C. R. Acad. Sci. Paris, Ser. I 355 (2017) 336–344

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Harmonic analysis

Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function ☆

Caractérisation des espaces de Lipschitz via les commutateurs de l'opérateur maximal de Hardy–Littlewood

Pu Zhang

Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, PR China

A R T I C L E   I N F O

Article history:
Received 6 November 2016
Accepted after revision 1 February 2017
Available online 20 February 2017

Presented by the Editorial Board

A B S T R A C T

Let M be the Hardy–Littlewood maximal function and b be a locally integrable function. Denote by M_b and $[b, M]$ the maximal commutator and the (nonlinear) commutator of M with b. In this paper, the author considers the boundedness of M_b and $[b, M]$ on Lebesgue spaces and Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

R É S U M É

Soit M l'opérateur maximal de Hardy–Littlewood et b une fonction localement intégrable. Notons M_b et $[b, M]$ le commutateur maximal et le commutateur (non linéaire) de M et b. Dans cette Note, l'auteur étudie la finitude de M_b et $[b, M]$ sur les espaces de Lebesgue et les espaces de Morrey lorsque b appartient à l'espace de Lipschitz. Cela conduit à de nouvelles caractérisations de l'espace de Lipschitz.

1. Introduction and results

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b, T]f = bT(f) - T(bf). \tag{1.1}$$

A well-known result due to Coifman, Rochberg and Weiss [6] (see also [13]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [13] gave some characterizations of the Lipschitz

☆ Supported by the National Natural Science Foundation of China (Grant Nos. 11571160 and 11471176).
E-mail address: puzhang@sohu.com.

http://dx.doi.org/10.1016/j.crma.2017.01.022
1631-073X/

space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ (see Definition 1.1 below) via commutator $[b, T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)(0 < \beta < 1)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [18]).

For a locally integrable function f, the Hardy–Littlewood maximal function M is given by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

the maximal commutator of M with a locally integrable function b is defined by

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The mapping properties of the maximal commutator M_b have been studied intensively by many authors. See [3,9,11,12, 20,21] and [25] for instance. The following result is proved by García-Cuerva et al. [9]. See also [20] and [21].

Theorem A *([9]). Let b be a locally integrable function and $1 < p < \infty$. Then the maximal commutator M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

The first part of this paper is to study the boundedness of M_b when the symbol b belongs to a Lipschitz space. Some characterizations of the Lipschitz space via such commutator are given.

Definition 1.1. Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \le C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

Our first result can be stated as follows.

Theorem 1.1. *Let b be a locally integrable function and $0 < \beta < 1$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$;
(2) M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$;
(3) M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$;
(4) M_b satisfies the weak-type $(1, n/(n - \beta))$ estimates, namely, there exists a positive constant C such that for all $\lambda > 0$,

$$\left|\{x \in \mathbb{R}^n : M_b(f)(x) > \lambda\}\right| \le C\left(\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}\right)^{n/(n-\beta)}; \tag{1.2}$$

(5) M_b is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Morrey spaces were originally introduced by Morrey in [17] to study the local behavior of solutions to second-order elliptic partial differential equations. Many classical operators of harmonic analysis were studied in Morrey-type spaces during the last decades. We refer the readers to Adams [2] and references therein.

Definition 1.2. Let $1 \le p < \infty$ and $0 \le \lambda \le n$. The classical Morrey space is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < \infty\right\},$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_Q \left(\frac{1}{|Q|^{\lambda/n}} \int_Q |f(x)|^p \mathrm{d}x\right)^{1/p}.$$

It is well known that if $1 \le p < \infty$ then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Theorem 1.2. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

Theorem 1.3. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$, $1/q = 1/p - \beta/n$ and $\lambda/p = \mu/q$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.*

On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the Hardy–Littlewood maximal function M with a locally integrable function b by

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Using real interpolation techniques, Milman and Schonbek [16] established a commutator result. As an application, they obtained the L^p-boundedness of $[b, M]$ when $b \in BMO(\mathbb{R}^n)$ and $b \geq 0$. This operator can be used in studying the product of a function in H^1 and a function in BMO (see [5] for instance). In 2000, Bastero, Milman and Ruiz [4] studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on L^p spaces when $1 < p < \infty$. Zhang and Wu obtained similar results for the fractional maximal function in [24] and extended the mentioned results to variable exponent Lebesgue spaces in [25] and [26]. Recently, Agcayazi et al. [3] gave the end-point estimates for the commutator $[b, M]$. Zhang [23] extended these results to the multilinear setting.

