C. R. Acad. Sci. Paris, Ser. I 355 (2017) 299–303

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

Instability of an integrable nonlocal NLS

De l'instabilité d'une équation NLS non locale intégrable

François Genoud

Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

ARTICLE INFO

Article history:
Received 9 October 2016
Accepted after revision 26 January 2017
Available online 9 February 2017

Presented by Haïm Brézis

ABSTRACT

In this note, we discuss the global dynamics of an integrable nonlocal NLS on $\mathbb{R}$, which has been the object of a recent investigation by integrable systems methods. We prove two results that are in striking contrast with the case of the local cubic focusing NLS. First, finite-time blow-up solutions exist with arbitrarily small initial data in $H^s(\mathbb{R})$, for any $s \geqslant 0$. On the other hand, the solitons of the local NLS, which are also solutions to the nonlocal equation, are unstable by blow-up for the latter.

RÉSUMÉ

Nous discutons dans cette note la dynamique globale d'une équation NLS intégrable non locale sur $\mathbb{R}$, qui a été étudiée récemment par des méthodes de systèmes intégrables. Nous démontrons deux résultats qui contrastent fortement avec le cas de l'équation NLS focalisante cubique locale. Premièrement, il existe des solutions qui explosent en temps fini, avec condition initiale arbitrairement petite dans $H^s(\mathbb{R})$, pour tout $s \geqslant 0$. Par ailleurs, les solitons de l'équation NLS locale, qui sont aussi solutions de l'équation non locale, sont instables par explosion pour cette dernière.

1. Introduction

The nonlocal nonlinear Schrödinger equation

$$iu_t(t,x) + u_{xx}(t,x) + u^2(t,x)\bar{u}(t,-x) = 0, \qquad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \tag{1.1}$$

has recently been shown to be a completely integrable system, with infinitely many conservation laws [1,2]. The equation is related to two different areas of physics: gain/loss systems in optics and so-called PT-symmetric quantum mechanics, see [3–6] and references therein. Mathematically, the feature connecting (1.1) to these areas is the PT-symmetry of the

E-mail address: s.f.genoud@tudelft.nl.

http://dx.doi.org/10.1016/j.crma.2017.01.018

'nonlinear potential' $u(t,x)\overline{u}(t,-x)$. Namely, this quantity is invariant under the joint transformation $x \to -x$ and $\mathrm{i} \to -\mathrm{i}$ (parity and time reversal).

The inverse-scattering transform was applied in [1,2] to produce a variety of solutions to (1.1). In particular, a 'one-soliton solution' is obtained, which blows up in finite time (actually, up to rescaling, at all times $t = 2m+1$, $m \in \mathbb{Z}$). The purpose of this note is to use this peculiar solution to prove some results about the global dynamics of (1.1), which are in striking contrast with the case of the local focusing cubic equation

$$\mathrm{i}u_t(t,x) + u_{xx}(t,x) + |u(t,x)|^2 u(t,x) = 0, \qquad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}. \tag{1.2}$$

We will first show that (1.1) is locally well posed (in $H^1(\mathbb{R})$), but then we prove that there exist solutions that blow up in finite time (in $L^\infty(\mathbb{R})$), with arbitrarily small initial data in $H^s(\mathbb{R})$, for any $s \geqslant 0$. This shows in particular that the trivial solution, $u \equiv 0$, is unstable by blow-up.

Let us now observe that (1.1) reduces to (1.2) provided the discussion is restricted to even solutions. Hence, the well-known solitons $u_\omega(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x)$ of (1.2), where

$$\varphi_\omega(x) = \frac{2\sqrt{2\omega}}{\mathrm{e}^{\sqrt{w}x} + \mathrm{e}^{-\sqrt{w}x}}, \qquad \omega > 0, \tag{1.3}$$

are also solutions to (1.1). These standing waves are orbitally stable with respect to (1.2), but we show that they are unstable by blow-up with respect to (1.1).

The rest of the paper is organized as follows. In Section 2, we prove the local well-posedness and the blow-up instability of the zero solution. In Section 3 we prove the blow-up instability of the solitons (1.3). We conclude in Section 4 with some remarks on the 'defocusing' equation

$$\mathrm{i}u_t(t,x) + u_{xx}(t,x) - u^2(t,x)\overline{u}(t,-x) = 0, \qquad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}. \tag{1.4}$$

Notation

For non-negative quantities A, B, we write $A \lesssim B$ if $A \leqslant CB$ for some constant $C > 0$, whose exact value is not essential to the analysis.

