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Multipliers for Besov spaces on graded Lie groups

Multiplicateurs pour les espaces de Besov sur les groupes de Lie gradués

Duván Cardona [a], Michael Ruzhansky [b]

[a] *Mathematics Department, Universidad de los Andes, Carrera 1 No. 18a 10, Bogotá, Colombia*
[b] *Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom*

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This article is dedicated to Erlan Nursultanov on the occasion of his 60th birthday

ABSTRACT

In this note, we give embeddings and other properties of Besov spaces, as well as spectral and Fourier multiplier theorems, in the setting of graded Lie groups. We also present a Nikolskii-type inequality and the Littlewood–Paley theorem that play a role in this analysis and are also of interest on their own.

RÉSUMÉ

Dans cette note, nous étudions des espaces de Besov sur les groupes de Lie gradués et nous prouvons des théorèmes de multiplicateurs spectraux et de Fourier sur ces groupes. Nous présentons aussi une inégalité de type Nikolskii et le théorème de Littlewood–Paley, qui jouent un rôle dans cette analyse et sont également d'intérêt indépendant.

Version française abrégée

Dans cette note, nous procédons à une étude systématique de certains résultats d'analyse harmonique dans le cas des groupes de Lie gradués à partir du livre fondamental de Folland et Stein [8], ainsi que des développements les plus récents de ces dernières décennies, notamment ceux résumés par V. Fischer et le second auteur [5]. Les résultats obtenus peuvent être résumés comme suit : une version de l'inégalité de Nikolskii (ou de l'inégalité de Hölder inverse) dans le cadre des groupes de Lie gradués ; un théorème de Littlewood–Paley sur les groupes de Lie gradués pour les décompositions dyadiques associées aux opérateurs positifs de Rockland – ces opérateurs sur les groupes de Lie graduées apparaissent naturellement dans des questions concernant les opérateurs différentiels partiels généraux sur les variétés (cf. [15]) – ; une version des espaces de Besov homogènes et inhomogènes en termes d'opérateurs de Rockland. En utilisant les propriétés de plongement, nous montrons que les espaces de Besov dans ce contexte sont également les espaces d'interpolation entre espaces de Sobolev, et nous prouvons qu'ils sont indépendants du choix particulier de l'opérateur de Rockland utilisé pour les définir. Nous appliquons ces résultats afin d'établir des théorèmes de multiplicateurs pour les multiplicateurs spectraux et de

E-mail addresses: d.cardona@uniandes.edu.co, duvanc306@gmail.com (D. Cardona), m.ruzhansky@imperial.ac.uk (M. Ruzhansky).

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Fourier dans les espaces de Besov sur les groupes de Lie gradués. Plus précisément, nous démontrons un théorème de type Marcinkiewicz pour les multiplicateurs spectraux sur L^p et sur les espaces de Besov. Nous donnons aussi des résultats négatifs sur la finitude des opérateurs invariants dans les espaces de Besov. Pour les multiplicateurs de Fourier, nous montrons que la finitude entre espaces L^p implique la finitude sur les espaces de Besov.

1. Introduction

In this note, we do a systematic investigation of some results regarding notions of harmonic analysis in the setting of graded Lie groups, building up on the fundamental book [8] of Folland and Stein, as well as on more recent developments over the decades, in particular summarised in a recent book [5] by V. Fischer and the second author. The results of this investigation can be summarised as follows: a version of the Nikolskii (or the reverse Hölder) inequality in the setting of graded Lie groups; a Littlewood–Paley theorem on graded Lie groups for the dyadic decompositions associated with positive Rockland operators. These operators on graded Lie groups naturally appear when one is dealing with questions concerning general partial differential operators on manifolds (cf. [15]); a version of homogeneous and inhomogeneous Besov spaces in terms of Rockland operators; then we prove their embedding properties. We show that the Besov spaces in this context are also the interpolation spaces between Sobolev spaces, and prove that they are independent of a particular choice of the Rockland operator used to define them; we apply these results in order to establish multiplier theorems for spectral and Fourier multipliers in Besov spaces on graded Lie groups. More precisely, we prove a Marcinkiewicz-type theorem for spectral multipliers on L^p and on Besov spaces. We also give negative results on the boundedness of invariant operators in Besov spaces. For Fourier multipliers, we show that the boundedness between L^p-spaces implies the boundedness on Besov spaces.

