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The complex Monge–Ampère equation on weakly pseudoconvex domains ☆

L'équation de Monge–Ampère complexe sur les domaines faiblement pseudo-convexes

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A B S T R A C T

We show here a "weak" Hölder regularity up to the boundary of the solution to the Dirichlet problem for the complex Monge–Ampère equation with data in the L^p space and Ω satisfying an f-property. The f-property is a potential-theoretical condition that holds for all pseudoconvex domains of finite type and many examples of infinite-type ones.

R É S U M É

Nous montrons ici une régularité de Hölder « faible » jusqu'au bord d'une solution du problème de Dirichlet pour l'équation de Monge–Ampère complexe, de donnée dans l'espace L^p, sur un domaine satisfaisant une f-propriété. Cette f-propriété est une condition de théorie du potentiel qui est satisfaite par tous les domaines pseudo-convexes de type fini et de nombreux exemples de type infini.

1. Introduction

For a C^2, bounded, pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$, the Dirichlet problem for the Monge–Ampère equation consists of

$$\begin{cases} u \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega), & \\ (dd^c u)^n = \psi\, \mathrm{d}V & \text{in } \Omega, \\ u = \varphi & \text{on } b\Omega. \end{cases} \tag{1.1}$$

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A great deal of work has been done for the case where Ω is strongly pseudoconvex. Within this domain, we can divide the literature into three kinds of data ψ.

- The Hölder data: Bedford–Taylor prove in [2] that $u \in C^{\frac{\alpha}{2}}(\bar{\Omega})$ if $\varphi \in C^{\alpha}(b\Omega)$, $\psi^{\frac{1}{n}} \in C^{\frac{\alpha}{2}}(\bar{\Omega})$ for $0 < \alpha \le 2$.
- The smooth data: Caffarelli, Kohn and Nirenberg prove in [4] that $u \in C^{\infty}(\bar{\Omega})$, for $\varphi \in C^{\infty}(b\Omega)$ and $\psi \in C^{\infty}(\bar{\Omega})$, in case $\psi > 0$ in $\bar{\Omega}$ and $b\Omega$ is smooth.
- The L^p data: Guedj, Kolodziej and Zeriahi prove in [6] that if $\psi \in L^p(\Omega)$ with $p > 1$ and $\varphi \in C^{1,1}(b\Omega)$ then $u \in C^{\gamma}(\bar{\Omega})$ for any $\gamma < \gamma_p := \frac{2}{qn+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$.

When Ω is no longer strongly pseudoconvex but has a certain "finite type", there are some known results for this problem due to Blocki [3], Coman [5], and Li [11]. Recently, Ha and the second author gave a general related result to a Hölder data under the hypothesis that Ω satisfies an f-property (see Definition 2.1 below). The f-property is a consequence of the geometric "type" of the boundary. All pseudoconvex domains of finite type satisfy the f-property as well as many classes of domains of infinite type (see [9,7,8] for discussion on the f-property). Using the f-property, a "weak" Hölder regularity for the solution to the Dirichlet problem of the complex Monge–Ampère equation is obtained in [9]. Coming back to the case of Ω of finite type, in a recent paper with Zampieri [1], we prove the Hölder regularity for $\psi \in L^p$, with $p > 1$. The purpose of the present paper is to generalize the result in [1] to a pseudoconvex domain satisfying an f-property. For this purpose, we recall the definition of a weak Hölder space in [9,7]. Let f be an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$, $f(t) \lesssim t$. For a subset A of $\mathbb{C}^n$, define the f-Hölder space on A by

$$\Lambda^f(A) = \{u : \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1}) \cdot |u(z) - u(w)| < \infty\}$$

and set

$$\|u\|_{\Lambda^f(A)} = \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1}) \cdot |u(z) - u(w)|.$$

Note that the notion of the f-Hölder space includes the standard Hölder space Λ_α by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \le 1$. Here is our result

