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Partial differential equations

A short remark on a growth–fragmentation equation

Une brève remarque sur une équation de croissance–fragmentation

Miguel Escobedo

Departamento de Matemáticas, Universidad del País Vasco (UPV/EHU), 48080 Bilbao, Spain

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A B S T R A C T

An explicit solution for a growth fragmentation equation with constant dislocation measure is obtained. In this example the necessary condition for the general results in [5] about the existence of global solutions in the so-called self-similar case is not satisfied. The solution is local and blows up in finite time.

R É S U M É

Nous obtenons une solution explicite d'une équation de croissance–fragmentation avec mesure de dislocation constante. Dans cet exemple, la condition nécessaire sous laquelle les résultats généraux d'existence de solutions globales obtenus dans [5] pour le cas dit self-similaire n'est pas satisfaite. La solution est locale et explose en temps fini.

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1. Introduction

The purpose of this note is to present an explicit solution that blows up in finite time to the growth fragmentation equation

$$\frac{\partial u}{\partial t}(t,x)+\frac{\partial}{\partial x}\left(x^{1+\gamma}u(t,x)\right)+x^{\gamma}u(t,x)=\int_x^{\infty}\frac{1}{y}k_0\left(\frac{x}{y}\right)y^{\gamma}u(t,y)\mathrm{d}y,\ \ t>0, x>0 \tag{1}$$

$$\gamma>0,\quad k_0(x)=\theta H(1-x),\quad \theta>1,\ H:\text{Heaviside's function}, \tag{2}$$

Motivated by the study of growth–fragmentation stochastic processes [6], this type of equation was considered recently by J. Bertoin and A.R. Watson in [5], with the initial data

$$u(0,x)=\delta(x-1), \tag{3}$$

E-mail address: miguel.escobedo@ehu.es.

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for $\gamma \in \mathbb{R}$ and k_0 a measure's density, with support contained in $[0, 1]$, that satisfies:

$$k_0(x)\mathrm{d}x = k_0(1-x)\mathrm{d}x, \ \ \forall x \in [1/2, 1); \qquad \int\limits_{[1/2,1)} (1-x)^2 k_0(x)\mathrm{d}x = 1. \tag{4}$$

These equations have proved to be interesting for mathematical reasons (cf. [5,7]) and also because of the great variety of their applications in mathematical modeling (cf. [3,8]).

For $\gamma = 0$, the existence and uniqueness of a non-negative solution to (1), (3) is proved in [5] under conditions (4) only. When $\gamma \neq 0$, the existence of a global solution in [5] is proved with the supplementary hypothesis

$$\inf_{s\geq 0} \Phi(s) < 0, \quad \text{where:} \tag{5}$$

$$\Phi(s) = (K(s) + s - 2)\,, \ \ K(s) = \int\limits_0^\infty x^{s-1} k_0(s)\mathrm{d}s. \tag{6}$$

After the results in [5] and [4], the importance of condition (5), (6) is well established for growth fragmentation processes, but it remains to be better understood for the growth fragmentation equation.

We are considering in this note the simplest possible choice for k_0, given in (2). It satisfies the condition (4), and is such that:

$$K(s) = \frac{\theta}{s} \ \text{ and } \ \Phi(s) = \frac{\theta}{s} + s - 2 \equiv \frac{(s-\sigma_1)(s-\sigma_2)}{s}, \ \ \forall s \in \mathbb{C}; \ \Re e(s) > 0, \tag{7}$$

where $\sigma_1 = 1 - \sqrt{1-\theta}$ and $\sigma_2 = 1 + \sqrt{1-\theta}$. For $\theta \in (0, 1)$ the two roots of $\Phi(s)$ are positive real numbers and then condition (5) is satisfied. But, for $\theta > 1$, σ_1 and σ_2 are complex conjugated, then $\inf_{s>0} \Phi(s) = 2(\sqrt{\theta} - 1)$ and (5) is not satisfied.

Our main result is the following Theorem, where $\mathscr{D}'_1$ denotes the set of distributions of order one.