The Nikolskii inequality was considered for the first time in the seminal work of S. M. Nikolskii [12]. Some versions of this inequality on symmetric spaces of non-compact type and on compact homogeneous manifolds can be found in [14] and [13], respectively. The Nikolskii inequality is a fundamental tool in the proof of several properties of function spaces such as the Besov spaces. These spaces form scales $B^r_{p,q}(G)$ carrying three indices $r \in \mathbb{R}$, $0 < p, q \leq \infty$, and they can be obtained by interpolation of Sobolev spaces, which can be defined on $\mathbb{R}^n$, and on compact and non-compact Lie groups in various equivalent ways. In a recent work of the second author with V. Fischer, Sobolev spaces were introduced on graded Lie groups by using positive Rockland operators (see [6]). These function spaces on stratified Lie groups were introduced earlier by Folland in [7] by using sub-Laplacians. Folland's Sobolev spaces coincide with those introduced in [6] on graded Lie groups in the setting of stratified groups. The Besov spaces for certain parameters p, q and r have been considered in [3] in the setting of Heisenberg groups. Global characterisations and properties of Besov spaces on compact homogeneous manifolds were obtained in [13]. Here we consider it in the setting of graded Lie groups. Liouville theorems on graded Lie groups have been considered in [10].

In this note, we also present a version of the Littlewood–Paley theorem and, as a consequence of both results, the Nikolskii inequality and the Littlewood–Paley theorem; we give the boundedness of certain Fourier and spectral multipliers on Besov spaces. We note that in the case of the sub-Laplacian, a wealth of results in available, to mention only a few, see, e.g., Folland [7] and Saka [16] for Sobolev spaces and Besov spaces on stratified groups, respectively; Furioli, Melzi and Veneruso [9], and Alexopoulos [2] for the Littlewood–Paley theorem and the Besov spaces, and for spectral multiplier theorems for the sub-Laplacian on Lie groups of polynomial growth, respectively. The novelty of this paper is that we are working with Rockland operators; these are linear invariant homogeneous hypoelliptic partial differential operators, in view of the Helffer and Nourrigat's resolution in [11] of the Rockland conjecture. The existence of such operators on nilpotent Lie groups does characterise the class of graded Lie groups (see [5, Section 4.1] for precise references).

Thus, already in the setting of the Heisenberg group $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, our results provide new insights, since instead of the positive sub-Laplacian $-\sum_{j=1}^n (X_j^2 + Y_j^2)$ we can choose, e.g., the positive Rockland operator $\mathcal{R} = \sum_{j=1}^n (X_j^4 + Y_j^4)$.

This paper is organised as follows. In Theorem 2.1, we present our version of the Nikolskii inequality for functions defined on graded Lie groups. In Theorem 2.2, we present our version of the Littlewood–Paley theorem. In Section 3, we define Besov spaces and we show some embedding properties for these spaces and we also prove that our Besov spaces can be obtained by interpolation of Sobolev spaces in the nilpotent setting. Finally, in Section 4, we study the boundedness of Fourier multipliers and spectral multipliers in Besov spaces.

The main results here may be the Nikolskii inequality (which seems to be new even on the Heisenberg group), the description of Besov spaces in terms of Rockland operators and independence on them, embedding properties, Marcinkiewicz theorem on graded groups and the observation that the properties of the global description of Besov spaces can be used to deduce Besov space results from L^p-ones for general Fourier multipliers on graded Lie groups.

2. Nikolskii inequality and Littlewood–Paley theorem on graded Lie groups

Let G be a graded Lie group, i.e. G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$.

