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Ordinary differential equations

A simple proof of the Lyapunov finite-time stability theorem

Une démonstration simple du théorème de Lyapunov sur la stabilité en temps fini

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ABSTRACT

We offer a simple proof of the Lyapunov finite-time stability theorem for Filippov systems which does not use any generalized derivatives to differentiate the composition of the Lyapunov function with absolutely continuous solutions.

RÉSUMÉ

On propose une démonstration simple du théorème de Lyapunov sur la stabilité en temps fini pour des systèmes de Filippov sans utilisation de dérivées généralisées pour dériver la composition d'une fonction de Lyapunov et d'une solution absolument continue.

1. Introduction

Recall that an absolutely continuous function x is a Filippov solution to a system of differential equations continuous in t and piecewise continuous in x right-hand sides

$$\dot{x} = f(t, x), \tag{1}$$

if, for almost all t, one has

$$\begin{aligned} &\dot{x}(t) \in K[f](t, x(t)), \\ &K[f](t, x) = \bigcap_{\delta>0} \bigcap_{\mu(J)=0} \overline{\text{co}}\, f(t, B_\delta(x)\backslash J), \end{aligned} \tag{2}$$

where $B_r(x)$ is the ball of $\mathbb{R}^n$ of radius r and centered at x, and $\overline{\text{co}}\, C$ is the convex hull of C [4, p. 49, p. 61], $\mu(J)$ is the Lebesgue measure of the measurable set J. For example, the solution $x(t) \equiv \bar{x}$ which we get from the phase portrait of Fig. 1(left) is a Filippov solution to the simplest bang-bang control system given by Fig. 1(right) and governed by

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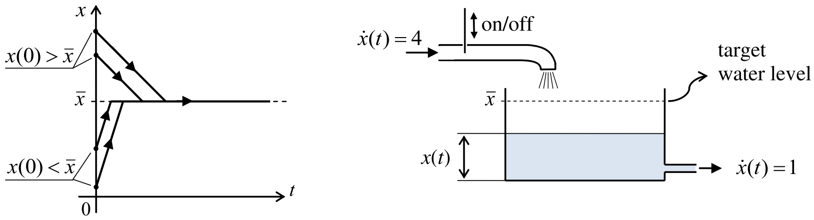

Fig. 1. Phase portrait of system (3) (left) and the respective model of a water tank with just one water level control sensor $\bar{h}$ (right).

$$\dot{x} = 1 - 2\,\mathrm{sign}(x - \bar{x}), \tag{3}$$

because (2) takes the form

$$\dot{x}(t) \in 1 - 2\,\mathrm{Sign}(x - \bar{x}),$$

$$\text{where } \mathrm{Sign}(s) = K[\mathrm{sign}](s) = \begin{cases} 1, & s > 0, \\ [-1, 1], & s = 0, \\ -1, & s < 0. \end{cases}$$

At the same time, $x(t) \equiv \bar{x}$ is not a solution to (3) as one gets $0 = 1$ when plugging $x(t) \equiv \bar{x}$ into (3).

As one can learn from Fig. 1, finite-time convergence to an equilibrium is a natural feature of discontinuous systems. The most widely used method to establish the finite-time stability (say, of the origin) in a discontinuous system is the direct Lyapunov method with the Lyapunov function given by

$$V(x) = |x_1| + \ldots + |x_n|. \tag{4}$$

The Lyapunov function (4) is used for stabilization of robotic manipulators [9] in mechanical engineering, formation control [11], consensus [6] and synchronization [7] in network science, current control in power electronics [2], and other applied problems. However, the available proofs are based on viewing $V(x)$ as a Clarke's regular function and utilizing the calculus for Clarke's generalized gradients (see [9, Theorem 2] and [10, Theorem 2.2]).

In the next section of the paper, we offer a simple proof of the Lyapunov stability theorem for Lyapunov functions of the form (4), which does not need any new concepts beyond what we introduced in this section already. In section 3, we give an example (Example 3.2) that can serve as a guideline for the rigorous application of the result of section 2, which did not appear in the literature so explicitly. The interested reader will be able to extend our proof to any function V given by a linear combination (with positive multipliers) of functions of the form $\max_{i \in I} g_i(x)$ (max-functions). Here I is a finite set of indices and $g_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Max-functions seem to be the central class of nonsmooth Lipschitz continuous Lyapunov functions that is used in applications to fulfill the regularity property of Clarke, under which assumption the Lyapunov finite-time stability theorem was originally proven by Paden and colleagues ([9, Theorem 2], [10, Theorem 2.2]). That is why we believe that, despite of its simplicity, our proof carries sufficient generality as far as applications are concerned.

