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Calculus of variations

Pathological solutions to the Euler–Lagrange equation and existence/regularity of minimizers in one-dimensional variational problems ☆

Solutions pathologiques à l'équation d'Euler–Lagrange et existence/régularité des minimiseurs des problèmes variationnels en dimension un

Richard Gratwick [a], Aidys Sedipkov [b,c], Mikhail Sychev [b,c], Aris Tersenov [b,c]

[a] *Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK*
[b] *Laboratory of Differential Equations and Related Problems of Analysis, Sobolev Institute of Mathematics, Koptuyg Avenue, 4, Novosibirsk 630090, Russia*
[c] *Novosibirsk State University, Russia*

A R T I C L E   I N F O

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A B S T R A C T

In this paper, we prove that if $L(x,u,v) \in C^3(R^3 \to R)$, $L_{vv} > 0$ and $L \geq \alpha|v| + \beta$, $\alpha > 0$, then all problems (1), (2) admit solutions in the class $W^{1,1}[a,b]$, which are in fact C^3-regular provided there are no pathological solutions to the Euler equation (5). Here $u \in C^3[c,d[$ is called a pathological solution to equation (5) if the equation holds in $[c,d[$, $|\dot{u}(x)| \to \infty$ as $x \to d$, and $||u||_{C[c,d]} < \infty$. We also prove that the lack of pathological solutions to the Euler equation results in the lack of the Lavrentiev phenomenon, see Theorem 9; no growth assumptions from below are required in this result.

R É S U M É

Dans cette Note, nous démontrons que si $L(x,u,v) \in C^3(R^3 \to R)$, $L_{vv} > 0$ et $L \geq \alpha|v| + \beta$, $\alpha > 0$, alors tous les problèmes (1)–(2) admettent des solutions dans la classe $W^{1,1}[a,b]$, qui sont en fait C^3-régulières pourvu que l'équation d'Euler (5) n'ait pas de solution pathologique. Ici, une solution $u \in C^3[c,d[$ de (5) est dite pathologique si l'équation est satisfaite dans $[c,d[$, $|\dot{u}(x)| \to \infty$ lorsque $x \to d$ et $\|u\|_{C[c,d]} < \infty$. Nous montrons également (voir Théorème 9), que l'absence de solution pathologique à l'équation d'Euler entraîne

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E-mail address: masychev@math.nsc.ru (M. Sychev).

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l'absence de phénomène de Lavrentiev ; aucune hypothèse de croissance minimale n'est requise pour ce résultat.

In this paper we consider classical one-dimensional variational problems

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))\mathrm{d}x \to \min, \tag{1}$$

$$u(a) = A, u(b) = B. \tag{2}$$

We assume that $L(x, u, v) : R \times R \times R \to R$ is of class C^3 and $L_{vv}(x, u, v) > 0$ everywhere. These assumptions on the integrand L will be regarded as *basic* throughout this article.

Under these assumptions, given a compact set $G \subset R^2$, we have that

$$L(x, u, v) \geq -\alpha|v| + \beta,\ \alpha > 0, \tag{3}$$

for $(x, u) \in G$, $v \in R$. Therefore, given a function $u \in W^{1,1}[a, b]$, we have that the function $L(\cdot, u(\cdot), \dot{u}(\cdot))$ is measurable and its negative part is integrable. Therefore, the integral $J(u)$ is defined and is either a finite value or $+\infty$.

In the case when the solution $u : [a, b] \to R$ is Lipschitz and $L \in C^1$ only, the Euler–Lagrange equation holds:

$$L_v(x, u(x), \dot{u}(x)) = \int_a^x L_u(t, u(t), \dot{u}(t))\mathrm{d}t + c, \tag{4}$$

see, e.g., [2]. In case additionally L satisfies the basic assumptions, we have $u \in C^3[a, b]$ and the equation (4) can be resolved with respect to the second derivative of the function u:

$$u'' = \frac{L_u - L_{xv} - L_{uv}\dot{u}}{L_{vv}}, \tag{5}$$

which is the Euler equation, see again [2].

The basic update approach to studying the existence and regularity of minimizers is Tonelli's theory.