If on the Lie algebra $\mathfrak{g}$ of G there exists a family of dilations D_r, $r > 0$, that is, a family of linear mappings from $\mathfrak{g}$ to itself satisfying the following two conditions: one is that, for every $r > 0$, D_r is a map of the form $D_r = \mathrm{Exp}(rA)$, for some diagonalisable linear operator A on $\mathfrak{g}$; the other one is that $\forall X, Y \in \mathfrak{g}$, and $r > 0$, $[D_r X, D_r Y] = D_r[X, Y]$, then we say that G is a homogeneous Lie group.

We call the eigenvalues of A, $\nu_1, \nu_2, \cdots, \nu_n$, the dilations weights or weights of G. The homogeneous dimension of a homogeneous Lie group G is given by $Q = \mathrm{Tr}(A) = \sum_{l=1}^{s} l \cdot \dim \mathfrak{g}_l$. It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator $\mathcal{R}$ on G. In view of Hellfer and Nourigat's resolution in [11] of the Rockland conjecture, (left-)invariant homogeneous hypoelliptic differential operators on G are called Rockland operators.

We denote by $\widehat{G}$ the unitary dual of G, that is, the set of all strongly continuous irreducible unitary representations of G and $\mathcal{F}_G$ the Fourier transform on G. We denote by $\pi(\mathcal{R})$ the symbol of $\mathcal{R}$ that, in this case (of left invariant differential operators), equals the infinitesimal representation of $\mathcal{R}$ as an element of the universal enveloping algebra of $\mathfrak{g}$. We refer to [5] for an exposition of the further properties of Rockland operators and the Fourier analysis on these groups, as well as for further references.

For every $L > 0$, let us consider the operator $\chi_L(\mathcal{R})$, defined by the functional calculus, where χ_L is the characteristic function of $[0, L]$. Functions of the form $\chi_L(\mathcal{R})f$ give analogues of trigonometric functions in this setting, having compactly supported Fourier transform with respect to the spectral resolution associated with positive Rockland operators.

Theorem 2.1 *(Nikolskii (or the reverse Hölder) inequality). Let G be a graded Lie group of homogeneous dimension Q. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. If $1 \le p \le q \le \infty$ then*

$$\|\chi_L(\mathcal{R})f\|_{L^q} \le \|\mathcal{F}_G^{-1}[E_\pi(1)]\|_{L^r} L^{\frac{Q}{\nu}(\frac{1}{p}-\frac{1}{q})} \|\chi_L(\mathcal{R})f\|_{L^p}, \tag{1}$$

where $1 \le r := (1 + (1/q - 1/p))^{-1} \le \infty$, $E_\pi(\lambda)$ is the spectral measure of $\pi(\mathcal{R})$, $\pi \in \widehat{G}$, and $E_\pi(1) = \int_{[0,1]} \mathrm{d}E_\pi(\lambda)$ is the E_π-spectral measure of the unit interval $[0, 1]$. In this case, since $\mathcal{F}_G^{-1}[E_\pi(1)] \in \mathcal{S}(G)$, its L^r-norm is finite.

We now present the Littlewood–Paley theorem on graded Lie groups that can be obtained as a consequence of the Mikhlin–Hörmander theorem from [4]. The main notion in the formulation of this result is the concept of a dyadic decomposition $\{\psi_l\}_{l\in\mathbb{N}_0}$ defined as follows: we choose a function $\psi \in C_0^\infty(G)$ supported in $[1/4, 2]$, $\psi = 1$ on $[1/2, 1]$. Denote by ψ_l the function $\psi_l(t) := \psi(2^{-l}t)$, $t \in \mathbb{R}$. For some smooth compactly supported function ψ_0 we have:

$$\sum_{l\in\mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0. \tag{2}$$

Theorem 2.2 *(Littlewood–Paley theorem). Let $1 < p < \infty$ and let G be a graded Lie group. If $\mathcal{R}$ is a positive Rockland operator, then there exist constants $0 < c_p, C_p < \infty$ depending only on p such that*

$$c_p\|f\|_{L^p(G)} \le \left\| \left[\sum_{l=0}^{\infty} |\psi_l(\mathcal{R})f|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \le C_p\|f\|_{L^p} \tag{3}$$

holds for every $f \in L^p(G)$.