2. Instability of the trivial solution

We start with a local well-posedness result.

Theorem 1. *Given any initial data $u_0 \in H^1(\mathbb{R})$, there exists a unique maximal solution $u \in C\big([0, T_{\max}), H^1(\mathbb{R})\big)$ of (1.1) such that $u(0,\cdot) = u_0$, where $T_{\max} = T_{\max}(\|u_0\|_{H^1(\mathbb{R})})$.*

Proof. The theorem is proved by a fixed point argument, similar to the case of the local equation (1.2). However, some calculations are different due to the nonlocal nonlinearity, so we give the proof here for completeness.

Fix $u_0 \in H^1(\mathbb{R})$, define $F(u)(x) = u^2(x)\overline{u}(-x)$ and a map $\tau : X_T \to X_T$ by

$$\tau(u)(t) = S(t)u_0 + \mathrm{i}\int_0^t S(t-s)F(u)(s)\,\mathrm{d}s,$$

where $X_T = L^\infty\big((0,T); H^1(\mathbb{R})\big)$ for some $T > 0$ and $S(t)$ is the free Schrödinger group. We shall prove the existence of a unique fixed point of τ in the ball

$$B_R = \{u \in X_T : \|u\|_{X_T} < R\},$$

for suitable values of $T, R > 0$. That this fixed point can be extended to a maximal solution $u \in C\big([0, T_{\max}), H^1(\mathbb{R})\big)$ of (1.1) then follows by standard arguments.

First observe that, for any $p \geqslant 2$, the Sobolev embedding theorem yields

$$\begin{aligned}
\|F(u)\|_{L^p}^p &= \int_{\mathbb{R}} |u(x)|^{2p}|u(-x)|^p\,\mathrm{d}x \\
&\leqslant \left\{\int_{\mathbb{R}} |u(x)|^{2pr}\,\mathrm{d}x\right\}^{1/r}\left\{\int_{\mathbb{R}} |u(-x)|^{ps}\,\mathrm{d}x\right\}^{1/s} \\
&= \|u\|_{L^{2pr}}^{2p}\,\|u\|_{L^{ps}}^p \lesssim \|u\|_{H^1}^{3p},
\end{aligned}$$

where $r, s \geqslant 1$ are arbitrary Hölder conjugate exponents. It follows that

$$\|F(u)\|_{L^p} \lesssim \|u\|_{H^1}^3 \quad \text{for any } p \geqslant 2, \tag{2.1}$$

and a similar estimate yields

$$\|F(u)_x\|_{L^p} \lesssim \|u\|_{H^1}^3 \quad \text{for any } p \geqslant 2, \tag{2.2}$$

where

$$F(u)_x = \left[u^2(x)\overline{u}(-x)\right]_x = 2u(x)u_x(x)\overline{u}(-x) - u^2(x)\overline{u}_x(-x). \tag{2.3}$$

By Strichartz's estimate and (2.1)–(2.2) with $p = 2$, we see in particular that τ indeed maps X_T into X_T. Furthermore, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\|\tau(u)\|_{X_T} &\leqslant C_1 \|u_0\|_{H^1} + T\,\|F(u)\|_{L^\infty(0,T;H^1)} \\ &\leqslant C_1 \|u_0\|_{H^1} + TC_2 \|u\|_{X_T}^3 .\end{aligned}$$

Choosing $R = 2C_1 \|u_0\|_{H^1}$ and $T > 0$ such that $C_1 T R^2 = 1/2$, it follows that, for any $u \in B_R$,

$$\|\tau(u)\|_{X_T} \leqslant \frac{R}{2} + TC_1 \|u\|_{X_T}^2 \|u\|_{X_T} \leqslant \frac{R}{2} + \frac{1}{2}\|u\|_{X_T} < R.$$

Hence, for these values of $T, R > 0$, τ maps the ball B_R into itself.