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain admitting the f-property. Suppose that $\int_1^\infty \frac{\mathrm{d}a}{af(a)} < \infty$ and denote by $g(t) := \left(\int_t^\infty \frac{\mathrm{d}a}{af(a)}\right)^{-1}$ for $t \ge 1$. If $0 < \alpha \le 2$, $\varphi \in \Lambda^{t^\alpha}(b\Omega)$, and $\psi \ge 0$ on Ω with $\psi \in L^p$ with $p > 1$, then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution $u \in \Lambda^{g^\beta}(\bar{\Omega})$. Here $\beta = \min(\alpha, \gamma)$, for any $\gamma < \gamma_p = \frac{2}{nq+1}$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

The proof follows immediately from Theorem 2.2 and 2.5 below. Throughout the paper we use $\lesssim$ and $\gtrsim$ to denote an estimate up to a positive constant, and $\approx$ when both of them hold simultaneously. Finally, the indices p, α, β, γ and γ_p only take ranges as in Theorem 1.1.

2. Hölder regularity of the solution

We start this section by defining the f-property as in [7,8].

Definition 2.1. For a smooth, monotonic, increasing function $f : [1, +\infty) \to [1, +\infty)$ with $f(t)t^{-1/2}$ decreasing, we say that Ω has the f-property if there exist a neighborhood U of $b\Omega$ and a family of functions $\{\varphi_\delta\}$ such that

(i) the functions φ_δ are plurisubharmonic, C^2 on U, and satisfy $-1 \le \varphi_\delta \le 0$,
(ii) $i\partial\bar{\partial}\varphi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\varphi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2-defining function of Ω.

In [7], using the f-property, the second author constructed a family of plurisubharmonic peak functions with good estimates. This family of plurisubharmonic peak functions yields the existence of a defining function ρ which is uniformly strictly plurisubharmonic and weakly Hölder (see [9]).

Theorem 2.2 *(Khanh [7] and Ha–Khanh [9]). Assume that Ω is a bounded, pseudoconvex domain admitting the f-property as in Theorem 1.1. Then there exists a uniformly strictly-plurisubharmonic defining function of Ω that belongs to the g^2-Hölder space of $\overline{\Omega}$, which means that*

$$\rho \in \Lambda^{g^2}(\bar\Omega), \quad \Omega = \{\rho < 0\} \quad \textit{and} \quad \mathrm{i}\partial\bar\partial\rho \geq Id. \tag{2.1}$$

The existence and uniqueness of the solution $u \in L^\infty(\Omega)$ to the equation (1.1) need a weaker condition, in particular, one only need $\rho \in C^0(\bar\Omega)$, as shown by [10].

Theorem 2.3 *(Kolodziej [10]). Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that there exists a function ρ such that*

$$\rho \in C^0(\bar\Omega), \quad \Omega = \{\rho < 0\} \quad \textit{and} \quad \mathrm{i}\partial\bar\partial\rho \geq Id.$$

Then, for any $\varphi \in C^0(b\Omega)$, $\psi \in L^p(\Omega)$, there is a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in C^0(\bar\Omega)$.

To improve the smoothness of u, we increase the smoothness of ρ and ψ.

Theorem 2.4 *(Ha–Khanh [9]). Let ρ satisfy (2.1). If $\varphi \in \Lambda^{t^\alpha}(b\Omega)$ and $\psi^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\bar\Omega)$, then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in \Lambda^{g^\alpha}(\bar\Omega)$.*

Now we focus on lowering the smoothness of ψ and prove the following theorem.

Theorem 2.5. *Let ρ satisfy (2.1). If $\varphi \in \Lambda^{t^\alpha}(b\Omega)$ and $\psi \in L^p(\Omega)$, then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution $u(\Omega, \varphi, \psi) \in \Lambda^{g^\beta}(\bar\Omega)$.*

In order to prove this theorem, we need to construct a subsolution with L^p data. Here, v is a subsolution to (1.1) in the sense that v is plurisubharmonic, $v|_{b\Omega} = \varphi$ and $(dd^c v)^n \geq \psi\, \mathrm{d}V$ in Ω.