Both the Nikolskii inequality and the Littlewood–Paley theorem prove an important role in proving embedding properties of the Besov spaces.

3. Homogeneous and inhomogeneous Besov spaces

Let $\mathcal{R}$ be a (left-invariant) positive Rockland operator on a graded Lie group G. In order to define the family of Besov spaces on G, let us assume that $\mathcal{R}$ is homogeneous of degree $\nu > 0$, and let us fix a dyadic decomposition of its spectrum $(\psi_l)_{l\in\mathbb{N}_0}$, as above. With the notations above, we define (left) Besov spaces associated with a (left-invariant) positive Rockland operator as follows.

Definition 3.1. Let $r \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. The homogeneous Besov space $\dot{B}^r_{p,q,\psi,\mathcal{R}}(G)$ associated with $(\mathcal{R}, (\psi_l)_l)$ consists of all $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{\dot{B}^r_{p,q,\psi,\mathcal{R}}(G)} := \left(\sum_{l\in\mathbb{N}_0} 2^{\frac{l}{\nu}rq} \|\psi_l(\mathcal{R})f\|^q_{L^p(G)} \right)^{\frac{1}{q}} < \infty, \tag{4}$$

for $0 < q < \infty$, and for $q = \infty$,

$$\|f\|_{\dot{B}^r_{p,\infty,\psi,\mathcal{R}}(G)} := \sup_{l\in\mathbb{N}_0} 2^{\frac{l}{\nu}r} \|\psi_l(\mathcal{R})f\|_{L^p(G)} < \infty. \tag{5}$$

Analogously, the inhomogeneous Besov space $B^r_{p,q,\psi,\mathcal{R}}(G)$ is defined as the space of distributions $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{B^r_{p,q,\psi,\mathcal{R}}(G)} := \left(\sum_{l\in\mathbb{N}_0} 2^{\frac{l}{\nu}rq}\|\psi_l(I+\mathcal{R})f\|^q_{L^p(G)}\right)^{\frac{1}{q}} < \infty, \tag{6}$$

if $0 < q < \infty$ and, for $q = \infty$,

$$\|f\|_{B^r_{p,\infty,\psi,\mathcal{R}}(G)} := \sup_{l\in\mathbb{N}_0} 2^{\frac{l}{\nu}r}\|\psi_l(I+\mathcal{R})f\|_{L^p(G)} < \infty. \tag{7}$$

Homogeneous and inhomogeneous Besov spaces do not depend on a particular choice of a positive Rockland operator $\mathcal{R}$ and of the sequence of smooth functions ψ_l, because they can be obtained by interpolation from inhomogeneous and homogeneous Sobolev spaces $H^{r,p}(G)$ and $\dot{H}^{r,p}(G)$ analysed recently in [5, Section 4], [6], which are known to be independent of a particular choice of $\mathcal{R}$:

Theorem 3.2. *Let G be a graded Lie group, and let $\mathcal{R}$ and $\mathcal{R}'$ be two positive Rockland operators with homogeneity degrees $\nu > 0$ and $\nu' > 0$, respectively. If $(\psi_l)_l$ and $(\psi'_l)_l$ are sequences satisfying (2), $1 < p < \infty$ and $1 \leq q < \infty$, then the spaces $\dot{B}^r_{p,q,\psi,\mathcal{R}}(G)$ and $\dot{B}^r_{p,q,\psi',\mathcal{R}'}(G)$ have equivalent norms, as well as the spaces $B^r_{p,q,\psi,\mathcal{R}}(G)$ and $B^r_{p,q,\psi',\mathcal{R}'}(G)$. We also have the following real interpolation properties:*

$$B^r_{p,q}(G) = (H^{b,p}(G), H^{a,p}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1-\theta) + a\theta, \tag{8}$$

and

$$\dot{B}^r_{p,q}(G) = (\dot{H}^{b,p}(G), \dot{H}^{b,a}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1-\theta) + a\theta. \tag{9}$$

As the referee of this paper pointed out, the independence of the Besov spaces of the particular choice of the Rockland operator may allow also for their description in terms of the Calderón decompositions of the delta measure as in [8, Thm. 1.61, p. 47].

