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Logarithmic Sobolev inequality revisited

L'inégalité de Sobolev logarithmique revisitée

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A R T I C L E I N F O

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A B S T R A C T

We provide a new characterization of the logarithmic Sobolev inequality.

R É S U M É

Nous donnons une nouvelle caractérisation de l'inégalité de Sobolev logarithmique.

1. Introduction

The classical Sobolev inequality translates information about the derivatives of a function into information about the size of the function itself. Precisely, for a function u with square summable gradient in dimension N, one obtains that u is $2N/(N-2)$-summable, that is a gain in summability that depends on N and tends to deteriorate as $N \to \infty$. On the other hand, since the middle 1950s, people have started looking at possible replacements of the Sobolev inequality in order to provide an improvement in the summability *independent* of the dimension N, which can be done in terms of the integrability properties of $u^2 \log u^2$. This was firstly done by Stam [23], who proved the logarithmic Sobolev inequality with Gauss measure $d\mathscr{G}$

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|^2_{2,d\mathscr{G}}} \, d\mathscr{G} \le \frac{1}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 \, d\mathscr{G}, \qquad d\mathscr{G} = e^{-\pi|x|^2} dx.$$

The formula was originally discovered in quantum field theory in order to handle estimates that are uniform in the space dimension, for systems with a large number of variables. A different proof and further insight was obtained by Gross in [17]. See also the work of Adams and Clarke [1] for an elementary proof of the previous inequality. These properties are

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widely used in statistical mechanics, quantum field theory and differential geometry. A variant of the logarithmic Sobolev inequality with Gauss measure is given by the following one-parameter family of Euclidean inequalities [18, Theorem 8.14]

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_2^2}\,\mathrm{d}x + N(1+\log a)\|u\|_2^2 \le \frac{a^2}{\pi}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x,$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. A version of this inequality for fractional Sobolev spaces $H^s(\mathbb{R}^N)$ can be found in [13]. Recently, some new characterization of the Sobolev spaces were provided in [2,19,21] (see also [3–9,20]) in terms of the following family of nonlocal functionals

$$I_\delta(u) := \int \int_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\mathrm{d}x\,\mathrm{d}y, \quad \delta > 0,$$

where u is a measurable function on $\mathbb{R}^N$. In particular, if $N \ge 3$ and $I_\delta(u) < \infty$ for some $\delta > 0$, then in [21] it was proved that

$$\int_{\{|u|>\lambda_N\delta\}} |u|^{2N/(N-2)}\mathrm{d}x \le C_N I_\delta(u)^{N/(N-2)}, \tag{1.1}$$

for some positive constants C_N and λ_N. This is a sort of nonlocal improvement of the classical Sobolev inequality, and it is also possible to show that in the singular limit $\delta \searrow 0$ one recovers the classical Sobolev result, since I_δ converges to the Dirichlet energy up to a normalization constant. The aim of this note is to remark that in this context also a logarithmic type estimate holds. Thus we have that the summability gain independent of N can be controlled in terms of $I_\delta(u)$.

More precisely, we have the following theorem.

Theorem 1.1. *Let $u \in L^2(\mathbb{R}^N)$ ($N \ge 3$). There is a positive constant C_N such that*

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2}\,\mathrm{d}x + \frac{N}{2}\log\|u\|_2^2 \le \frac{N}{2}\log\left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta(u)\right),$$

for all $\delta > 0$. In particular, if $u \in L^2(\mathbb{R}^N)$ is such that $I_\delta(u) < \infty$ for some $\delta > 0$, then

$$\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x < +\infty. \tag{1.2}$$

Proof. By a simple normalization argument, we may reduce the assertion to proving that

$$\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x \le \frac{N}{2}\log\left(C_N \delta^{\frac{4}{N}} + C_N I_\delta(u)\right), \qquad \text{for all } \delta > 0, \tag{1.3}$$

for any $u \in L^2(\mathbb{R}^N)$ such that $\|u\|_2 = 1$. Considering the normalized outer measure

$$\mu(E) := \int_E u^2(x)\mathrm{d}x, \qquad \mu(\mathbb{R}^N) = 1,$$

and using Jensen's inequality for concave nonlinearities and with measure μ, we have

$$\log\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\mathrm{d}x\right) = \log\left(\int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}}\mathrm{d}\mu\right) \ge \int_{\mathbb{R}^N} \log|u|^{\frac{4}{N-2}}\mathrm{d}\mu = \frac{2}{N-2}\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x. \tag{1.4}$$