Theorem 1.1. *For all $\gamma > 0$ the measure on $\left(0, \gamma^{-1}\right) \times (0, \infty)$, defined by:*

$$u(t,x) = u^S(t,x) + u^R(t,x) \tag{8}$$

$$u^S(t,x) = (1-\gamma t)^{\frac{1}{\gamma}}\, \delta\left(x - (1-\gamma t)^{-\frac{1}{\gamma}}\right) \tag{9}$$

$$u^R(t,x) = \theta\,(1-\gamma t)^{\frac{2}{\gamma}}\, tF\left(1 + \frac{\sigma_1}{\gamma}, 1 + \frac{\sigma_2}{\gamma}, 2, \gamma t\left(1 + (\gamma t - 1)\, x^\gamma\right)\right) H\left(1 - (1-\gamma t)^{\frac{1}{\gamma}}\, x\right), \tag{10}$$

is non-negative and satisfies equations (1), (2) in $\mathscr{D}'_1\left(\left(0, \gamma^{-1}\right) \times (0, \infty)\right)$. It also satisfies $u(t) \rightharpoonup \delta(x-1)$ in the weak sense of measures as $t \to 0$.

As a corollary we deduce the following.

Corollary 1.2. *The solution u defined in (8), (10) satisfies:*

$$\lim_{\gamma t \to 1^-} u(t,x) = \frac{\gamma\Gamma\left(\frac{2}{\gamma}\right)}{\Gamma\left(\frac{\sigma_1}{\gamma}\right)\Gamma\left(\frac{\sigma_2}{\gamma}\right)}(1+x^\gamma)^{-\frac{2}{\gamma}}, \ \ \forall x > 0 \tag{11}$$

and it blows up in finite time in the following sense:

$$\forall r > 1: \ \lim_{t\to\gamma^{-1}} (1-\gamma t)^{\frac{r-1}{\gamma}} \int\limits_0^\infty x^r u(t,x)\mathrm{d}x = \frac{\Gamma\left(\frac{r+1}{\gamma}\right)\Gamma\left(\frac{r-1}{\gamma}\right)}{\Gamma\left(\frac{r+1-\sigma_1}{\gamma}\right)\Gamma\left(\frac{r+1-\sigma_2}{\gamma}\right)}, \tag{12}$$

$$\lim_{t\to\gamma^{-1}} \frac{-1}{\log(1-\gamma t)} \int\limits_0^\infty x u(t,x)\mathrm{d}x = \frac{\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(\frac{\sigma_1}{\gamma}\right)\Gamma\left(\frac{\sigma_2}{\gamma}\right)}, \tag{13}$$

$$\forall r \in (0,1): \ \lim_{t\to\gamma^{-1}} \int\limits_0^\infty x^r u(t,x)\mathrm{d}x = \frac{\Gamma\left(\frac{r+1}{\gamma}\right)\Gamma\left(\frac{1-r}{\gamma}\right)}{\Gamma\left(\frac{\sigma_1}{\gamma}\right)\Gamma\left(\frac{\sigma_2}{\gamma}\right)}. \tag{14}$$

The question of the possible extension for $t > \gamma^{-1}$ is beyond the scope of this note. More general dislocation measures like $k_m(x) = (x^m + (1-x)^m)H(1-x)$ for $m = 1, 2, 3, \cdots$ may also be considered (cf. [9]), although the solutions are not always so explicit.

2. Mellin variables

If u were a suitable solution to (1), (3), applying the Mellin transform to both sides of (1) and (3), we would obtain for $\mathcal{M}_u$, the Mellin transform of u:

$$\frac{\partial}{\partial t}\mathcal{M}_u(t,s) = (K(s)+s-2)\mathcal{M}_u(t,s+\gamma) \tag{15}$$

$$\mathcal{M}_u(0,s) = 1. \tag{16}$$

Solutions to (15), (16) may be obtained by a general method, based on Wiener Hopf arguments (cf. [9] for details). For a description and applications of that method, the reader may consult [2]. However, in our case, the problem (15), (16) has a particularly simple explicit solution.