Now, if we use the Nikolskii inequality and the Littlewood–Paley theorem, we can show the following embedding properties of Besov spaces. We use the simplified notation motivated by Theorem 3.2:

$$(\dot{B}^r_{p,q}(G), \|\cdot\|_{\dot{B}^r_{p,q}(G)}) = (\dot{B}^r_{p,q,\psi,\mathcal{R}}(G), \|\cdot\|_{\dot{B}^r_{p,q,\psi,\mathcal{R}}(G)})$$

and

$$(B^r_{p,q}(G), \|\cdot\|_{B^r_{p,q}(G)}) = (B^r_{p,q,\psi,\mathcal{R}}(G), \|\cdot\|_{B^r_{p,q,\psi,\mathcal{R}}(G)}).$$

Theorem 3.3. *Let G be a graded Lie group of homogeneous dimension Q and let $r \in \mathbb{R}$. Then*

(1) $\dot{B}^{r+\varepsilon}_{p,q_1}(G) \hookrightarrow \dot{B}^r_{p,q_1}(G) \hookrightarrow \dot{B}^r_{p,q_2}(G) \hookrightarrow \dot{B}^r_{p,\infty}(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.
(2) $\dot{B}^{r+\varepsilon}_{p,q_1}(G) \hookrightarrow \dot{B}^r_{p,q_2}(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
(3) $\dot{B}^{r_1}_{p_1,q}(G) \hookrightarrow \dot{B}^{r_2}_{p_2,q}(G)$, $1 \leq p_1 \leq p_2 \leq \infty$, $0 < q < \infty$, $r_1 \in \mathbb{R}$ *and* $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$.
(4) $\dot{H}^r(G) = \dot{B}^r_{2,2}(G)$ *and* $\dot{B}^r_{p,p}(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}^r_{p,2}(G)$, $1 < p \leq 2$.
(5) $\dot{B}^r_{p,1}(G) \hookrightarrow L^q(G)$, $1 \leq p \leq q \leq \infty$, $r = Q(\frac{1}{p} - \frac{1}{q})$.

If, in this theorem, we change the homogeneous Besov spaces and homogeneous Sobolev spaces by their inhomogeneous versions, every embedding above still holds true. Also, we have considered Besov spaces associated with (left-invariant) positive Rockland operators, but a similar formulation of homogeneous and inhomogeneous (right) Besov spaces can be obtained if we choose (right-invariant) positive Rockland operators, and these spaces satisfy (right) versions of the statements of Theorem 3.3.

Localisations of Besov spaces on a graded Lie group can be defined in the standard way:

$$B^r_{p,q}(G, \text{loc}) = \{f \in \mathcal{D}'(G) : \phi f \in B^r_{p,q}(G), \text{ for all } \phi \in C^\infty_0(G)\}. \tag{10}$$

Then we have the following embeddings between $B^r_{p,q}(G, \text{loc})$ and $B^r_{p,q}(\mathbb{R}^n, \text{loc})$, where n is the topological dimension of G, and the latter space is defined in the same way from the (graded) group $\mathbb{R}^n$ equipped with the usual Abelian group law.

Proposition 3.4. *Let $r \in \mathbb{R}$, $1 < p < \infty$ and $1 < q \le \infty$. If $B^r_{p,q}(G, \mathrm{loc})$ denotes the local Besov space defined above, then for all $r \in \mathbb{R}$, $1 < p < \infty$ and $0 < q \le \infty$ we have:*

$$B^{\frac{r}{\nu_1}}_{p,q}(G, \mathrm{loc}) \subset B^r_{p,q}(\mathbb{R}^n, \mathrm{loc}) \subset B^{\frac{r}{\nu_n}}_{p,q}(G, \mathrm{loc}), \tag{11}$$

where ν_1 and ν_n are respectively the smallest and the largest weights of the dilations.

