C. R. Acad. Sci. Paris, Ser. I 355 (2017) 296–298

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

A stochastic Hamilton–Jacobi equation with infinite speed of propagation

Une équation de Hamilton–Jacobi stochastique à vitesse de propagation infinie

Paul Gassiat

Ceremade, Université Paris-Dauphine, PSL Research University, place du Maréchal-de-Lattre-de-Tassigny, 75775, Paris cedex 16, France

A R T I C L E I N F O

Article history:
Received 27 September 2016
Accepted 1 February 2017
Available online 9 February 2017

Presented by the Editorial Board

A B S T R A C T

We give an example of a stochastic Hamilton–Jacobi equation $\mathrm{d}u = H(\mathrm{D}u)\mathrm{d}\xi$ which has an infinite speed of propagation as soon as the driving signal ξ is not of bounded variation.

R É S U M É

Nous présentons un exemple d'équation d'Hamilton–Jacobi stochastique $\mathrm{d}u = H(\mathrm{D}u)\mathrm{d}\xi$ dont la vitesse de propagation est infinie dès que le signal ξ n'est pas à variation bornée.

1. Introduction

An important feature of (deterministic) Hamilton–Jacobi equations

$$\partial_t u = H(\mathrm{D}u) \quad \text{on } (0,T)\times\mathbb{R}^N \tag{1.1}$$

is the so-called *finite speed of propagation*: assuming for instance that $H:\mathbb{R}^N\to\mathbb{R}$ is C-Lipschitz, then if u^1 and u^2 are two (viscosity) solutions to (1.1), one has

$$u^1(0,\cdot)=u^2(0,\cdot) \text{ on } B(R) \;\Rightarrow\; \forall t\geq 0,\; u^1(t,\cdot)=u^2(t,\cdot) \text{ on } B(R-Ct), \tag{1.2}$$

where by $B(R)$ we mean the ball of radius R centered at 0.

In this note, we are interested in Hamilton–Jacobi equations with rough time dependence of the form

$$\partial_t u = H(\mathrm{D}u)\dot{\xi}(t) \quad \text{on } (0,T)\times\mathbb{R}^N, \tag{1.3}$$

E-mail address: gassiat@ceremade.dauphine.fr.

http://dx.doi.org/10.1016/j.crma.2017.01.021
1631-073X/

where ξ is only assumed to be continuous. Of course, the above equation only makes classical (viscosity) sense for ξ in C^1, but Lions and Souganidis [2] have shown that if H is the difference of two convex functions, the solution map can be extended continuously (with respect to supremum norm) to any continuous ξ. (In typical applications, one wants to take ξ as the realization of a random process such as Brownian motion.)

In fact, the Lions–Souganidis theory also gives the following result: if $H = H_1 - H_2$ where H_1, H_2 are convex, C-Lipschitz, with $H_1(0) = H_2(0) = 0$, then for any constant A,

$$u(0,\cdot) \equiv A \text{ on } B(R) \;\Rightarrow u(t,\cdot) \equiv A \text{ on } B(R(t)),$$

where $R(t) = R - C(\max_{s\in[0,t]} \xi(s) - \min_{s\in[0,t]} \xi(s))$.

However, this does not imply a finite speed of propagation for (1.3) for arbitrary initial conditions, and a natural question (as mentioned in lecture notes by Souganidis [3]) is to know whether a property analogous to (1.2) holds in that case. The purpose of this note is to show that in general it does not: we present an example of an H such that, if the total variation of ξ on $[0,T]$ is strictly greater than R, one may find initial conditions u_0^1, u_0^2 that coincide on $B(R)$, but such that, for the associated solutions u^1 and u^2, one has $u^1(T,0) \neq u^2(T,0)$.

For instance, if ξ is a (realization of a) Brownian motion, then (almost surely), one may find initial conditions coinciding on balls of arbitrary large radii, but such that $u^1(t,0) \neq u^2(t,0)$ for all $t > 0$.

It should be noted that the Hamiltonian H in our example is not convex (or concave). When H is convex, some of the oscillations of the path cancel out at the PDE level,[1] so that one cannot hope for simple bounds such as (2.2) below. Whether one has finite speed of propagation in this case remains an open question.

