g'_1, \dots, g'_z$ and g from $\mathcal{D}_d$.

We know that

$$\dim\left(<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}\right) = a_z = a_{z+1} = \dim\left(<g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}\right).$$

We have

$$\dim\left(<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}\right) = \dim\left(<g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}\right).$$

The intersection in the left-hand side is a subspace of that in the right-hand side. Hence, they should coincide; we get

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}.$$

Similarly, we have

$$<g_1, \dots g_{z-1}, g'_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k},$$

which implies

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g'_z>_{d+k} \cap <g>_{d+k}.$$

Similarly, if we change g_{z-1} in right-hand side by the form g'_{z-1}, we get the same space. Repeating this procedure with g_{z-2}, g_{z-3}, etc., we obtain

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g'_1, \dots g'_{z-1}, g'_z>_{d+k} \cap <g>_{d+k}.$$

Hence for generic $g_1, \dots, g_z, g \in \mathcal{D}_d$, the linear space $V_g := <g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}$ depends only on g.

Fix any generic g, and choose a form $h \in V_g$. Hence for any generic $g_1, \ldots, g_z$, the form h belongs to the ideal. For a linear coordinate transformation A, denote by h_A the form h after this coordinate transformation. Consider coordinate transformations $A_1, \ldots, A_b$ (b is finite) such that the linear span of $h_{A_1}, \ldots, h_{A_b}$ has the maximal dimension.

For generic $g_1, \ldots, g_z$ (generic with these b coordinate transformations), the forms $h_{A_1}, \ldots, h_{A_b}$ belong to the ideal I generated by $\{g_1, \ldots, g_z\}$. Hence, the linear span of $h_{A_1}, \ldots, h_{A_b}$ belongs to the ideal I. Since this linear space has the maximal dimension, it is closed under the change of coordinates.

Hence, there is a nonempty linear subspace $H \subset S_{d+k}$ closed under the change of coordinates such that it belongs to any ideal generated by generic $\{g_1, \ldots, g_z\}$. By Lemma 1, H is the whole S_{d+k}. Then, the $(d+k)$-th graded component of the ideal is the whole S_{d+k}. Therefore, $a_z = \dim\left(S_{d+k} \cap <g>_{d+k}\right) = \dim(S_k)$. This proves the lemma. □

Let z_0 be the minimal z such that $a_z \neq 0$, and z_1 be the minimal z such that $a_{z_1} = \dim(S_k)$. By Lemma 1, the dimension a_z is strictly growing between z_0 and z_1, thus

$$z_1 - z_0 \leqslant \dim(S_k).$$

It is clear that

- for $z \leqslant z_0$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = z \cdot \dim(S_k)$;
- for $z \geqslant z_1$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = \dim(S_{d+k})$.

Since $z_0 \leqslant \frac{\dim(S_{d+k})}{\dim(S_k)}$ and $z_1 \geqslant \frac{\dim(S_{d+k})}{\dim(S_k)}$, we have

$$z_1 \leqslant z_0 + \dim(S_k) \leqslant \frac{\dim(S_{d+k})}{\dim(S_k)} + \dim(S_k);$$
$$z_0 \geqslant z_1 - \dim(S_k) \geqslant \frac{\dim(S_{d+k})}{\dim(S_k)} - \dim(S_k),$$

which gives the proof of the theorem. □

Remark 2. In fact, we proved that $HF_{(\mathcal{D}_d,z)}(d+k) = \min(z \cdot \dim(S_k), \dim(S_{d+k}))$ except for at most $\dim(S_k)$ possible values of z. However, we do not know these $\dim(S_k)$ values, we know the suspect interval of length $2\dim(S_k)$.

Proof of Proposition 2. By Theorem 1, we know that $HF_{(\mathcal{D}_d,z)}(d+r+1) = \dim(S_{d+r+1})$ and $HF_{(\mathcal{D}_d,z)}(d+r) = z \cdot \dim(S_r)$. From the first claim, we get that the $(d+r+1)$-th graded component of the ideal is S_{d+r+1}; hence for $k \geqslant r+1$, the $(d+k)$-th graded component of ideal is S_{d+k}.

From the second claim, we get that for generic $g_1, \ldots, g_z$ from $\mathcal{D}_d$, there are no $f_1, \ldots, f_z \in S_r$ (not all zeroes) such that $g_1 f_1 + \ldots + g_z f_z = 0$. Hence, there are no such $f_1, \ldots, f_z \in S_k$, for $k \leqslant r$. Then for $k \leqslant r$, we have $HF_{(\mathcal{D}_d,z)}(d+k) = z \cdot \dim(S_k)$. Hence in this case, the whole Hilbert series is given by Fröberg's conjecture. □

Proof of Proposition 3. Take an integer k. Then for large d, we know the Hilbert series for at least

$$\sum_{r=0}^{k} \left(\left(\frac{\dim(S_{d+r})}{\dim(S_r)} - \dim(S_r) \right) - \left(\frac{\dim(S_{d+r+1})}{\dim(S_{r+1})} + \dim(S_{r+1}) \right) \right)$$

different values of z (some of these summands can be negative). Then we have

$$1 - p_d \leq \frac{\dim(S_d) - \sum_{r=0}^{k} \left(\left(\frac{\dim(S_{d+r})}{\dim(S_r)} - \dim(S_r) \right) - \left(\frac{\dim(S_{d+r+1})}{\dim(S_{r+1})} + \dim(S_{r+1}) \right) \right)}{\dim(S_d)},$$
$$1 - p_d \leqslant \frac{\left(\frac{\dim(S_{d+k+1})}{\dim(S_{k+1})} \right)}{\dim(S_d)} + \frac{\sum_{r=0}^{k} (\dim(S_r) + \dim(S_{r+1}))}{\dim(S_d)}.$$

The fist summand tends to $\frac{1}{\dim(S_{k+1})}$ and the second one tends to zero as d increases. Hence, limsup of $(1 - p_d)$ is at most $\frac{1}{\dim(S_{k+1})}$. Therefore, $\lim_{d \to \infty}(1 - p_d) = 0$, because we have such a bound for any integer k. □

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