4. Spectral multipliers and Fourier multipliers

In this section, we give results for the boundedness of spectral and of Fourier multipliers in Besov spaces on graded Lie groups. First, we present a generalisation of a classical result by Marcinkiewicz, its proof relying on an application of our Littlewood–Paley theorem, which asserts that, if $m \in \mathrm{BV}(\mathbb{R})$ is a bounded function such that

$$\sup_{j\in\mathbb{Z}} \|m\|_{\mathrm{BV}[2^{j-1},2^j]} < \infty,$$

(here, $\|m\|_{\mathrm{BV}[a,b]}$ denotes the bounded variation norm on $[a, b]$), then $m(D)$ is a multiplier on $L^p(\mathbb{R})$ for every $1 < p < \infty$. Now we present our version of the Marcinkiewicz theorem on graded Lie groups.

Theorem 4.1. *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. If $m \in C^\infty$ is a bounded function which has uniformly bounded variation on every dyadic interval of $\mathbb{R}$, then $m(\mathcal{R})$ extends to a bounded spectral multiplier on $L^p(G)$ for every $1 < p < \infty$. Moreover, for every $0 < q \le \infty$, and $r \in \mathbb{R}$, $m(\mathcal{R})$ is bounded on $B^r_{p,q}(G)$.*

The following theorem shows that in the problem of the boundedness of spectral multipliers (or of more general left-invariant operators) on Besov spaces, not all indices p and $\tilde{p}$ are available.

Theorem 4.2. *Let G be a graded Lie group and let T be a linear left-invariant operator bounded from $B^r_{p,q}(G)$ (respectively, $\dot{B}^r_{p,q}(G)$) into $B^{\tilde{r}}_{\tilde{p},\tilde{q}}(G)$, (respectively, $\dot{B}^{\tilde{r}}_{\tilde{p},\tilde{q}}(G)$), for $1 \le p, \tilde{p} < \infty$, $-\infty < r, \tilde{r} < \infty$, and $0 < q, \tilde{q} \le \infty$. If $1 \le \tilde{p} < p < \infty$, then $T = 0$.*

Finally, one can show that there is an interesting connection between L^p-multipliers and Besov multipliers. However, curiously, the right Besov spaces appear here.

Theorem 4.3. *Let G be a graded Lie group. Let $\sigma = \{\sigma(\pi) : \pi \in \widehat{G}\}$ be a μ-measurable field of operators in $L^2(\widehat{G})$. Let us assume that the corresponding operator $T = T_\sigma$, given by*

$$T_\sigma u(x) = \int_{\widehat{G}} \mathrm{Tr}(\pi(x)\sigma(\pi)\widehat{u}(\pi))\, \mathrm{d}\mu(\pi),$$

is a bounded operator from $L^{p_1}(G)$ into $L^{p_2}(G)$, $1 \le p_i \le \infty$. Then T is a bounded operator from the (right) Besov space $\dot{B}^r_{p_1,q}(G)$ into the (right) Besov space $\dot{B}^r_{p_2,q}(G)$, for all $-\infty < r < \infty$ and $0 < q \le \infty$. Moreover, T is also a bounded operator from the (right) Besov space $B^r_{p_1,q}(G)$ into the (right) Besov space $B^r_{p_2,q}(G)$.

The space $L^2(\widehat{G})$ above is the space on the Fourier transform side that corresponds to $L^2(G)$ via the Plancherel theorem on G (see, e.g., [5, Theorem 1.8.11]).

We refer to [4] for conditions of Mihlin–Hörmander type that ensure the L^p-boundedness of Fourier multipliers on graded Lie groups, and to [1] for conditions on the boundedness from L^{p_1} to L^{p_2} for $1 < p_1 \le 2 \le p_2 < \infty$.

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