If $F(a,b,c,z)$ denotes the Gauss hypergeometric function ${}_2F_1(a,b,c,z)$ (see for example [1]), it follows from the identities 15.2.1 and 15.3.3 in [1], that the function:

$$\Omega(t,s) = F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \equiv (1-\gamma t)^{\frac{2-s}{\gamma}} F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \tag{17}$$

satisfies (15), (16) for t and s in the domain of analyticity of $F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right)$ such that $t \neq \gamma^{-1}$.

Our purpose is to define the function u as the inverse Mellin transform of Ω, to prove that the Mellin transform of u is Ω and then to prove that u solves (1), (3).

3. The inverse Mellin transform of $\Omega(t,s)$

We first show the following Proposition, where $\mathscr{M}(0,\infty)$ denotes the space of non-negative locally bounded measures on $(0,\infty)$.

Proposition 3.1. *For all $t \in \left(0,\gamma^{-1}\right)$ the function $\Omega(t,s)$ defined in (17) has an inverse Mellin transform that belongs to $\mathscr{M}(0,\infty)$, that we denote $u(t,x) = \mathcal{M}^{-1}(\Omega)$, and that satisfies*

$$\mathcal{M}_u(t,s) = \Omega(t,s), \quad \forall s \in \mathbb{C};\ \Re e(s) > 0,\ t \in \left(0,\gamma^{-1}\right). \tag{18}$$

Proof. For $0 < t < \gamma^{-1}$ fixed, the hypergeometric function $F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right)$ is analytic in the domain $D = \{s \in \mathbb{C}; \Re e(s) > 0\}$ and by 15.7.1 in [1]

$$\left|F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1 - \frac{2t}{s}\right| \le C(1+|s|)^{-2}, \quad \forall s \in D,$$

for some constant $C = C(t,\sigma_1,\sigma_2,\gamma)$. Then, by Theorem 11.10.1 in [10], for all $t \in \left(0,\gamma^{-1}\right)$, the function $\Omega(t)$ has an inverse Mellin transform $u(t) \in \mathscr{M}(0,\infty)$, given by

$$u(t,x) = \frac{1}{2i\pi}\int_{\Re e(s)=s_0} x^{-s}\Omega(t,s)ds,$$

for an arbitrary $s_0 > 0$, and such that for all $s \in D$, $\mathcal{M}_{u(t)}(s) = \Omega(t,s)$. $\square$

We may obtain now the explicit expression of u. Let us prove first the following Proposition.

Proposition 3.2. *Suppose $\sigma_1 \in \mathbb{C}$, $\sigma_2 \in \mathbb{C}$, $\gamma > 0$ and $t \in \left(0,\gamma^{-1}\right)$ and define the function*

$$v(t,x) = F\left(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma}, 2, \gamma t\left(1+(\gamma t-1)x^{\gamma}\right)\right) H\left(1-(1-\gamma t)^{\frac{1}{\gamma}}x\right) \tag{19}$$

for $x > 0$. Then, for all $t \in \left(0,\gamma^{-1}\right)$, the Mellin transform of v is:

$$\mathcal{M}_v(t,s) = (1-\gamma t)^{-\frac{s}{\gamma}} \frac{F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1}{\theta t}, \quad \forall s \in \mathbb{C};\ \Re e(s) > 0. \tag{20}$$

Proof. Since $\gamma > 0$ and $1 - \gamma t > 0$, it follows that for all $x > 0$, $\left(1 + (\gamma t - 1)x^{\gamma}\right) < 1$. Then:

$$F\left(1 + \frac{\sigma_1}{\gamma}, 1 + \frac{\sigma_2}{\gamma}, 2, \gamma t\left(1 + (\gamma t - 1)x^{\gamma}\right)\right) = \sum_{n=0}^{\infty} \frac{\Gamma\left(1 + \frac{\sigma_1}{\gamma} + n\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma} + n\right)\Gamma(2)(\gamma t)^n}{\Gamma\left(1 + \frac{\sigma_1}{\gamma}\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma}\right)\Gamma(2+n)\Gamma(n+1)}(1 + (\gamma t - 1)x^{\gamma})^n,$$

$$\mathcal{M}_v(t,s) = \int_0^{\infty} v(t,x)x^{s-1}\mathrm{d}x = \sum_{n=0}^{\infty} \frac{\Gamma\left(1 + \frac{\sigma_1}{\gamma} + n\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma} + n\right)\Gamma(2)(\gamma t)^n}{\Gamma\left(1 + \frac{\sigma_1}{\gamma}\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma}\right)\Gamma(2+n)\Gamma(n+1)} \int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} (1 + (\gamma t - 1)x^{\gamma})^n x^{s-1}\mathrm{d}x. \tag{21}$$

