



Differential geometry

Formality theorem for \mathfrak{g} -manifolds [☆]Théorème de formalité pour les \mathfrak{g} -variétés

Hsuan-Yi Liao, Mathieu Stiénon, Ping Xu

Department of Mathematics, Pennsylvania State University, United States

ARTICLE INFO

Article history:

Received 20 January 2017

Accepted after revision 13 March 2017

Available online 30 March 2017

Presented by Michèle Vergne

À la mémoire de notre ami Jacky Michéa

ABSTRACT

With any \mathfrak{g} -manifold M are associated two dglas $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ and $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$, whose cohomologies $H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ and $H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$ are Gerstenhaber algebras. We establish a formality theorem for \mathfrak{g} -manifolds: there exists an L_∞ quasi-isomorphism $\Phi : \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M)) \rightarrow \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$ whose first ‘Taylor coefficient’ (1) is equal to the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd cocycle of the \mathfrak{g} -manifold M , and (2) induces an isomorphism of Gerstenhaber algebras on the level of cohomology. Consequently, the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd class of the \mathfrak{g} -manifold M is an isomorphism of Gerstenhaber algebras from $H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ to $H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

À toute \mathfrak{g} -variété M sont associées deux algèbres de Lie différentielles graduées $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ et $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$, dont les cohomologies $H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ et $H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$ sont des algèbres de Gerstenhaber. Nous obtenons un théorème de formalité pour les \mathfrak{g} -variétés : il existe un quasi-isomorphisme $\Phi : \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M)) \rightarrow \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$ d’algèbres L_∞ dont le premier « coefficient de Taylor » (1) est égal à l’application de Hochschild–Kostant–Rosenberg tordue par la racine carrée du cocycle de Todd de la \mathfrak{g} -variété M et (2) induit un isomorphisme d’algèbre de Gerstenhaber au niveau des cohomologies. Par conséquent, l’application de Hochschild–Kostant–Rosenberg tordue par la racine carrée de la classe de Todd de la

[☆] Research partially supported by NSF grants DMS-1406668 and DMS-1101827, and NSA grant H98230-14-1-0153.

E-mail addresses: hul170@psu.edu (H.-Y. Liao), stienon@psu.edu (M. Stiénon), ping@math.psu.edu (P. Xu).

\mathfrak{g} -variété M est un isomorphisme d'algèbres de Gerstenhaber de $H_{\text{CE}}^{\bullet}(\mathfrak{g}, T_{\text{poly}}^{\bullet}(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ sur $H_{\text{CE}}^{\bullet}(\mathfrak{g}, D_{\text{poly}}^{\bullet}(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

À une variété différentiable M donnée sont associées deux algèbres de Lie différentielles graduées canoniques : l'algèbre de Lie différentielle graduée des champs de multivecteurs $T_{\text{poly}}^{\bullet}(M) = \bigoplus_{k=-1}^{\infty} \Gamma(\Lambda^{k+1} T_M)$ munie de la différentielle triviale et du crochet de Schouten $[\cdot, \cdot]$ d'une part, et l'algèbre de Lie différentielle graduée des opérateurs multidifférentiels $D_{\text{poly}}^{\bullet}(M) = \bigoplus_{k=-1}^{\infty} D_{\text{poly}}^k(M)$ munie de la différentielle de Hochschild d_H et du crochet de Gerstenhaber $\llbracket \cdot, \cdot \rrbracket$, de l'autre. Le symbole $D_{\text{poly}}^{-1}(M)$ désigne l'algèbre $R = C^{\infty}(M)$ des fonctions différentiables sur M , le symbole $D_{\text{poly}}^0(M)$ désigne l'algèbre des opérateurs différentiels sur M , et le symbole $D_{\text{poly}}^k(M)$ (pour $k \geq 0$) désigne l'espace des opérateurs $(k+1)$ -différentiels sur M , c'est-à-dire le produit tensoriel $D_{\text{poly}}^0(M) \otimes_R \cdots \otimes_R D_{\text{poly}}^0(M)$ de $(k+1)$ copies du R -module à gauche $D_{\text{poly}}^0(M)$. Le théorème de Hochschild–Kostant–Rosenberg [6,7] classique affirme que le plongement canonique $\text{hkr} : T_{\text{poly}}^{\bullet}(M) \hookrightarrow D_{\text{poly}}^{\bullet}(M)$ définit par l'Éq. (4) détermine un isomorphisme d'algèbres de Gerstenhaber $\text{hkr} : T_{\text{poly}}^{\bullet}(M) \xrightarrow{\cong} H^{\bullet}(D_{\text{poly}}^{\bullet}(M), d_H)$ au niveau des cohomologies – les multiplications dans $T_{\text{poly}}^{\bullet}(M)$ et $D_{\text{poly}}^{\bullet}(M)$ sont respectivement le produit extérieur et le cup-produit. Cependant, l'application de Hochschild–Kostant–Rosenberg $\text{hkr} : T_{\text{poly}}^{\bullet}(M) \hookrightarrow D_{\text{poly}}^{\bullet}(M)$ n'est pas un morphisme d'algèbres de Lie différentielles graduées. Le célèbre théorème de formalité de Kontsevich affirme que l'application hkr s'étend en un L_{∞} quasi-isomorphisme de $T_{\text{poly}}^{\bullet}(M)$ vers $D_{\text{poly}}^{\bullet}(M)$ [7,12]. Le théorème de formalité est hautement non trivial et se prête à maintes applications, y compris la quantification par déformation des variétés de Poisson.