We note that the asymptotic stability of a Filippov solution to discontinuous system (1) can always be investigated over smooth Lyapunov functions, see Clarke–Ledyaev–Stern [3] and Bacciotti–Rosier [1, Ch. 4]. Asymptotic stability further implies finite-time stability, e.g., for homogeneous systems (1), see, e.g., Orlov [8]. Direct Lyapunov finite-time stability theorems, such as the above-mentioned [9, Theorem 2] and [10, Theorem 2.2] (as well as Theorem 2.1 below), do not require homogeneity.

2. The main result

We say that $(t, x) \in \mathbb{R} \times S$, $S = \cup_{i=1}^{n} \{x \in \mathbb{R}^n : x_i = 0,\ x \neq 0\}$, is a point of crossing if

$$x + \xi \notin S, \quad \text{for all} \quad \xi \in K[f](t, x). \tag{5}$$

Theorem 2.1. *Let $(t, x) \mapsto f(t, x)$ be continuous except probably on $\mathbb{R} \times S$ where $(t, x) \mapsto f(t, x)$ experiences discontinuities of the first kind with respect to x. Let W be a set of the form $W = \{x : |x_1| + \ldots + |x_n| < const\}$. Assume that there exists $k > 0$ such that*

$$\sum_{i=1}^{n} \mathrm{sign}(x_i)\xi_i \le -k < 0 \quad \text{for all } \xi \in K[f](t, x), \quad (t, x) \in \mathbb{R} \times W \backslash \{0\}, \tag{6}$$

excluding those $(t, x) \in S$, which are the points of crossing for (1). Then the origin is a finite-time stable equilibrium for (1). Specifically, any Filippov solution x of (1) with the initial condition $x(t_0) \in W$ never leaves W in the future and approaches the origin in finite time.

Remark 2.1. If $x \notin S$, then (6) just coincides with

$$\sum_{i=1}^{n} \operatorname{sign}(x_i) f(t, x(t)),$$

which is the derivative of $V(x(t))$ with V given by

$$V(x) = |x_1| + |x_2| + \ldots + |x_n|. \tag{7}$$

The core of the proof is the following lemma. This lemma is the most essential ingredient of the Clarke's theory-based proofs, but it never appeared in the literature in so simple form before.

Lemma 2.1. *Assume that $b : \mathbb{R} \to \mathbb{R}$ is differentiable at t_* and $b(t_*) = 0$. If the function $a(t) = |b(t)|$ is differentiable at t_*, then $a'(t_*) = b'(t_*) = 0$.*

Proof. If $a'(t_*)$ exist, then

$$a'(t_*) = \lim_{\Delta t \to 0^-} \frac{a(t_* + \Delta t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{a(t_* + \Delta t)}{\Delta t}, \tag{8}$$

where we used that $a(t_*) = 0$. Computing the one-sided limits one gets

$$\lim_{\Delta t \to 0^-} \frac{a(t_* + \Delta t)}{\Delta t} = \lim_{\Delta t \to 0^-} \left| \frac{b(t_* + \Delta t)}{\Delta t} \right| \operatorname{sign} \Delta t = |b'(t_*)| \cdot (-1)$$

and, analogously,

$$\lim_{\Delta t \to 0^+} \frac{a(t_* + \Delta t)}{\Delta t} = |b'(t_*)| \cdot (+1).$$

Substituting the computed formulas for one-sided limits to (8) gives $|b'(t_*)| \cdot (-1) = |b'(t_*)| \cdot (+1)$, which is only possible when $b'(t_*) = 0$. The proof of the lemma is complete. $\square$

Proof of Theorem 2.1. Let x be a Filippov solution to (1) with the initial condition $x(t_0) \in W$. Consider

$$v(t) = V(x(t)) = v_1(t) + \ldots + v_n(t), \text{ where } v_i(t) = |x_i(t)|, \ i = 1, \ldots, n.$$

Since a composition of a Lipschitz function and of an absolutely continuous function is an absolutely continuous function, we have that there exists a zero measure set J_i such that both $\dot{v}_i(t)$ and $\dot{x}_i(t)$ exist for any $t \in [t_0, t_1) \backslash J_i$. In this instance, $t_1 = \infty$ if $x(t)$ exists on $[t_0, \infty)$, and t_1 is finite or $x(t)$ blows up at t_1. We will show that $t_1 = \infty$ later in the proof. By applying the above offered Lemma we can now conclude that

$$\begin{array}{l} \text{if } t \in [t_0, t_1) \backslash J_i, \\ \text{then either } [x_i(t) = 0 \text{ and } \dot{v}_i(t) = 0] \text{ or } x_i(t) \neq 0. \end{array} \tag{9}$$