We now show that, if $T > 0$ is small enough, then τ is a contraction in B_R. We have

$$\|\tau(u) - \tau(v)\|_{X_T} \leqslant T\,\|F(u) - F(v)\|_{L^\infty(0,T;H^1)}, \quad u, v \in X_T. \tag{2.4}$$

Writing $|F(u) - F(v)| = |\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} F(\theta u + (1-\theta)v)\,\mathrm{d}\theta|$, we obtain

$$|F(u) - F(v)| \lesssim |u(x) + v(x)||u(x) - v(x)||u(-x) + v(-x)| + |u(x) + v(x)|^2 |u(-x) - v(-x)|$$

and it follows that

$$\|F(u) - F(v)\|_{L^2} \lesssim \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right) \|u - v\|_{L^2} . \tag{2.5}$$

On the other hand, in view of (2.3), letting

$$G(u)(x) = 2u(x)u_x(x)\overline{u}(-x) \quad \text{and} \quad H(u)(x) = u^2(x)\overline{u}_x(-x)$$

we have

$$|[F(u) - F(v)]_x| \leqslant |G(u) - G(v)| + |H(u) - H(v)|,$$

where

$$\begin{aligned}|G(u) - G(v)| \lesssim\; &|u(x) - v(x)||u_x(x) + v_x(x)||u(-x) + v(-x)| \\ &+ |u(x) + v(x)||u_x(x) - v_x(x)||u(-x) + v(-x)| \\ &+ |u(x) + v(x)||u_x(x) + v_x(x)||u(-x) - v(-x)|\end{aligned}$$

and

$$\begin{aligned}|H(u) - H(v)| \lesssim\; &|u(x) + v(x)||u(x) - v(x)||u_x(-x) + v_x(-x)| \\ &+ |u(x) + v(x)|^2 |u_x(-x) + v_x(-x)|.\end{aligned}$$

It follows that

$$\|[F(u) - F(v)]_x\|_{L^2} \lesssim \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right) \|u - v\|_{H^1} . \tag{2.6}$$

By (2.4), (2.5) and (2.6), there is a constant $C > 0$ such that

$$\|\tau(u) - \tau(v)\|_{X_T} \leqslant CT\left(\|u\|_{X_T}^2 + \|v\|_{X_T}^2\right) \|u - v\|_{X_T} .$$

Hence, if $u, v \in B_R$ we have

$$\|\tau(u) - \tau(v)\|_{X_T} \leqslant 2CTR^2 \|u - v\|_{X_T} ,$$

showing that τ is a contraction in B_R provided $T < (2CR^2)^{-1}$. The contraction mapping principle now yields a unique fixed point of τ in B_R, which concludes the proof. □

For the local equation (1.2), the next chapter of the story is well known. One proves that, for any $u_0 \in H^1(\mathbb{R})$, the maximal solution is global, i.e. that $T_{\max} = \infty$. This is usually done by means of the energy and charge functionals

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} |u|^4 \,\mathrm{d}x, \qquad Q(u) = \frac{1}{2}\int_{\mathbb{R}} |u|^2 \,\mathrm{d}x.$$

Using the conservation of these quantities along the flow and the Gagliardo–Nirenberg inequality, one shows that the first term in E is controlled by the second one, and must remain bounded. Hence, global existence in $H^1(\mathbb{R})$ is ensured by the blow-up alternative.

The corresponding conservation laws for (1.1) are [1,2]

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} u_x(x)\overline{u}_x(-x) \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} u^2(x)\overline{u}^2(-x) \,\mathrm{d}x \quad \text{and} \quad Q(u) = \frac{1}{2}\int_{\mathbb{R}} u(x)\overline{u}(-x) \,\mathrm{d}x.$$

Even though each of these integrals is real, in general none of the three terms has a definite sign, unless u is even (or odd), in which case we recover the energy and charge of the local equation (1.2). This predicament wipes away any hope of proving a global well-posedness result for (1.1), even for small initial data. In fact, we have the following result.