We would like to remark that operators M_b and $[b, M]$ essentially differ from each other. For example, M_b is positive and sublinear, but $[b, M]$ is neither positive nor sublinear.

The second part of this paper aims to study the mapping properties of the (nonlinear) commutator $[b, M]$ when b belongs to some Lipschitz space. To state our results, we recall the definition of the maximal operator with respect to a cube. For a fixed cube Q_0, the Hardy–Littlewood maximal function with respect to Q_0 of a function f is given by

$$M_{Q_0}(f)(x) = \sup_{Q_0 \supseteq Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all the cubes Q with $Q \subseteq Q_0$ and $Q \ni x$.

Theorem 1.4. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Then the following statements are equivalent:*
(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$;
(2) $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;
(3) there exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \leq C. \tag{1.3}$$

Theorem 1.5. *Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Then there is a positive constant C such that, for all $\lambda > 0$,*

$$\left|\{x \in \mathbb{R}^n : |[b, M](f)(x)| > \lambda\}\right| \leq C\left(\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}\right)^{n/(n-\beta)}.$$

Theorem 1.6. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. Then the following statements are equivalent:*
(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.
(2) $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

Theorem 1.7. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$, $1/q = 1/p - \beta/n$ and $\lambda/p = \mu/q$. Then the following statements are equivalent:*
(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$,
(2) $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1–1.3. Section 4 is devoted to proving Theorems 1.4–1.7.

2. Preliminaries and lemmas

For a measurable set E, we denote by $|E|$ the Lebesgue measure and by χ_E the characteristic function of E. For $p \in [1, \infty]$, we denote by p' the conjugate index of p, namely, $p' = p/(p-1)$. For a locally integrable function f and a cube Q, we denote by $f_Q = (f)_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$.

To prove the theorems, we need some known results. It is known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey–Companato space (see [14] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [7] and Janson, Taibleson and Weiss [14] (see also Paluszyński [18]).

Lemma 2.1. *Let $0 < \beta < 1$ and $1 \le q < \infty$. Define*

$$\dot{\Lambda}_{\beta,q}(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{\Lambda}_{\beta,q}} = \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \mathrm{d}x \right)^{1/q} < \infty \right\}.$$

Then, for all $0 < \beta < 1$ and $1 \le q < \infty$, $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms.

Let $0 < \alpha < n$ and f be a locally integrable function, the fractional maximal function of f is given by

$$\mathfrak{M}_\alpha(f)(x) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \mathrm{d}y$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The following strong and weak-type boundednesses of $\mathfrak{M}_\alpha$ are well known, see [10] and [8].

Lemma 2.2. *Let $0 < \alpha < n$, $1 \le p \le n/\alpha$ and $1/q = 1/p - \alpha/n$.*

(1) If $1 < p < n/\alpha$ then there exists a positive constant $C(n, \alpha, p)$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^q(\mathbb{R}^n)} \le C(n, \alpha, p) \|f\|_{L^p(\mathbb{R}^n)}.$$

(2) If $p = n/\alpha$ then there exists a positive constant $C(n, \alpha)$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^\infty(\mathbb{R}^n)} \le C(n, \alpha) \|f\|_{L^{n/\alpha}(\mathbb{R}^n)}.$$

(3) If $p = 1$ then there exists a positive constant $C(n, \alpha)$ such that for all $\lambda > 0$

$$\left| \left\{ x \in \mathbb{R}^n : \mathfrak{M}_\alpha(f)(x) > \lambda \right\} \right| \le C(n, \alpha) \left(\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)} \right)^{n/(n-\alpha)}.$$

Spanne (see [19]) and Adams [1] studied the boundedness of the fractional integral I_α in classical Morrey spaces. We note that the fractional maximal function enjoys the same boundedness as that of the fractional integral since the pointwise inequality $\mathfrak{M}_\alpha(f)(x) \le I_\alpha(|f|)(x)$. These results can be summarized as follows (see also [22]).