Theorem 1 *(Tonelli, [14]). If, in addition to the basic assumptions, $L(x, u, v)$ has superlinear growth in v, i.e. $L \geq \theta(v)$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$, then each problem (1), (2) admits a solution in the class $W^{1,1}[a, b]$.*

Theorem 2 *(Tonelli, [15]; Ball–Mizel [1]). If in the problem (1), (2) with L satisfying the basic assumptions, there exists a solution u_0 in the class $W^{1,1}[a, b]$, then each such solution has everywhere a classical derivative (possibly infinite) which is continuous as a function with values in $\bar{R} = R \cup \{-\infty, \infty\}$. In particular, u_0 is of class C^3 in an open set of full measure where it also satisfies the Euler equation (5).*

Corollary 3. *Suppose L satisfies the conditions of Theorem 1. Suppose also that there are no pathological solutions to the Euler equation (5) on the interval $[a, b]$, i.e. ones such that $u \in C^3[c, d[$ ($[c, d[\subset [a, b]$ and possibly $d < c$), u satisfies (5) in $[c, d[$, and $|\dot{u}(x)| \to \infty$ as $x \to d$. Then each problem (1), (2) admits a solution in the class $W^{1,1}[a, b]$ and all such solutions are C^3-regular functions.*

Therefore the assumptions of Tonelli's theory are the basic assumptions on L, the superlinear growth of $L(x, u, v)$ in v, and the lack of pathological solutions to the Euler equation (5). Superlinear growth is needed to state weak compactness in $W^{1,1}$ of minimizing sequences; existence then follows from lower semicontinuity of the functional J with respect to weak convergence in $W^{1,1}$, which is guaranteed by convexity of L in v, see, e.g., [13] for a modern proof of this fact. This existence/regularity theory became the basic one in the literature, see, e.g., [3]. Singular solutions to minimization problems were constructed comparably recently, see the papers of Ball–Mizel [1], Clarke–Vinter [4], Davie [5], Sychev [10,11], Gratwick [6,7].

The discovery of this paper is that the lack of pathological solutions to the Euler equation (5) is by itself sufficient both for existence and regularity of minimizers in the class $W^{1,1}$. The following theorem holds.

Theorem 4. *Let L satisfy the basic assumptions and let*

$$L(x, u, v) \geq \alpha|v| + \beta,\ \alpha > 0.$$

Assume also that there are no pathological solutions $u \in C^3[c,d[$ ($[c,d[\subset [a,b]$, where possibly $d<c$) to the Euler equation such that $\dot{u}(c) = (B-A)/(b-a)$. Then the problem (1), (2) admits a solution in the class $W^{1,1}$, and all such solutions are equi-bounded in C^3-norm.

Theorem 4 is a consequence of Theorems 8 and 9 stated below.

We will use the following definition.

Definition 5. Consider the class of functions $\Xi = \{\xi : [a_\xi, b_\xi] \to \bar{R} = R \cup \{-\infty, +\infty\}\}$ such that each function $\xi : [a_\xi, b_\xi] \to \bar{R}$ is continuous. We say that the family Ξ is a conditionally equa-continuous family (CEF) if for every $M>0$, $\epsilon>0$ there exists $\delta = \delta(M,\epsilon) > 0$ such that if $|\xi(x_0)| \le M$ then $|\xi(x) - \xi(x_0)| \le \epsilon$ for $|x - x_0| \le \delta$.

Lemma 6. *Let L satisfy the basic assumptions and let $c>0$. For each $M>0$, consider a solution u_M of the problem (1), (2) in the class of Lipschitz functions such that $||u||_{C[a,b]} \le c$ and $||\dot{u}||_{L^\infty[a,b]} \le M$. Then $\dot{u}_M$, $M>0$, is a conditionally equa-continuous family (CEF).*

For a proof see, e.g., the proofs of Theorems 1.1 of [12] or [7]. CEF was introduced by Sychev in [10], and the basic properties of CEF are stated, e.g., in [8, §2].