A straightforward calculation gives, using $\gamma > 0$:

$$\int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} (1 + (\gamma t - 1)x^{\gamma})^n x^{s-1}\mathrm{d}x = (1 - \gamma t)^{-\frac{s}{\gamma}} \frac{\Gamma(n+1)\Gamma\left(\frac{s}{\gamma}\right)}{\gamma\Gamma\left(1 + \frac{s}{\gamma} + n\right)},$$

and then, for all $s \in \mathbb{C}$ such that $-s/\gamma \notin \mathbb{N}$:

$$\begin{aligned}\mathcal{M}_v(t,s) &= (1 - \gamma t)^{-\frac{s}{\gamma}} \sum_{n=0}^{\infty} \frac{\Gamma\left(1 + \frac{\sigma_1}{\gamma} + n\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma} + n\right)\Gamma(2)(\gamma t)^n}{\Gamma\left(1 + \frac{\sigma_1}{\gamma}\right)\Gamma\left(1 + \frac{\sigma_2}{\gamma}\right)\Gamma(2+n)\Gamma(n+1)} \frac{\Gamma(n+1)\Gamma\left(\frac{s}{\gamma}\right)}{\gamma\Gamma\left(1 + \frac{s}{\gamma} + n\right)} \\ &= (1 - \gamma t)^{-\frac{s}{\gamma}} \frac{F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1}{\sigma_1\sigma_2 t}. \quad \square\end{aligned}$$

The next Corollary follows from Proposition 3.2 and Theorem 11.10.1 in [10] on the uniqueness of the inverse Mellin transform:

Corollary 3.1. *For all $\sigma_1 \in \mathbb{C}$, $\sigma_2 \in \mathbb{C}$, suppose that $\gamma > 0$, $0 < \gamma t < 1$ and let ω be the measure:*

$$\omega(t,x) = (1 - \gamma t)^{\frac{1}{\gamma}}\delta\left(x - (1 - \gamma t)^{-\frac{1}{\gamma}}\right) + \theta t(1 - \gamma t)^{\frac{2}{\gamma}} v(t,x).$$

Then, for all $t \in \left(0, \gamma^{-1}\right)$:

$$\mathcal{M}_{\omega}(t,s) = \Omega(t,s) \ \ \textit{for all} \ \ s \in \mathbb{C}, \ \Re e(s)s > 0$$

and $u(t) = \omega(t)$ for all $t \in \left(0, \gamma^{-1}\right)$.

We may prove now our main result.

Proof of Theorem 1.1. It is easy to check that $u(t) \rightharpoonup \delta(x-1)$ as $t \to 0$ in the weak sense of measures. We already know that $\Omega(t,s) = \mathcal{M}_u(t,s)$ solves the problem (15), (16). Applying the inverse Mellin transform to both sides of the equation (15), we deduce the following equation in $\mathscr{D}'\left(\left(0, \gamma^{-1}\right) \times (0, \infty)\right)$:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2i\pi} \int_{\Re es = \sigma_0} \left(\frac{\theta}{s} + (s-1) - 1\right)\mathcal{M}_u(t, s+\gamma)x^{-s}\mathrm{d}s. \tag{22}$$

We consider now each of the terms in the right and side separately. Since $\sigma_0 > 0$, $\gamma > 0$, using that $\mathcal{M}_u(t,s) = \Omega(t,s)$ for all $\Re e(s) > 0$ we write $\gamma > 0$:

$$\begin{aligned}\frac{1}{2i\pi} \int_{\Re es = \sigma_0} \mathcal{M}_u(t, s+\gamma)x^{-s}\mathrm{d}s &= \frac{1}{2i\pi} \int_{\Re es = \sigma_0} \int_0^{\infty} u(t,y)y^{s+\gamma-1}\mathrm{d}y x^{-s}\mathrm{d}s \\ &= \int_0^{\infty} u(t,y)y^{\gamma-1}\left(\frac{1}{2i\pi} \int_{\Re es = \sigma_0} \left(\frac{x}{y}\right)^{-s}\mathrm{d}s\right) = x^{\gamma}u(t,x).\end{aligned} \tag{23}$$