Lemma 2.3. *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $0 < \lambda < n - \alpha p$.*

(1) If $1/q = 1/p - \alpha/(n - \lambda)$, then there is a constant $C > 0$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^{q,\lambda}(\mathbb{R}^n)} \le C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \quad \text{for every } f \in L^{p,\lambda}(\mathbb{R}^n).$$

(2) If $1/q = 1/p - \alpha/n$ and $\lambda/p = \mu/q$. Then there is a constant $C > 0$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^{q,\mu}(\mathbb{R}^n)} \le C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \quad \text{for every } f \in L^{p,\lambda}(\mathbb{R}^n).$$

Lemma 2.4 *([15]). Let $1 \le p < \infty$ and $0 < \lambda < n$, then there is a constant $C > 0$ that depends only on n such that*

$$\|\chi_Q\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C |Q|^{\frac{n-\lambda}{np}}.$$

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$\begin{aligned} M_b(f)(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| \mathrm{d}y \\ &\le C \|b\|_{\dot{\Lambda}_\beta} \sup_{Q \ni x} \frac{1}{|Q|^{1-\beta/n}} \int_Q |f(y)| \mathrm{d}y \\ &= C \|b\|_{\dot{\Lambda}_\beta} \mathfrak{M}_\beta(f)(x). \end{aligned} \tag{3.1}$$

Obviously, (2), (3), (4) and (5) follow from Lemma 2.2, Lemma 2.3 and (3.1).

(3) $\Longrightarrow$ (1): Assume M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. For any cube $Q \subset \mathbb{R}^n$, by Hölder's inequality and noting that $1/p + 1/q' = 1 + \beta/n$, one gets

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\mathrm{d}x &\le \frac{1}{|Q|^{1+\beta/n}}\int_Q\left(\frac{1}{|Q|}\int_Q |b(x)-b(y)|\mathrm{d}y\right)\mathrm{d}x\\
&= \frac{1}{|Q|^{1+\beta/n}}\int_Q\left(\frac{1}{|Q|}\int_Q |b(x)-b(y)|\chi_Q(y)\mathrm{d}y\right)\mathrm{d}x\\
&\le \frac{1}{|Q|^{1+\beta/n}}\int_Q M_b(\chi_Q)(x)\mathrm{d}x\\
&\le \frac{1}{|Q|^{1+\beta/n}}\left(\int_Q [M_b(\chi_Q)(x)]^q\mathrm{d}x\right)^{1/q}\left(\int_Q \chi_Q(x)\mathrm{d}x\right)^{1/q'}\\
&\le \frac{C}{|Q|^{1+\beta/n}}\|M_b\|_{L^p\to L^q}\|\chi_Q\|_{L^p(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'}(\mathbb{R}^n)}\\
&\le C\|M_b\|_{L^p\to L^q}.
\end{aligned}$$

This together with Lemma 2.1 gives $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$.

(4) $\Longrightarrow$ (1): We assume (1.2) is true and will verify $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q_0\subset\mathbb{R}^n$, since for any $x\in Q_0$,

$$|b(x)-b_{Q_0}|\le\frac{1}{|Q_0|}\int_{Q_0}|b(x)-b(y)|\mathrm{d}y,$$

then, for all $x\in Q_0$,

$$\begin{aligned}
M_b(\chi_{Q_0})(x) &= \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(x)-b(y)|\chi_{Q_0}(y)\mathrm{d}y\\
&\ge \frac{1}{|Q_0|}\int_{Q_0}|b(x)-b(y)|\chi_{Q_0}(y)\mathrm{d}y\\
&= \frac{1}{|Q_0|}\int_{Q_0}|b(x)-b(y)|\mathrm{d}y\\
&\ge |b(x)-b_{Q_0}|.
\end{aligned}$$

This together with (1.2) gives

$$\begin{aligned}
\left|\left\{x\in Q_0: |b(x)-b_{Q_0}|>\lambda\right\}\right| &\le \left|\left\{x\in Q_0: M_b(\chi_{Q_0})(x)>\lambda\right\}\right|\\
&\le C\left(\lambda^{-1}\|\chi_{Q_0}\|_{L^1(\mathbb{R}^n)}\right)^{n/(n-\beta)}\\
&= C\left(\lambda^{-1}|Q_0|\right)^{n/(n-\beta)}.
\end{aligned}$$

Let $t>0$ be a constant to be determined later, then

$$\begin{aligned}
\int_{Q_0}|b(x)-b_{Q_0}|\mathrm{d}x &= \int_0^\infty \left|\left\{x\in Q_0: |b(x)-b_{Q_0}|>\lambda\right\}\right|\mathrm{d}\lambda\\
&= \int_0^t \left|\left\{x\in Q_0: |b(x)-b_{Q_0}|>\lambda\right\}\right|\mathrm{d}\lambda\\
&\quad + \int_t^\infty \left|\left\{x\in Q_0: |b(x)-b_{Q_0}|>\lambda\right\}\right|\mathrm{d}\lambda\\
&\le t|Q_0| + C\int_t^\infty \left(\lambda^{-1}|Q_0|\right)^{n/(n-\beta)}\mathrm{d}\lambda
\end{aligned}$$