On the other hand, applying (1.1), we derive that, for all $\delta > 0$,

$$\frac{2}{N-2}\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x \le \log\left(D_N \delta^{\frac{4}{N-2}} + C_N I_\delta(u)^{\frac{N}{N-2}}\right),$$

for some positive constant D_N, which implies (1.3). Here we used the fact that

$$\int_{\{|u|\le\lambda_N\delta\}} |u|^{\frac{2N}{N-2}}\mathrm{d}x \le \lambda_N^{\frac{4}{N-2}}\delta^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2\,\mathrm{d}x = 1$. □

Defining a notion of *entropy* as typical in statistical mechanics:

$$\mathrm{Ent}_\mu(f) := \int_{\mathbb{R}^N} \frac{f}{\|f\|_{1,\mu}} \log \frac{f}{\|f\|_{1,\mu}}\,\mathrm{d}\mu + \frac{N}{2}\log\|f\|_{1,\mu}, \qquad f \ge 0, \qquad \|f\|_{1,\mu} := \int f\,\mathrm{d}\mu,$$

the conclusion of the previous results reads as

$$u \in L^2(\mathbb{R}^N),\ \exists\delta > 0 : I_\delta(u) < +\infty \implies \mathrm{Ent}_{\mathcal{L}^N}(u^2) < +\infty.$$

Remark 1.2 *(Logarithmic NLS).* If $u \in H^1(\mathbb{R}^N)$, then the results of [19] show that

$$\lim_{\delta\searrow 0} I_\delta(u) = Q_N \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x, \tag{1.5}$$

for some constant $Q_N > 0$. Hence, passing to the limit as $\delta \searrow 0$ in the inequality of Theorem 1.1, one recovers classical forms of the logarithmic inequality. The logarithmic Schrödinger equation

$$\mathrm{i}\partial_t\phi + \Delta\phi + \phi\log|\phi|^2 = 0, \quad \phi : [0,\infty)\times\mathbb{R}^N \to \mathbb{C}, \quad N \ge 3, \tag{1.6}$$

admits applications to quantum mechanics, quantum optics, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation (see [25] and [10–12]). The *standing waves* solutions to (1.6) solve the following semi-linear elliptic problem

$$-\Delta u + \omega u = u\log u^2, \qquad u \in H^1(\mathbb{R}^N). \tag{1.7}$$

These equations were recently investigated in [15,24]. From a variational point of view, the search for solutions to (1.7) can be associated with the study of critical points (in a nonsmooth sense) of the lower semi-continuous functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{\omega+1}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x,$$

which is well defined by the logarithmic Sobolev inequality. Due to Theorem 1.1 and (1.5), one could handle a kind of *nonlocal approximations* of (1.7), formally defined for $\delta > 0$ by

$$I'_\delta(u) + \omega u = u\log u^2,$$

which are associated with the energy functional $J_\delta : H^1(\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$J_\delta(u) = I_\delta(u) + \frac{\omega+1}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x.$$

Since there holds $I_\delta(u) \le C_N \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x$ for all $\delta > 0$ and $u \in H^1(\mathbb{R}^N)$ (cf. [19, Theorem 2]), the energy functional J_δ is well defined, for every $\delta > 0$.

Remark 1.3 *(Magnetic case).* If $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded and $u : \mathbb{R}^N \to \mathbb{C}$, we set

$$\Psi_u(x,y) := \mathrm{e}^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y), \quad x, y \in \mathbb{R}^N.$$

It was observed in [14] that the following *diamagnetic inequality* holds

$$||u(x)| - |u(y)|| \le \left|\Psi_u(x,x) - \Psi_u(x,y)\right|, \quad \text{for a.e. } x, y \in \mathbb{R}^N.$$

In turn, by defining

$$I_\delta^A(u) := \int_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\mathrm{d}x\,\mathrm{d}y,$$

we have

$$I_\delta(|u|) \le I_\delta^A(u), \quad \text{for all } \delta > 0 \text{ and all measurable } u : \mathbb{R}^N \to \mathbb{C}. \tag{1.8}$$