The second term in the right-hand side of (22) is given by the classical formula:

$$\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}(s-1)\mathcal{M}_u(t,s+\gamma)x^{-s}ds=\frac{\partial}{\partial x}\left(\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}\mathcal{M}_u(t,s+\gamma)x^{1-s}ds\right) \tag{24}$$

In the last term in the right-hand side of (22), we write as above:

$$\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}\frac{\theta}{s}\mathcal{M}_u(t,s+\gamma)x^{-s}ds=\int_0^\infty u(t,y)\left(\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}\frac{\theta}{s}y^{s+\gamma-1}x^{-s}ds\right)dy.$$

Using that for $\sigma_0>0$:

$$\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}\frac{1}{s}y^{s+\gamma-1}x^{-s}ds=\begin{cases}0, & \text{if } y<x\\ y^{\gamma-1}, & \text{if } y>x\end{cases}$$

we deduce

$$\frac{1}{2i\pi}\int_{\Re e s=\sigma_0}\frac{\theta}{s}\mathcal{M}_u(t,s+\gamma)x^{-s}ds=\theta\int_x^\infty u(t,y)y^{\gamma-1}dy. \tag{25}$$

Since $u\in C\left([0,\gamma^{-1});\mathscr{M}(0,\infty)\right)\cap C^1\left((0,\gamma^{-1});\mathscr{D}_1'(0,\infty)\right)$ it follows from (22)–(25) that both sides of the equation (1) are equal in $C\left((0,\gamma^{-1});\mathscr{D}_1'(0,\infty)\right)$ and then, for all $\varphi\in C_c^1\left(\left(0,\frac{1}{\gamma}\right)\times(0,\infty)\right)$:

$$\left\langle u_t(t)+\frac{\partial}{\partial x}\left(x^{\gamma+1}u(t)\right)+x^\gamma u(t),\varphi(t)\right\rangle=\theta\left\langle\int_x^\infty y^{\gamma-1}u(t,y)dy,\varphi(t)\right\rangle \tag{26}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between $\mathscr{D}_1'\left((0,\gamma^{-1})\times(0,\infty)\right)$ and $C_c^1\left((0,\gamma^{-1})\times(0,\infty)\right)$.

Since $\sigma_1=\overline{\sigma_2}$, $\Gamma\left(1+\frac{\sigma_1}{\gamma}+n\right)=\overline{\Gamma\left(1+\frac{\sigma_2}{\gamma}+n\right)}$ for all $n\in\mathbb{N}$, and the positivity of u follows. □

Proof of Corollary 1.2. Properties (12)–(14) follow from the explicit expressions of $\Omega(t,s)$ and u^R in (10), (17) and formulas 15.4 (ii) in [11]. □

Remark 1. Due to the particular form of the measure u, it is easy to check that (26) is satisfied for all $\varphi\in C_c^1\left(\left[0,\frac{1}{\gamma}\right)\times(0,\infty)\right)$. We also deduce from (26) that u^S and u^R satisfy:

$$\frac{\partial u^S}{\partial t}+\frac{\partial}{\partial x}\left(x^{\gamma+1}u^S\right)+x^\gamma u^S=0,\ \text{ in } \mathscr{D}_1'(0,\infty) \tag{27}$$

and for all $t\in(0,\gamma^{-1})$ and $x\in(0,(1-\gamma t)^{-1/\gamma})$:

$$\frac{\partial u^R}{\partial t}(t,x)+\frac{\partial}{\partial x}\left(x^{\gamma+1}u^R\right)(t,x)+x^\gamma u^R(t,x)=\theta(1-\gamma t)^{-1}+\theta\int_x^\infty u^R(t,y)y^{\gamma-1}dy. \tag{28}$$

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