$$\leq t|Q_0| + C|Q_0|^{n/(n-\beta)} \int_t^\infty \lambda^{-n/(n-\beta)} \mathrm{d}\lambda$$

$$\leq C(n,\beta)\big(t|Q_0| + |Q_0|^{n/(n-\beta)} t^{1-n/(n-\beta)}\big).$$

Set $t = |Q_0|^{\beta/n}$ in the above estimate, we have

$$\int_{Q_0} |b(x) - b_{Q_0}| \mathrm{d}x \leq C|Q_0|^{1+\beta/n}.$$

It follows from Lemma 2.1 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ since Q_0 is an arbitrary cube in $\mathbb{R}^n$.

(5) $\Longrightarrow$ (1): If M_b is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \mathrm{d}x &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) \mathrm{d}y \right) \mathrm{d}x \\ &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q M_b(\chi_Q)(x) \mathrm{d}x \\ &\leq \frac{1}{|Q|^{\beta/n}} \|M_b(\chi_Q)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \frac{C}{|Q|^{\beta/n}} \|M_b\|_{L^{n/\beta} \to L^\infty} \|\chi_Q\|_{L^{n/\beta}(\mathbb{R}^n)} \\ &\leq C\|M_b\|_{L^{n/\beta} \to L^\infty}.\end{aligned}$$

This together with Lemma 2.1 gives $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

The proof of Theorem 1.1 is completed since (2) $\Longrightarrow$ (1) follows from (3) $\Longrightarrow$ (1). □

Proof of Theorem 1.2. Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. By (3.1) and Lemma 2.3 (1), we have

$$\|M_b(f)\|_{L^{q,\lambda}} \leq \|b\|_{\dot{\Lambda}_\beta} \|\mathfrak{M}_\beta(f)\|_{L^{q,\lambda}} \leq C\|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\lambda}}.$$

Conversely, if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned}\frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q \mathrm{d}x \right)^{1/q} &\leq \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q \left[\frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) \mathrm{d}y \right]^q \mathrm{d}x \right)^{1/q} \\ &\leq \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q [M_b(\chi_Q)(x)]^q \mathrm{d}x \right)^{1/q} \\ &= \frac{1}{|Q|^{\beta/n}} \left(\frac{|Q|^{\lambda/n}}{|Q|} \right)^{1/q} \left(\frac{1}{|Q|^{\lambda/n}} \int_Q [M_b(\chi_Q)(x)]^q \mathrm{d}x \right)^{1/q} \\ &\leq |Q|^{-\beta/n - 1/q + \lambda/(nq)} \|M_b(\chi_Q)\|_{L^{q,\lambda}(\mathbb{R}^n)} \\ &\leq C|Q|^{-\beta/n - 1/q + \lambda/(nq)} \|M_b\|_{L^{p,\lambda} \to L^{q,\lambda}} \|\chi_Q\|_{L^{p,\lambda}(\mathbb{R}^n)} \\ &\leq C\|M_b\|_{L^{p,\lambda} \to L^{q,\lambda}},\end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(n-\lambda)$ and Lemma 2.4.

It follows from Lemma 2.1 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. This completes the proof. □

Proof of Theorem 1.3. By a similar proof to the one of Theorem 1.2, we can obtain Theorem 1.3. □

4. Proof of Theorems 1.4–1.7

Proof of Theorem 1.4. (1) $\Longrightarrow$ (2): For any fixed $x \in \mathbb{R}^n$ such that $M(f)(x) < \infty$, since $b \geq 0$ then

$$\begin{aligned} |[b, M](f)(x)| &= |b(x)M(f)(x) - M(bf)(x)| \\ &= \left| \sup_{Q \ni x} \frac{1}{|Q|} \int_Q b(x)|f(y)|\mathrm{d}y - \sup_{Q \ni x} \frac{1}{|Q|} \int_Q b(y)|f(y)|\mathrm{d}y \right| \\ &\leq \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|\mathrm{d}y \\ &= M_b(f)(x). \end{aligned} \tag{4.1}$$