Proposition 2.6. *Let ρ satisfy (2.1). Then there is a subsolution $v \in \Lambda^{g^\beta}(\bar\Omega)$ to (1.1) for $\varphi \in C^\alpha(b\Omega)$ and $\psi \in L^p(\Omega)$.*

Proof. For a large ball $\mathbb{B}$ containing Ω, we set $\tilde\psi(z) := \begin{cases} \psi(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{B} \setminus \Omega. \end{cases}$ First, we apply Theorem 1 in [6] on $\mathbb{B}$ with $\tilde\psi \in L^p(\mathbb{B})$ and zero-valued boundary condition; it follows $u_1 = u(\mathbb{B}, 0, \tilde\psi) \in \Lambda^{t^\gamma}(\bar{\mathbb{B}})$. Second, we apply Theorem 2.4 on Ω twice: first for $u_2 := u(\Omega, -u_1|_{b\Omega}, 0) \in \Lambda^{g^\gamma}$, since $u_1|_{b\Omega} \in \Lambda^{t^\gamma}$, and second for $u_3 := u(\Omega, \varphi, 0) \in \Lambda^{g^\alpha}$ by the hypothesis $\varphi \in \Lambda^{t^\alpha}$. Finally, taking the summation $v = u_1 + u_2 + u_3$, we have the conclusion. □

Proof of Theorem 2.5. Keeping the notation of Theorem 2.3, let $u(\Omega, \varphi, \psi) \in C^0(\bar\Omega)$ be the solution to (1.1). What follows is dedicated to showing that this C^0 plurisubharmonic solution $u(\Omega, \varphi, \psi)$ is in fact in $\Lambda^{g^\beta}(\bar\Omega)$. By Theorem 2.4 we have that $w := u(\Omega, \varphi, 0)$ is in $\Lambda^{g^\alpha}(\bar\Omega)$. Let v be as in Proposition 2.6 then the comparison principle yields at once

$$v \leq u(\Omega, \varphi, \psi) \leq w. \tag{2.2}$$

By (2.2) and the g^β-Hölder regularity of v and w, we get

$$|u(z) - u(\zeta)| \lesssim [g(|z - \zeta|^{-1})]^{-\beta} \quad z \in \bar\Omega,\ \zeta \in b\Omega,$$

and therefore for δ suitably small

$$|u(z) - u(z')| \lesssim [g(\delta^{-1})]^{-\beta}, \quad z,\ z' \in \Omega \setminus \Omega_\delta \text{ and } |z - z'| < \delta \tag{2.3}$$

where $\Omega_\delta := \{z \in \mathbb{C}^n : r(z) < -\delta\}$ and r is the C^2 defining function for Ω with $|\nabla r| = 1$ on $b\Omega$. We have to prove that (2.3) also holds for $z,\ z' \in \Omega_\delta$. For $z \in \bar\Omega_\delta$, we use the notation

$$u_{\frac{\delta}{2}}(z) := \sup_{|\zeta| < \frac{\delta}{2}} u(z + \zeta), \qquad \tilde u_{\frac{\delta}{2}}(z) := \frac{1}{\sigma_{2n-1}\left(\frac{\delta}{2}\right)^{2n-1}} \int\limits_{b\mathbb{B}(z, \frac{\delta}{2})} u(\zeta)\, \mathrm{d}S(\zeta),$$

and

$$\hat u_{\frac{\delta}{2}}(z) := \frac{1}{\sigma_{2n}(\frac{\delta}{2})^{2n}} \int\limits_{\mathbb{B}(z, \frac{\delta}{2})} u(\zeta)\, \mathrm{d}V(\zeta),$$

where $\sigma_{2n-1}\left(\frac{\delta}{2}\right)^{2n-1} = \mathrm{Vol}(b\mathbb{B}(z, \frac{\delta}{2}))$ and $\sigma_{2n}\left(\frac{\delta}{2}\right)^{2n} = \mathrm{Vol}(\mathbb{B}(z, \frac{\delta}{2}))$. It is obvious that

$$\hat u_{\frac{\delta}{2}} \leq \tilde u_{\frac{\delta}{2}} \leq u_{\frac{\delta}{2}} \quad \text{in} \quad \Omega_\delta. \tag{2.4}$$

Furthermore, we have an L^1 estimate of the difference between u and $\tilde u_{\frac{\delta}{2}}$ and of the stability estimate in the following theorems (2.7 and 2.8).