Dans cette Note, nous étudions les algèbres de Gerstenhaber associées à une \mathfrak{g} -variété, et nous obtenons un théorème de formalité pour les \mathfrak{g} -variétés. Par \mathfrak{g} -variété, nous entendons une variété différentiable et une action infinitésimale d'une algèbre de Lie \mathfrak{g} sur celle-ci. Dans cette situation, les complexes de Chevalley–Eilenberg $\text{tot}(\Lambda^{\bullet} \mathfrak{g}^{\vee} \otimes_{\mathbb{k}} T_{\text{poly}}^{\bullet}(M))$ et $\text{tot}(\Lambda^{\bullet} \mathfrak{g}^{\vee} \otimes_{\mathbb{k}} D_{\text{poly}}^{\bullet}(M))$ jouent respectivement les rôles d'espace de champs de multivecteurs et d'espace d'opérateurs multidifférentiels. L'un et l'autre sont naturellement des algèbres de Lie différentielles graduées (voir les Lemmes 3.1 et 3.2), et leurs cohomologies sont des algèbres de Gerstenhaber.

Pour énoncer ce théorème de formalité et la relation précise entre ces deux algèbres de Gerstenhaber, il nous faut prendre en considération l'obstruction à l'existence d'une connexion affine \mathfrak{g} -invariante sur M , un 1-cocycle de Chevalley–Eilenberg $R_{1,1}^{\nabla} \in \mathfrak{g}^{\vee} \otimes \Gamma(T_M^{\vee} \otimes \text{End } T_M)$ à valeurs dans le \mathfrak{g} -module $\Gamma(T_M^{\vee} \otimes \text{End } T_M)$, que nous appelons cocycle d'Atiyah. Plus précisément, nous avons recours à sa classe de cohomologie, la classe d'Atiyah $\alpha_{M/\mathfrak{g}} \in H_{\text{CE}}^1(\mathfrak{g}, \Gamma(T_M^{\vee} \otimes \text{End } T_M))$, que nous introduisons à la Proposition 4.1.

Le cocycle de Todd $\text{td}_{M/\mathfrak{g}} \in \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{g}^{\vee} \otimes \Omega^k(M)$ d'une \mathfrak{g} -variété M est défini à partir du cocycle d'Atiyah à l'Éq. (5). La classe correspondante $\text{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0}^{\infty} H_{\text{CE}}^k(\mathfrak{g}, \Omega^k(M))$ en cohomologie de Chevalley–Eilenberg est appelée classe de Todd. Voir l'Éq. (6).

Les principaux résultats de cette Note sont un théorème de formalité pour les \mathfrak{g} -variétés et sa conséquence : un théorème de type Kontsevich–Duflou pour les \mathfrak{g} -variétés.

Théorème 0.1 (Théorème de formalité). *Étant donné une \mathfrak{g} -variété M et une connexion affine sans torsion ∇ sur M , il existe un L_{∞} quasi-isomorphisme Φ de la dgla $\text{tot}(\Lambda^{\bullet} \mathfrak{g}^{\vee} \otimes_{\mathbb{k}} T_{\text{poly}}^{\bullet}(M))$ sur la dgla $\text{tot}(\Lambda^{\bullet} \mathfrak{g}^{\vee} \otimes_{\mathbb{k}} D_{\text{poly}}^{\bullet}(M))$, dont le premier « coefficient de Taylor » Φ_1 satisfait les deux propriétés suivantes :*

- (i) Φ_1 est, à une homotopie près, un isomorphisme d'algèbres associatives ;
- (ii) Φ_1 est l'application composée $\text{hkr} \circ \text{td}_{M/\mathfrak{g}}^{\frac{1}{2}}$ – la racine carrée $\text{td}_{M/\mathfrak{g}}^{\frac{1}{2}} \in \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{g}^{\vee} \otimes \Omega^k(M)$ du cocycle de Todd agissant sur $\text{tot}(\Lambda^{\bullet} \mathfrak{g}^{\vee} \otimes_{\mathbb{k}} T_{\text{poly}}^{\bullet}(M))$ par contraction.