Lemma 7. *Let L satisfy the basic assumptions and let $c_1>0$. Assume that there are no pathological solutions to the Euler equation (5) such that $u \in C^3[c,d[$ ($[c,d[\subset [a,b]$, possibly with $d<c$), $\dot{u}(c) = (B-A)/(b-a)$, $||u||_{C[c,d[} \le c_1$. Then there exists $N_1>0$ such that if $[c,d] \subset [a,b]$, and $u \in C^3[c,d[$ is a solution to the Euler equation (5) with the properties that $\dot{u}(c) = (B-A)/(b-a)$ and $||u||_{C[c,d[} \le c_1$, then $||\dot{u}||_{C[c,d]} \le N_1$.*

Proof. If the conclusion is not the case, we can find $[c_k, d_k[\subset [a,b]$ and solutions $u_k \in C^3[c_k,d_k]$ of (5) such that $\dot{u}(c_k) = (B-A)/(b-a)$, $||\dot{u}_k||_{C[c_k,d_k]} \le k$ and $\lim_{x\to d_k} |\dot{u}_k(x)| = k$, $k \in N$. But then, since $\dot{u}_k : [c_k,d_k] \to R$, $k\in N$, form a CEF, we can isolate a subsequence u_k (not relabelled) such that as $k\to\infty$, $[c_k,d_k] \to [c,d]$, and u_k converges in $C^1[c,d']$ for each $d'<d$ to some $u \in C^1[c,d[$. Then u has the properties that $u \in C^3[c,d[$ is a solution to the Euler equation (5), $||u||_{C[c,d]} \le c_1$, $\dot{u}(c) = (B-A)/(b-a)$, and $|\dot{u}(x)| \to \infty$ as $x \to d$, which is a contradiction. This proves Lemma 7. □

Theorem 8. *Let L satisfy the basic assumptions. Assume there is $c_1>0$ such that there is a minimizing sequence u_k in the class $W^{1,\infty}[a,b]$ of the problem (1), (2), and $||u_k||_{C[a,b]} \le c_1$, $k\in N$. Assume also that there are no pathological solutions to the Euler equation (5) with the following properties: $u \in C^3[c,d[$ ($[c,d[\subset [a,b]$, possibly with $d<c$), $\dot{u}(c) = (B-A)/(b-a)$, $||u||_{C[c,d[} \le c_1$, and $|\dot{u}(x)| \to \infty$ as $x\to d$. Then the problem (1), (2) has a solution in the class $W^{1,\infty}[a,b]$, and this solution is C^3-regular and satisfies $||\dot{u}||_{C[a,b]} \le N_1$, where N_1 is as given by Lemma 7.*

Proof. Consider the solutions u_M given by Lemma 6, i.e. in the class of M-Lipschitz functions with the property $||u||_{C[a,b]} \le c_1$. Then $\dot{u}_M$, $M\in N$, form a CEF and u_M is a minimizing sequence for the problem (1)–(2) in the class of Lipschitz functions. We claim that $\limsup_{M\to\infty} ||\dot{u}_M||_{C[a,b]} \le N_1$, where N_1 is given by Lemma 7. Otherwise there exists $\epsilon>0$ and a subsequence u_{M_k} ($M_k\to\infty$ as $k\to\infty$) such that $\lim_{k\to\infty} ||\dot{u}_{M_k}||_C \ge N_1+\epsilon$. Using the CEF property, it is possible to find $[c,d[\subset [a,b]$ such that a subsequence u_k of the sequence u_{M_k} converges in C^1 to $u \in C^3[c,d[$ with the properties $\dot{u}(c) = (B-A)/(b-a)$, $||\dot{u}||_{C[c,d]} \ge N_1+\epsilon$, $||u||_{C[c,d]} \le c_1$, and u is a solution to the Euler equation (5) in $[c,d[$. This gives a contradiction with Lemma 7. Therefore indeed $\limsup_{M\to\infty} ||\dot{u}_M||_{C[a,b]} \le N_1$ and, therefore, the limit function u in C^1-norm of a subsequence u_{M_k} is a solution to the minimization problem in the class of Lipschitz functions. Then $u\in C^3[a,b]$ as well and u satisfies the Euler equation (5). □

Theorem 9. *Let L satisfy the basic assumptions. Assume also that there are no pathological solutions on the interval $[a,b]$ such that $u \in C^3[c,d[$ ($[c,d[\subset [a,b]$, $d<c$ possibly), $\dot{u}(c) = (B-A)/(b-a)$, and $|\dot{u}(x)| \to \infty$ as $x\to d$. Then there is no Lavrentiev phenomenon in the problem (1), (2), i.e.*

$$I_1 = I_\infty,$$

where

$$I_1 = \inf\{J(u) : u(a) = A, u(b) = B, u \in W^{1,1}[a,b]\},$$
$$I_\infty = \inf\{J(u) : u(a) = A, u(b) = B, u \in W^{1,\infty}[a,b]\}.$$