2. Main result and proof

We fix $T > 0$ and denote $\mathcal{P} = \{(t_0,\ldots,t_n),\ 0 = t_0 \le t_1 \le \ldots \le t_n = T\}$ the set of partitions of $[0,T]$. Recall that the total variation of a continuous path $\xi : [0,T] \to \mathbb{R}$ is defined by

$$V_{0,T}(\xi) = \sup_{(t_0,\ldots,t_n)\in\mathcal{P}} \sum_{i=0}^{n-1} |\xi(t_{i+1}) - \xi(t_i)|\,.$$

Our main result is then:

Theorem 1. *Given $\xi \in C([0,T])$, let $u : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ be the viscosity solution to*

$$\partial_t u = \big(|\partial_x u| - \big|\partial_y u\big|\big)\,\dot{\xi}(t) \text{ on } (0,T)\times\mathbb{R}^2 \tag{2.1}$$

with initial condition

$$u(0,x,y) = |x-y| + \Theta(x,y),$$

where $\Theta \ge 0$ is such that $\Theta(x,y) \ge 1$ if $\min\{x,y\} \ge R$.

One then has the estimate

$$u(T,0,0) \ge \left(\sup_{(t_0,\ldots,t_n)\in\mathcal{P}} \frac{\sum_{j=0}^{n-1} \big|\xi(t_{j+1}) - \xi(t_j)\big|}{n} - \frac{R}{n} \right)_+ \wedge 1. \tag{2.2}$$

In particular, $u(T,0,0) > 0$ as soon as $V_{0,T}(\xi) > R$.

Note that since $|x-y|$ is a stationary solution to (2.1), the claims from the introduction about the speed of propagation follow.

The proof of Theorem 1 is based on the differential game associated with (2.1). Informally, the system is constituted of a pair (x,y) and the two players take turn controlling x or y depending on the sign of $\dot{\xi}$, with speed up to $|\dot{\xi}|$. The minimizing player wants x and y to be as close as possible to each other, while keeping them smaller than R. The idea is then that if the minimizing player keeps x and y stuck together, the maximizing player can lead x and y to be greater than R as long as $V_{0,T}(\xi) > R$.

Proof of Theorem 1. By approximation, we can consider $\xi \in C^1$, and in fact we consider the backward equation:

$$\begin{cases} -\partial_t v &= \big(|\partial_x v| - \big|\partial_y v\big|\big)\,\dot{\xi}(t), \\ v(T,x,y) &= |x-y| + \Theta(x,y). \end{cases} \tag{2.3}$$

[1] For example: for $\delta \ge 0$, $S_H(\delta)\circ S_{-H}(\delta)\circ S_H(\delta) = S_H(\delta)$, where S_H, S_{-H} are the semigroups associated with H, $-H$.

We then need a lower bound on $v(0,0,0)$. Note that

$$\left(|\partial_x v| - \left|\partial_y v\right|\right)\dot{\xi}(t) = \sup_{|a|\le 1}\inf_{|b|\le 1}\left\{\dot{\xi}_+(t)\left(a\partial_x u + b\partial_y u\right) + \dot{\xi}_-(t)\left(a\partial_y u + b\partial_s u\right)\right\},$$

so that by classical results (e.g., [1]) one has the representation

$$v(0,0,0) = \sup_{\delta(\cdot)\in\Delta}\inf_{\beta\in\mathcal{U}} J(\delta(\beta),\beta), \tag{2.4}$$

where $\mathcal{U}$ is the set of controls (measurable functions from $[0,T]$ to $[-1,1]$) and Δ the set of progressive strategies (i.e. maps $\delta\colon \mathcal{U}\to\mathcal{U}$ such that if $\beta=\beta'$ a.e. on $[0,t]$, then $\delta(\beta)(t)=\delta(\beta')(t)$). Here for $\alpha,\beta\in\mathcal{U}$, the payoff is defined by

$$J(\alpha,\beta) = |x^{\alpha,\beta}(T) - y^{\alpha,\beta}(T)| + \Theta(x^{\alpha,\beta}(T), y^{\alpha,\beta}(T)),$$

where

$$x^{\alpha,\beta}(0) = y^{\alpha,\beta}(0) = 0,\ \ \dot{x}^{\alpha,\beta}(s) = \dot{\xi}_+(s)\alpha(s) + \dot{\xi}_-(s)\beta(s),\quad \dot{y}^{\alpha,\beta}(s) = \dot{\xi}_-(s)\alpha(s) + \dot{\xi}_+(s)\beta(s).$$