Theorem 2.7 *(Baracco–Khanh–Pinton–Zampieri [1]). For any $0 < \epsilon < 1$, we have*

$$\|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_\delta)} \lesssim \delta^{1-\epsilon}. \tag{2.5}$$

Theorem 2.8 *(Guedj–Kolodziej–Zeriahi [6]). Fix $0 \le f \in L^p(\Omega)$, $p > 1$. Let U, W be two bounded plurisubharmonic functions in Ω such that $(dd^c U)^n = f\,dV$ in Ω and let $U \ge W$ on $\partial\Omega$. Fix $s \ge 1$ and $0 \le \eta < \frac{s}{nq+s}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a uniform constant $C = C(\eta, \|f\|_{L^p(\Omega)}) > 0$ such that*

$$\sup_\Omega (W - U) \le C\|(W-U)_+\|^\eta_{L^s(\Omega)},$$

where $(W-U)_+ := \max(W-U, 0)$.

By (2.3), we have

$$\tilde{u}_{\frac{\delta}{2}} \le u_{\frac{\delta}{2}} \le u + c[g(\delta^{-1})]^{-\beta}, \quad \text{on } b\Omega_\delta \text{ for suitable constant } c.$$

Thus, we can apply Theorem 2.8 for Ω_δ with $U := u + c[g(\delta^{-1})]^{-\beta}$, $W := \tilde{u}_{\frac{\delta}{2}}$ and $s := 1$; thus we get

$$\begin{aligned}\sup_{\Omega_\delta}\left(\tilde{u}_{\frac{\delta}{2}} - (u + c[g(\delta^{-1})]^{-\beta})\right) &\underset{\text{Theorem 2.8}}{\lesssim} \left\|\left(\tilde{u}_{\frac{\delta}{2}} - (u + c[g(\delta^{-1})]^{-\beta})\right)_+\right\|^\eta_{L^1(\Omega_\delta)} \\ &\lesssim \|\tilde{u}_{\frac{\delta}{2}} - u\|^\eta_{L^1(\Omega_\delta)} \underset{\text{Theorem 2.7}}{\lesssim} \delta^{(1-\epsilon)\eta},\end{aligned} \tag{2.6}$$

for any $\eta < \frac{1}{2}\gamma_p = \frac{1}{nq+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$. Taking $\gamma < \gamma_p$, $\beta = \min(\alpha, \gamma)$, $\epsilon = \frac{\gamma_p - \gamma}{\gamma_p + \gamma} > 0$ and $\eta = \frac{1}{4}(\gamma_p + \gamma) < \frac{1}{2}\gamma_p$ so that $(1-\varepsilon)\eta = \frac{\gamma}{2}$, it follows

$$\sup_{\Omega_\delta}\left(\tilde{u}_{\frac{\delta}{2}} - u\right) \lesssim \delta^{(1-\epsilon)\eta} + [g(\delta^{-1})]^{-\beta} \lesssim \delta^{\frac{\gamma}{2}} + [g(\delta^{-1})]^{-\beta} \lesssim [g(\delta^{-1})]^{-\beta}, \tag{2.7}$$

where the last inequality of (2.7) follows by $g(\delta^{-1}) \lesssim \delta^{-\frac{1}{2}}$ (by the conditions on f in the f-property).

Similarly to [6, Lemma 4.2] by using the fact that $g(c\delta^{-1})) \approx g(\delta^{-1})$ for any constant $c > 0$, one can state the equivalence between

$$\sup_{\Omega_\delta}(u_\delta - u) \lesssim [g(\delta^{-1})]^{-\beta} \quad \text{and} \quad \sup_{\Omega_\delta}(\hat{u}_\delta - u) \lesssim [g(\delta^{-1})]^{-\beta}.$$

Using this equivalence together with the inequalities in (2.4), it follows that (2.7) is equivalent to

$$\sup_{\Omega_\delta}(u_{\frac{\delta}{2}} - u) \lesssim [g(\delta^{-1})]^{-\beta}. \tag{2.8}$$

From (2.3) and (2.8), it is easy to prove that

$$|u(z) - u(z')| \lesssim [g(|z - z'|^{-1})]^{-\beta} \quad \text{for any } z, z' \in \bar{\Omega}. \qquad \square$$

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