Theorem 2. *For any $0 < \alpha < 1$, there exists a solution $u^\alpha(t, x)$ to (1.1), defined on $[0, T_\alpha) \times \mathbb{R}$, where $T_\alpha = \pi/3\alpha^2$, with the following properties:*

(i) *u^α blows up in $L^\infty(\mathbb{R})$ as $t \to T_\alpha$, with $\lim_{t\to T_\alpha} |u^\alpha(t, 0)| = \infty$;*
(ii) *$u_0^\alpha = u^\alpha(0, \cdot)$ satisfies $\|u_0^\alpha\|_{H^k(\mathbb{R})} \lesssim \alpha^{1/2}$, for all $k \in \mathbb{N}$.*

Proof. The result is obtained from the explicit solution

$$u^{\alpha,\beta}(t, x) = \frac{2\sqrt{2}(\alpha + \beta)}{\mathrm{e}^{-4\mathrm{i}\alpha^2 t}\mathrm{e}^{2\alpha x} + \mathrm{e}^{-4\mathrm{i}\beta^2 t}\mathrm{e}^{-2\beta x}}. \tag{2.7}$$

For any $\alpha, \beta > 0, \alpha \neq \beta$, this function blows up at all times

$$T_m = \frac{(2m + 1)\pi}{4(\alpha^2 - \beta^2)}, \qquad m \in \mathbb{Z},$$

with $\lim_{t\to T_m} |u^{\alpha,\beta}(t, 0)| = \infty$, and is a solution to (1.1) in the sense of Theorem 1 between these times, i.e. $u^{\alpha,\beta} \in C\big((T_m, T_{m+1}), H^1(\mathbb{R})\big)$, $m \in \mathbb{Z}$. To simplify the analysis, we choose $\beta = \alpha/2$, so that $u^{\alpha,\beta}$ reduces to

$$u^{\alpha}(t, x) = \frac{3\sqrt{2}\alpha}{\mathrm{e}^{-4\mathrm{i}\alpha^2 t}\mathrm{e}^{2\alpha x} + \mathrm{e}^{-4\mathrm{i}\beta^2 t}\mathrm{e}^{-\alpha x}}, \tag{2.8}$$

and the first blow-up time to the right of $t = 0$ becomes

$$T_\alpha = \frac{\pi}{3\alpha^2}.$$

For the initial condition $u_0^\alpha = u^\alpha(0, \cdot)$, direct calculations then show that

$$\|u_0^\alpha\|_{L^2}^2 = \frac{4\pi\alpha}{3}, \quad \|(u_0^\alpha)_x\|_{L^2}^2 = \frac{8\pi\alpha^3}{3\sqrt{3}}, \quad \|(u_0^\alpha)_{xx}\|_{L^2}^2 = \frac{8\pi\alpha^5}{\sqrt{3}}. \tag{2.9}$$

Upon inspection of the integrals involved, one easily sees that, for all $k \in \mathbb{N}$, there is a constant $C_k > 0$, independent of α, such that

$$\left\| \frac{\mathrm{d}^k u_0^\alpha}{\mathrm{d}x^k} \right\|_{L^2}^2 = C_k \alpha^{2k+1}. \tag{2.10}$$

For $\alpha \in (0, 1)$, this completes the proof. □

Remark 1. (a) If $\alpha = \beta = \sqrt{\omega}/2$, then $u^{\alpha,\beta}(t, x)$ reduces to the usual soliton $\mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x)$, with φ_ω defined in (1.3).

(b) A direct verification shows that the solution $u^{\alpha,\beta}(t, x)$ only blows up at $x = 0$, i.e. the denominator in (2.7) never vanishes if $x \neq 0$.

(c) The particular choice $\beta = \alpha/2$ enables one to compute explicitly the norms in (2.9). In fact, the relations (2.10) are easily derived by choosing $\beta = \gamma\alpha$ with, say, $\gamma \in (0, 1)$, and using the change of variables $y = \alpha x$ in the integrals.

3. Instability of the solitons

The blow-up instability of the solitons (1.3) is now a consequence of Remark 1(a). More precisely, fixing $\alpha = \sqrt{\omega}/2$ and letting $\beta = \sqrt{\omega+\delta}/2$ with $0 < \delta \ll 1$, we obtain finite-time blow-up solutions $u^{\alpha,\beta}$ as close as we want to $e^{i\omega t}\varphi_\omega(x)$.