Then, Theorem 1.1 yields the following *magnetic logarithmic Sobolev inequality*. For $u \in L^2(\mathbb{R}^N)$, there is a positive constant C_N such that

$$\int_{\mathbb{R}^N} \frac{|u|^2}{\|u\|_2^2} \log \frac{|u|^2}{\|u\|_2^2} \, dx + \frac{N}{2} \log \|u\|_2^2 \le \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta^A(u) \right).$$

Notice that, since $I_\delta(|u|) \approx \|\nabla |u|\|_2^2$ as $\delta \searrow 0$ [19] and $I_\delta^A(u) \approx \|\nabla u - \mathrm{i}Au\|_2^2$ as $\delta \searrow 0$ [22], from inequality (1.8) one recovers $\|\nabla |u|\|_2 \le \|\nabla u - \mathrm{i}Au\|_2$, which follows from the well-know diamagnetic inequality for the gradients $|\nabla |u|| \le |\nabla u - \mathrm{i}Au|$, see [18].

As a companion to Theorem 1.1, we also have the following theorem.

Theorem 1.4. *Let $u \in L^2(\mathbb{R}^N)$ $(N \ge 3)$. Assume that there exists a non-decreasing function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $F(ts) \le t^\beta F(s)$ for any $s, t \ge 0$ and some $\beta > 0$ and*

$$\int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy < +\infty. \tag{1.9}$$

Then there exists a positive constant $C_{N,F}$ such that

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dx + \frac{N}{2} \log \|u\|_2^\beta \le \frac{N}{2} \log \left(C_{N,F} \|u\|_2^\beta + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy \right).$$

In particular, condition (1.2) holds.

Proof. Consider the statement when $\|u\|_2 = 1$. In light of inequality (1.4), since by [21, Proposition 6] there exists $C_N > 0$ and $\lambda_N > 0$ such that

$$\int_{\{|u| > \lambda_N F(1/2)\}} |u|^{2N/(N-2)} dx \le C_N \left(\frac{1}{F(1/2)} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy \right)^{N/(N-2)}, \tag{1.10}$$

by arguing as in the previous proof, we get

$$\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \le \log \left(D_{N,F} + D_{N,F} \left(\int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy \right)^{N/(N-2)} \right),$$

where we used the fact that

$$\int_{\{|u| \le \lambda_N F(1/2)\}} |u|^{\frac{2N}{N-2}} dx \le \lambda_N^{\frac{4}{N-2}} F(1/2)^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2 \, dx = 1$. Then, we get

$$\int_{\mathbb{R}^N} u^2 \log u^2 \le \frac{N}{2} \log \left(C_{N,F} + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy \right).$$

In the general case, using the sub-homogeneity condition on F yields

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \le \frac{N}{2} \log \left(C_{N,F} + \frac{C_{N,F}}{\|u\|_2^\beta} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x-y|^{N+2}} \, dx \, dy \right),$$

which yields the desired conclusion. □

Remark 1.5 *($L^p(\mathbb{R}^N)$-version).* If $p > 1$ and $N > p$, one has a variant of (1.4), namely

$$\log\left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}}\,\mathrm{d}x\right) \geq \frac{p}{N-p}\int_{\mathbb{R}^N} |u|^p \log|u|^p\,\mathrm{d}x. \tag{1.11}$$

Then, by arguing as in the proofs of Theorems 1.1 and 1.4 with

$$u \mapsto \int \int_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y, \quad u \mapsto \int_{\mathbb{R}^{2N}} \frac{F(|u(x)-u(y)|)}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.12}$$

in place of $I_\delta(u)$ and (1.9) respectively, it is possible to get the corresponding log-Sobolev inequalities as for the case $p = 2$, via the results of [21]. In particular, if $u \in L^p(\mathbb{R}^N)$ and the functionals in (1.12) are finite at u for some $\delta > 0$, then

$$\int_{\mathbb{R}^N} |u|^p \log|u|^p\,\mathrm{d}x < +\infty.$$

The Euclidean logarithmic Sobolev inequalities for the p-case have been intensively studied, see, e.g., the work of Del Pino and Dolbeault [16] and the references therein.

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