Assume $v(0,0,0)<1$ (otherwise there is nothing to prove) and fix $\varepsilon\in(0,1)$ such that $v(0,0,0)<\varepsilon$. Consider the strategy δ^ε for the maximizing player defined as follows: for $\beta\in\mathcal{U}$, let

$$\tau_\varepsilon^\beta = \inf\left\{t\ge 0,\ |x^{1,\beta}(t) - y^{1,\beta}(t)|\ge\varepsilon\right\},$$

and then

$$\delta^\varepsilon(\beta)(t) = \begin{cases} 1, & t<\tau_\varepsilon^\beta \\ \beta(t), & t\ge\tau_\varepsilon^\beta \end{cases}.$$

In other words, the maximizing player moves to the right at maximal speed, until the time when $|x-y|=\varepsilon$, at which point he moves in a way such that x and y stay at distance ε.

Now by (2.4), there exists $\beta\in\mathcal{U}$ with $J(\delta^\varepsilon(\beta),\beta)<\varepsilon$. Clearly, for the corresponding trajectories $x(\cdot), y(\cdot)$, this means that $|x(T)-y(T)|<\varepsilon$, and by definition of δ^ε this implies $|x(t)-y(t)|\le\varepsilon$ for $t\in[0,T]$. We now fix $(t_0,\dots,t_n)\in\mathcal{P}$ and prove by induction that for $i=0,\dots,n$,

$$\min\{x(t_i), y(t_i)\} \ge \sum_{j=0}^{i-1}\left|\xi(t_{j+1}) - \xi(t_j)\right| - i\varepsilon.$$

Indeed, if it is true for some index i, then assuming that for instance $\xi(t_{i+1})-\xi(t_i)\ge 0$, one has

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + \int_{t_i}^{t_{i+1}}\dot{\xi}_+(s)\mathrm{d}s - \int_{t_i}^{t_{i+1}}\beta(s)\dot{\xi}_-(s)\mathrm{d}s \\ &\ge x(t_i) + \xi(t_{i+1}) - \xi(t_i) \ \ge\ \sum_{j=0}^{i}\left|\xi(t_{j+1}) - \xi(t_j)\right| - i\varepsilon \end{aligned}$$

and since $y(t_{i+1})\ge x(t_{i+1})-\varepsilon$, one also has $y(t_{i+1})\ge\sum_{j=0}^{i}\left|\xi(t_{j+1})-\xi(t_j)\right| - (i+1)\varepsilon$. The case when $\xi(t_{i+1})-\xi(t_i)\le 0$ is similar.

Since $J(\delta^\varepsilon(\beta),\beta)\le 1$, one must necessarily have $\min\{x(T),y(T)\}\le R$, so that

$$\varepsilon\ge\frac{1}{n}\left(\sum_{j=0}^{n}\left|\xi(t_{j+1})-\xi(t_j)\right| - R\right).$$

Letting $\varepsilon\to v(0,0,0)$ and taking the supremum over $\mathcal{P}$ on the r.h.s. we obtain (2.2). □

Acknowledgements

The author acknowledges the support of the ANR, via the ANR project ANR-16-CE40-0020-01.

References

[1] L.C. Evans, P.E. Souganidis, Differential games and representation formulas for solutions of Hamilton–Jacobi–Isaacs equations, Indiana Univ. Math. J. 33 (5) (1984) 773–797.

[2] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations: non-smooth equations and applications, C. R. Acad. Sci. Paris, Ser. I 327 (8) (1998) 735–741, http://dx.doi.org/10.1016/S0764-4442(98)80161-4.

[3] P.E. Souganidis, Fully nonlinear first- and second-order stochastic partial differential equations, in: Lecture Notes from the CIME Summer School "Singular random dynamics", 2016, available at http://php.math.unifi.it/users/cime/Courses/2016/course.php?codice=20162.