Proof. If $I_\infty = -\infty$ then $I_1 = I_\infty$. Assume then that $I_\infty > -\infty$, and assume that the Lavrentiev phenomenon occurs, i.e.

$$I_1 < I_\infty. \tag{6}$$

Then there is a function $u_0 \in W^{1,1}[a,b]$ such that

$$J(u_0) < I_\infty. \tag{7}$$

We have that $||u_0||_{C[a,b]} \le c_1$ for some $c_1 < \infty$. Let N_1 be the constant associated with c_1 from Lemma 7.

Consider a convex function $\theta \in C^\infty(R \to R)$ such that $\theta(v) = 0$ for $|v| \le N_1 + 1$, θ has superlinear growth, $\theta_{vv} \ge 0$ and $\int_a^b \theta(\dot{u}_0(x))\mathrm{d}x < \infty$. Consider the integrands $L_\mu = L + \mu\theta(v)$ for $\mu > 0$. We have that $I_\infty^\mu \ge I_\infty$. There exists a minimizer u_μ of the associated functional J_μ in the class of Sobolev functions with the property $||u||_{C[a,b]} \le c_1$ by Theorem 1, because of the superlinear growth of L_μ in v. We have that $J_\mu(u_\mu) < I_\infty^\mu$ for sufficiently small $\mu > 0$, because of (7). Also u_μ has Tonelli's regularity, i.e. $\dot{u}_\mu : [a,b] \to \bar{R}$ is continuous. Since $||\dot{u}_\mu||_{L^\infty[a,b]} = \infty$, we have that in some interval $[c,d[\subset [a,b]$ the following holds: $||u_\mu||_{C[c,d]} \le c_1$, $\dot{u}_\mu(c) = (B-A)/(b-a)$, and $|\dot{u}_\mu(x)| \to \infty$ as $x \to d$. Then for some $d' \in [c,d[$, we have that $||\dot{u}_\mu||_{C[c,d']} \le N_1 + 1/2$, and $|\dot{u}_\mu(d')| = N_1 + 1/2$. In $[c,d']$ the function u_μ satisfies the Euler equation $(5)_\mu$. For $x \in [c,d']$ we also have $L = L_\mu$. Therefore u_μ satisfies the original Euler equation (5) on $[c,d']$, which contradicts Lemma 7. This contradiction shows that (6) is incorrect. This proves the theorem. □

Now Theorem 4 follows from Theorems 8 and 9.

Proof of Theorem 4. Due to the inequality

$$L(x,u,v) \ge \alpha|v| + \beta, \ \alpha > 0,$$

the conditions of Theorem 8 are satisfied. Let u_0 be the solution in the class of Lipschitz functions given by Theorem 8. By Theorem 9, it is also a solution in the class $W^{1,1}[a,b]$.

Now we have to prove that if there is another solution $\tilde{u}$ in the class $W^{1,1}[a,b]$ other than u_0, then it is Lipschitz. Suppose not, then $||\dot{\tilde{u}}||_{L^\infty[a,b]} = \infty$. Such a solution has Tonelli's regularity, see, e.g., [8]. Therefore, there exists $[c,d[\subset [a,b]$ such that $\tilde{u} \in C^3[c,d[$, $\dot{\tilde{u}}(c) = (B-A)/(b-a)$, and $|\dot{\tilde{u}}(x)| \to \infty$ as $x \to d$. But then this is a pathological solution to the Euler equation (5). This contradiction shows that all solutions are Lipschitz functions. Then they are also equi-bounded in C^3-norm by Lemma 7. □

Note that the lack of pathological solutions to the Euler equation (5) is only a sufficient condition for regularity of minimizers. In [9] Sychev constructed examples of L satisfying the basic assumptions and having superlinear growth for which pathological solutions exist despite minimizers of all problems (1), (2) being C^3-regular functions. Under these assumptions on the integrands, a condition both necessary and sufficient for full regularity of minimizers is Lipschitz continuity of the cost-value function

$$S(a,A,b,B) := \inf\{J(u) : u(a) = A, u(b) = B, u \in W^{1,1}[a,b]\},$$

see [8] or [7].

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