It follows from Theorem 1.1 that $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ since $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $\Longrightarrow$ (3): For any fixed cube $Q \subset \mathbb{R}^n$ and all $x \in Q$, we have (see the proof of Proposition 4.1 in [4], see also (2.4) in [24])

$$M(\chi_Q)(x) = \chi_Q(x) \quad \text{and} \quad M(b\chi_Q)(x) = M_Q(b)(x).$$

Then,

$$\begin{aligned} &\frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q \left|b(x) - M_Q(b)(x)\right|^q \mathrm{d}x \right)^{1/q} \\ &= \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q \left|b(x)M(\chi_Q)(x) - M_Q(b\chi_Q)(x)\right|^q \mathrm{d}x \right)^{1/q} \\ &= \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q \left|[b, M](\chi_Q)(x)\right|^q \mathrm{d}x \right)^{1/q} \\ &\leq \frac{1}{|Q|^{\beta/n+1/q}} \left\|[b, M](\chi_Q)\right\|_{L^q(\mathbb{R}^n)} \\ &\leq \frac{C}{|Q|^{\beta/n+1/q}} \left\|\chi_Q\right\|_{L^p(\mathbb{R}^n)} \\ &\leq C, \end{aligned} \tag{4.2}$$

which implies (3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) $\Longrightarrow$ (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 2.1, it suffices to verify that there is a constant $C > 0$ such that for all cubes Q,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q|\mathrm{d}x \leq C. \tag{4.3}$$

For any fixed cube Q, let $E = \{x \in Q : b(x) \leq b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. The following equality is trivially true (see [4] page 3331):

$$\int_E |b(x) - b_Q|\mathrm{d}x = \int_F |b(x) - b_Q|\mathrm{d}x.$$

Since for any $x \in E$ we have $b(x) \leq b_Q \leq M_Q(b)(x)$, then for any $x \in E$,

$$|b(x) - b_Q| \leq |b(x) - M_Q(b)(x)|.$$

Thus,

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\mathrm{d}x &= \frac{1}{|Q|^{1+\beta/n}}\int_{E\cup F} |b(x)-b_Q|\mathrm{d}x\\ &= \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-b_Q|\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-M_Q(b)(x)|\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\mathrm{d}x.\end{aligned} \tag{4.4}$$

On the other hand, it follows from Hölder's inequality and (1.3) that

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q \big|b(x)-M_Q(b)(x)\big|\mathrm{d}x &\le \frac{1}{|Q|^{1+\beta/n}}\left(\int_Q \big|b(x)-M_Q(b)(x)\big|^q\mathrm{d}x\right)^{1/q}|Q|^{1/q'}\\ &\le \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q \big|b(x)-M_Q(b)(x)\big|^q\mathrm{d}x\right)^{1/q}\\ &\le C.\end{aligned}$$

This together with (4.4) gives (4.3), and so we achieve $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$.

In order to prove $b\ge 0$, it suffices to show $b^-=0$, where $b^-=-\min\{b,0\}$. Let $b^+=|b|-b^-$, then $b=b^+-b^-$. For any fixed cube Q, observe that

$$0\le b^+(x)\le |b(x)|\le M_Q(b)(x)$$

for $x\in Q$ and therefore we have that, for $x\in Q$,

$$0\le b^-(x)\le M_Q(b)(x)-b^+(x)+b^-(x)=M_Q(b)(x)-b(x).$$

Then, it follows from (1.3) that, for any cube Q,

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x &\le \frac{1}{|Q|}\int_Q |M_Q(b)(x)-b(x)|\\ &\le \left(\frac{1}{|Q|}\int_Q |b(x)-M_Q(b)(x)|^q\mathrm{d}x\right)^{1/q}\\ &= |Q|^{\beta/n}\left\{\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x)-M_Q(b)(x)|^q\mathrm{d}x\right)^{1/q}\right\}\\ &\le C|Q|^{\beta/n}.\end{aligned}$$

Thus, $b^-=0$ follows from Lebesgue's differentiation theorem.

The proof of Theorem 1.4 is completed. □

Proof of Theorem 1.5. Obviously, Theorem 1.5 follows from (4.1) and Theorem 1.1. □

Proof of Theorem 1.6. (1) $\Longrightarrow$ (2): Assume $b\ge 0$ and $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$, then by (4.1) and Theorem 1.2 we see that $[b,M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

(2) $\Longrightarrow$ (1): Assume that $[b,M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Similarly to (4.2), we have, for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}&\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q \big|b(x)-M_Q(b)(x)\big|^q\mathrm{d}x\right)^{1/q}\\ &= \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q \big|[b,M](\chi_Q)(x)\big|^q\mathrm{d}x\right)^{1/q}\end{aligned}$$

$$\begin{aligned}&\le \frac{|Q|^{\lambda/(nq)}}{|Q|^{\beta/n+1/q}}\big\|[b,M](\chi_Q)\big\|_{L^{q,\lambda}(\mathbb{R}^n)}\\&\le \frac{C|Q|^{\lambda/(nq)}}{|Q|^{\beta/n+1/q}}\|\chi_Q\|_{L^{p,\lambda}(\mathbb{R}^n)}\\&\le C,\end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(n-\lambda)$ and Lemma 2.4.