Since for $x_i(t) \neq 0$, such that $\dot{x}_i(t)$ exists, one has $\dot{v}_i(t) = \operatorname{sign}(x_i(t)) \dot{x}_i(t)$, then (9) implies:

$$\begin{array}{l} \text{if } t \in [t_0, t_1) \backslash J_i, \\ \text{then either } [x_i(t) = 0 \text{ and } \dot{v}_i(t) = 0] \\ \qquad \text{or } [x_i(t) \neq 0 \text{ and } \dot{v}_i(t) = \operatorname{sign}(x_i(t)) \dot{x}_i(t)], \end{array}$$

which compact formulation is

$$\text{if } t \in [t_0, t_1) \backslash J_i, \text{ then } \dot{v}_i(t) = \operatorname{sign}(x_i(t)) \dot{x}_i(t).$$

Summing this up from 1 to n and denoting $J = J_1 \cup \ldots \cup J_n$, one gets

$$\dot{v}(t) = \sum_{i=1}^{n} \operatorname{sign}(x_i(t)) \dot{x}_i(t), \quad \text{for any } t \in [t_0, t_1) \backslash J.$$

Since $\dot{x}(t) \in K[f](t, x(t))$ for almost all $t \in [t_0, t_1)$, then there exists $H \subset [t_0, t_1)$ of measure zero such that

$$\dot{v}(t) = \sum_{i=1}^{n} \operatorname{sign}(x_i(t)) \xi_i,$$

for some $\xi \in K[f](t, x(t))$ and any $t \in [t_0, t_1)\backslash(J \cup H)$. Using the assumption (6) of the theorem, we conclude

$$\dot{v}(t) \le -k \quad \text{for any } t \in [t_0, t_1)\backslash(J \cup H \cup C), \tag{10}$$

where $C \subset [t_0, t_1)$ is the set of times t for which either $x(t) = 0$ or $(t, x(t))$ is a point of crossing of (1). Since the discontinuities of $(t, x) \mapsto f(t, x)$ are discontinuities of first kind, then the sets $K[f](t, x)$ are compact and thus the property (5) holds in a neighborhood of each point of crossing. As a consequence, the times t of crossing are isolated in time. Therefore, the set of times of crossing is countable and has measure zero (as well as J and H). Thus, we finally get

$$\dot{v}(t) \le -k, \quad \text{for almost all } t \in [t_0, t_1) \text{ such that } x(t) \ne 0. \tag{11}$$

The rest of the proof is standard.

Step 1. Let us show that $x(t)$ reaches the origin on $[t_0, t_1)$ at least once. Assume the contrary, i.e. $x(t) \ne 0$ for all $t \in [t_0, t_1)$. Therefore, by taking the integral of (11),

$$v(t) \le v(t_0) - k(t - t_0), \quad \text{for all } t \in [t_0, t_1). \tag{12}$$

This implies that $x(t)$ cannot escape from W on $[t_0, t_1)$ and so $t_1 = \infty$. In particular, $v(t)$ is defined on $[t_0, t_0 + v(0)/k]$ and thus it vanishes at some point of $[t_0, t_0 + v(0)/k]$ by (12). The contradiction obtained proves that there exists $\tau \in [t_0, t_1)$ such that $v(\tau) = 0$ (i.e. $x(\tau) = 0$).

Step 2. Observe that $x(t) = 0$ for all $t \in [\tau, t_1)$. Indeed, otherwise there will exist $\tau_1 \in [\tau, t_1)$ such that $v(\tau_1) = 0$ and $v(t) \ne 0$ for all $t \in (\tau_1, t_1)$, so that we can integrate (11) once again as

$$v(t) \le -k(t - \tau_1), \quad \text{for all } t \in [\tau_1, t_1),$$

and obtain a contradiction with the positiveness of $v(t)$.

Finally, since $x(t) = 0$ for all $t \in [\tau, t_1)$, then $x(t)$ cannot escape from W on $[\tau, t_1)$ and so $t_1 = \infty$. The proof of the theorem is complete. □

3. Examples

The next example illustrates how Theorem 2.1 translates into an arbitrary target equilibrium $\bar{x}$.

Example 3.1. Prove the finite-time stability of $x = \bar{x}$ in the bang-bang-controlled water tank (Fig. 1).

To apply Theorem 2.1, we need to examine the expression $\operatorname{sign}(x - \bar{x})\xi$ with $\xi \in K[1 - 2\operatorname{sign}(\cdot - \bar{x})](x)$. This gives

$$\operatorname{sign}(x - \bar{x})\xi = \begin{cases} (-1)\cdot(1+2) = -3, & \text{if } x < \bar{x}, \\ 1 \cdot (1-2) = -1, & \text{if } x > \bar{x} \end{cases}$$

and Theorem 2.1 applies with $k = 1$ to prove the global finite-time stability of $x = \bar{x}$.