Theorem 3. *Fix $\omega > 0$. For any $\epsilon > 0$, there exists $q_{\omega,\epsilon} \in H^1(\mathbb{R})$ such that*

$$\|\varphi_\omega - q_{\omega,\epsilon}\|_{H^1(\mathbb{R})} < \epsilon$$

and the solution with initial data $u(0,\cdot) = q_{\omega,\epsilon}$ blows up in finite time.

Proof. Define $q_{\omega,\delta}(x)$ as $u^{\alpha,\beta}(0,x)$, with $\alpha = \sqrt{\omega}/2$ and $\beta = \sqrt{\omega+\delta}/2$, $\delta > 0$, namely

$$q_{\omega,\delta}(x) = \frac{\sqrt{2}(\sqrt{\omega} + \sqrt{\omega+\delta})}{e^{\sqrt{w}x} + e^{-\sqrt{w+\delta}x}}.$$

We only need to check that

$$\|\varphi_\omega - q_{\omega,\delta}\|_{H^1} \to 0 \quad \text{as } \delta \to 0. \tag{3.1}$$

To show that

$$\int_{\mathbb{R}} |\varphi_\omega(x) - q_{\omega,\delta}(x)|^2 \, dx \to 0 \quad \text{as } \delta \to 0, \tag{3.2}$$

we first observe that $|\varphi_\omega(x) - q_{\omega,\delta}(x)| \to 0$ as $\delta \to 0$ for all $x \in \mathbb{R}$. Furthermore, if $0 < \delta < 1$, we have, for $-\infty < x \leqslant 0$,

$$\frac{\sqrt{2}(\sqrt{\omega} + \sqrt{\omega+\delta})}{e^{\sqrt{w}x} + e^{-\sqrt{w+\delta}x}} \leqslant \frac{\sqrt{2}(\sqrt{\omega} + \sqrt{\omega+1})}{e^{\sqrt{w}x} + e^{-\sqrt{w}x}},$$

while, for $0 < x < \infty$,

$$\frac{\sqrt{2}(\sqrt{\omega} + \sqrt{\omega+\delta})}{e^{\sqrt{w}x} + e^{-\sqrt{w+\delta}x}} \leqslant \frac{\sqrt{2}(\sqrt{\omega} + \sqrt{\omega+1})}{e^{\sqrt{w}x} + e^{-\sqrt{w+1}x}},$$

and so (3.2) follows by dominated convergence. Applying similar estimates to the derivative

$$(q_{\omega,\delta})_x(x) = \sqrt{2}(\sqrt{\omega} + \sqrt{\omega+\delta}) \frac{\sqrt{w+\delta}\, e^{-\sqrt{w+\delta}x} - \sqrt{w}\, e^{\sqrt{w}x}}{(e^{\sqrt{w}x} + e^{-\sqrt{w+\delta}x})^2}$$

and using again dominated convergence, we also have

$$\int_{\mathbb{R}} |(\varphi_\omega)_x(x) - (q_{\omega,\delta})_x(x)|^2 \, dx \to 0 \quad \text{as } \delta \to 0,$$

from which the conclusion follows. □

4. Remarks on the defocusing case

The 'defocusing' equation (1.4) has also been considered in [1,2]. Our local well-posedness result, Theorem 1, carries over to (1.4), with an identical proof. On the other hand, it is shown in [2, p. 936] that 'one-soliton' solutions of the type (2.7) are not available in the defocusing case. Global well-posedness for (1.4) seems to be an open problem.

Acknowledgements

I am grateful to Stefan Le Coz for an interesting discussion about (1.1), and to the anonymous referee, whose remarks helped improve the presentation of the paper.

References

[1] M.J. Ablowitz, Z.H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett. 110 (2013) 064105.
[2] M.J. Ablowitz, Z.H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Nonlinearity 29 (2016) 915.
[3] C.M. Bender, D.C. Brody, H.F. Jones, Complex extension of quantum mechanics, Phys. Rev. Lett. 89 (2002) 270401.
[4] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, Int. J. Geom. Methods Mod. Phys. 07 (2010) 1191.
[5] Z.H. Musslimani, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, Optical solitons in $\mathcal{PT}$ periodic potentials, Phys. Rev. Lett. 100 (2008) 030402.
[6] S. Nixon, L. Ge, J. Yang, Stability analysis for solitons in $\mathcal{PT}$-symmetric optical lattices, Phys. Rev. A 85 (2012) 023822.