This shows by Theorem 1.4 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$. □

Proof of Theorem 1.7. By the same way of the proof of Theorem 1.6, Theorem 1.7 can be proven. We omit the details. □

Acknowledgements

The author would like to express his gratitude to the referee for his/her very valuable comments and suggestions, which greatly improved the final version of this paper.

References

[1] D.R. Adams, A note on Riesz potentials, Duke Math. J. 42 (1975) 765–778.
[2] D.R. Adams, Morrey Spaces, Lect. Notes Appl. Numer. Harmon. Anal., Springer International Publishing, Switzerland, 2015.
[3] M. Agcayazi, A. Gogatishvili, K. Koca, R. Mustafayev, A note on maximal commutators and commutators of maximal functions, J. Math. Soc. Jpn. 67 (2) (2015) 581–593.
[4] J. Bastero, M. Milman, F.J. Ruiz, Commutators for the maximal and sharp functions, Proc. Amer. Math. Soc. 128 (11) (2000) 3329–3334.
[5] A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister, On the product of functions in BMO and H^1, Ann. Inst. Fourier 57 (5) (2007) 1405–1439.
[6] R.R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. Math. 103 (2) (1976) 611–635.
[7] R.A. DeVore, R.C. Sharpley, Maximal functions measuring smoothness, Mem. Amer. Math. Soc. 47 (293) (1984) 1–115.
[8] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. Math., vol. 29, Amer. Math. Soc., Providence, RI, USA, 2001.
[9] J. García-Cuerva, E. Harboure, C. Segovia, J.L. Torrea, Weighted norm inequalities for commutators of strongly singular integrals, Indiana Univ. Math. J. 40 (1991) 1397–1420.
[10] L. Grafakos, Classical Fourier Analysis, third edition, Grad. Texts Math., vol. 249, Springer, New York, 2014.
[11] G. Hu, H. Lin, D. Yang, Commutators of the Hardy–Littlewood maximal operator with BMO symbols on spaces of homogeneous type, Abstr. Appl. Anal. (2008) 237937, 21 pages.
[12] G. Hu, D. Yang, Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type, J. Math. Anal. Appl. 354 (2009) 249–262.
[13] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Mat. 16 (1978) 263–270.
[14] S. Janson, M. Taibleson, G. Weiss, Elementary characterization of the Morrey–Campanato spaces, Lect. Notes Math. 992 (1983) 101–114.
[15] Y. Komori, T. Mizuhara, Notes on commutators and Morrey spaces, Hokkaido Math. J. 32 (2003) 345–353.
[16] M. Milman, T. Schonbek, Second order estimates in interpolation theory and applications, Proc. Amer. Math. Soc. 110 (4) (1990) 961–969.
[17] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938) 126–166.
[18] M. Paluszyński, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J. 44 (1) (1995) 1–17.
[19] J. Peetre, On the theory of $L_{p,\lambda}$ spaces, J. Funct. Anal. 4 (1969) 71–87.
[20] C. Segovia, J.L. Torrea, Vector-valued commutators and applications, Indiana Univ. Math. J. 38 (4) (1989) 959–971.
[21] C. Segovia, J.L. Torrea, Higher order commutators for vector-valued Calderón–Zygmund operators, Trans. Amer. Math. Soc. 336 (2) (1993) 537–556.
[22] S. Shirai, Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces, Hokkaido Math. J. 35 (2006) 683–696.
[23] P. Zhang, Multiple weighted estimates for commutators of multilinear maximal function, Acta Math. Sin. Engl. Ser. 31 (6) (2015) 973–994.
[24] P. Zhang, J.L. Wu, Commutators of the fractional maximal functions, Acta Math. Sin. 52 (6) (2009) 1235–1238.
[25] P. Zhang, J.L. Wu, Commutators of the fractional maximal function on variable exponent Lebesgue spaces, Czechoslov. Math. J. 64 (2014) 183–197.
[26] P. Zhang, J.L. Wu, Commutators for the maximal functions on Lebesgue spaces with variable exponent, Math. Inequal. Appl. 17 (4) (2014) 1375–1386.