Note the asymptotic stability of the equilibrium $\bar{x}$ in Example 3.1 follows by utilizing the smooth (quadratic) Lyapunov function $V(x) = (x - \bar{x})^2$.

The next example resembles switching properties typical for neural networks, see [6,5].

Example 3.2. Prove the finite-time stability of the origin in

$$\begin{aligned} \dot{x}_1 &= -3\operatorname{sign} x_1 - \operatorname{sign} x_2 &&=: f_1(x_1, x_2), \\ \dot{x}_2 &= \operatorname{sign} x_1 - 3\operatorname{sign} x_2 &&=: f_1(x_1, x_2). \end{aligned} \tag{13}$$

To prove the finite-time stability of the origin, we will find $k > 0$, such that

$$\operatorname{sign}(x_1)\xi_1 + \operatorname{sign}(x_2)\xi_2 \le -k, \quad \text{for all } \xi \in K[f](x),\ x \ne 0.$$

There are several cases to consider.

$\boldsymbol{x_1 \ne 0,\ x_2 \ne 0}$**:** In this case, $K[f](x) = \{f(x)\}$ and $\xi = f(x)$. Therefore

$$\operatorname{sign}(x_1)f_1(x) + \operatorname{sign}(x_2)f_2(x) = \operatorname{sign}(x_1)(-3\operatorname{sign}(x_1) - \operatorname{sign}(x_2)) + \operatorname{sign}(x_2)(\operatorname{sign}(x_1) - 3\operatorname{sign}(x_2)) \le -6.$$

$\boldsymbol{x_1 = 0,\ x_2 \ne 0}$**:** Here $K[f](x) \subset \begin{pmatrix} -3\operatorname{Sign}(0) - \operatorname{sign}(x_2) \\ \operatorname{Sign}(0) - 3\operatorname{sign}(x_2) \end{pmatrix}$ and so

$$\operatorname{sign}(x_2)\xi_2 = \operatorname{sign}(x_2)(a - 3\operatorname{sign}(x_2)), \quad \text{where } a \in \operatorname{Sign}(0).$$

Therefore, $\operatorname{sign}(x_2)\xi_2 \le -2$.

$\boldsymbol{x_1 \neq 0,\ x_2 = 0}$**:** Here $K[f](x) \subset \begin{pmatrix} -3\,\mathrm{sign}(x_1) - \mathrm{Sign}(0) \\ \mathrm{sign}(x_1) - 3\,\mathrm{Sign}(0) \end{pmatrix}$ and so

$$\mathrm{sign}(x_1)\xi_1 = \mathrm{sign}(x_1)(-3\,\mathrm{sign}(x_1) - a), \quad \text{where } a \in \mathrm{Sign}(0),$$

from where $\mathrm{sign}(x_1)\xi_1 \leq -2$.

To summarize, the finite-time stability of the origin in (13) follows by applying Theorem 2.1 with $k = 2$.

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References

[1] A. Bacciotti, L. Rosier, Liapunov Functions and Stability in Control Theory, second edition, Communications and Control Engineering Series, Springer-Verlag, Berlin, 2005, xiv+237 pp.
[2] A. Cavini, C. Rossi, A. Tilli, Comparison of hysteresis current controllers using nonsmooth Lyapunov functions, in: Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, Seville, Spain, 2005, pp. 3982–3987.
[3] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, Asymptotic stability and smooth Lyapunov functions, J. Differ. Equ. 149 (1) (1998) 69–114.
[4] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Mathematics and its Applications (Soviet Series), vol. 18, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1988.
[5] M. Forti, P. Nistri, Global convergence of neural networks with discontinuous neuron activations, IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. 50 (11) (2003) 1421–1435.
[6] Q. Hui, W.M. Haddad, S.P. Bhat, Semistability theory for differential inclusions with applications to consensus problems in dynamical networks with switching topology, in: Proceedings of 2008 American Control Conference, IEEE, 2008, pp. 3981–3986.
[7] O. Makarenkov, P. Nistri, D. Papini, Synchronization problems for unidirectional feedback coupled nonlinear systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 15 (4) (2008) 453–468.
[8] Y. Orlov, Finite time stability and robust control synthesis of uncertain switched systems, SIAM J. Control Optim. 43 (4) (2004–2005) 1253–1271.
[9] B.E. Paden, S.S. Sastry, A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators, IEEE Trans. Circuits Syst. 34 (1) (1987) 73–82.
[10] D. Shevitz, B. Paden, Lyapunov stability theory of nonsmooth systems, IEEE Trans. Autom. Control 39 (9) (1994) 1910–1914.
[11] S. Zhao, F. Lin, K. Peng, B.M. Chen, T.H. Lee, Finite-time stabilisation of cyclic formations using bearing-only measurements, Int. J. Control 87 (4) (2014) 715–727.