Théorème 0.2 (Théorème de type Kontsevich–Duflou). *Étant donné une \mathfrak{g} -variété M , l'application*

$$\text{hkr} \circ \text{td}_{M/\mathfrak{g}}^{\frac{1}{2}} : H_{\text{CE}}^{\bullet}(\mathfrak{g}, T_{\text{poly}}^{\bullet}(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M) \rightarrow H_{\text{CE}}^{\bullet}(\mathfrak{g}, D_{\text{poly}}^{\bullet}(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$$

est un isomorphisme d'algèbres de Gerstenhaber – la racine carrée $\text{Td}_{M/\mathfrak{g}}^{\frac{1}{2}}$ de la classe de Todd $\text{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0}^{\infty} H_{\text{CE}}^k(\mathfrak{g}, \Omega^k(M))$ agissant sur $H_{\text{CE}}^{\bullet}(\mathfrak{g}, T_{\text{poly}}^{\bullet}(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ par contraction.

Ce deuxième théorème est conceptuellement similaire à un résultat de géométrie complexe découvert par Kontsevich [7] et analogue au théorème de Duflo de la théorie de Lie classique. Kontsevich observa que, pour toute variété complexe X , la composée $\text{hkr} \circ (\text{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \rightarrow HH^\bullet(X)$ est un isomorphisme d'algèbres associatives. Le symbole Td_X représente ici la classe de Todd du fibré tangent T_X à la variété complexe X , tandis que la notation $HH^\bullet(X)$ désigne les groupes de cohomologie de Hochschild de X , c'est-à-dire les groupes $\text{Ext}_{\mathcal{O}_{X \times X}}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Les multiplications dans $H^\bullet(X, \Lambda^\bullet T_X)$ et $HH^\bullet(X)$ sont respectivement le produit extérieur et le produit de Yoneda. Une preuve détaillée de ce résultat dû à Kontsevich est disponible [2]. Soulignons encore qu'en cohomologie l'application $\text{hkr} \circ (\text{Td}_X)^{\frac{1}{2}}$ entrelace les structures d'algèbres de Gerstenhaber [2].

Les résultats de la présente Note découlent d'un théorème plus général, un théorème de formalité pour les paires de Lie, dont une preuve détaillée paraîtra prochainement dans une révision de notre prépublication [8].

1. Introduction

Two differential graded Lie algebras (dglas) are canonically associated with a given smooth manifold M : the dgla of polyvector fields $T_{\text{poly}}^\bullet(M) = \bigoplus_{k=-1}^\infty \Gamma(\Lambda^{k+1} T_M)$, which is endowed with the zero differential and the Schouten bracket $[\cdot, \cdot]$, and the dgla of polydifferential operators $D_{\text{poly}}^\bullet(M) = \bigoplus_{k=-1}^\infty D_{\text{poly}}^k(M)$, which is endowed with the Hochschild differential d_H and the Gerstenhaber bracket $[[\cdot, \cdot]]$. Here $D_{\text{poly}}^{-1}(M)$ denotes the algebra of smooth functions $\mathcal{R} = C^\infty(M)$, $D_{\text{poly}}^0(M)$ the algebra of differential operators on M , and $D_{\text{poly}}^k(M)$ (with $k \geq 0$) the space of $(k + 1)$ -differential operators on M , i.e. the tensor product $D_{\text{poly}}^0(M) \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} D_{\text{poly}}^0(M)$ of $(k + 1)$ copies of the left \mathcal{R} -module $D_{\text{poly}}^0(M)$. The classical Hochschild–Kostant–Rosenberg (HKR) theorem [6,7] states that the Hochschild–Kostant–Rosenberg map, the natural embedding $\text{hkr} : T_{\text{poly}}^\bullet(M) \hookrightarrow D_{\text{poly}}^\bullet(M)$ defined by Eq. (4), determines an isomorphism of Gerstenhaber algebras $\text{hkr} : T_{\text{poly}}^\bullet(M) \xrightarrow{\cong} H^\bullet(D_{\text{poly}}^\bullet(M), d_H)$ on the cohomology level – the products on $T_{\text{poly}}^\bullet(M)$ and $D_{\text{poly}}^\bullet(M)$ are the wedge product and the cup product respectively. However, the HKR map $\text{hkr} : T_{\text{poly}}^\bullet(M) \hookrightarrow D_{\text{poly}}^\bullet(M)$ is not a morphism of dglas. Kontsevich's celebrated formality theorem states that the HKR map hkr extends to an L_∞ quasi-isomorphism from $T_{\text{poly}}^\bullet(M)$ to $D_{\text{poly}}^\bullet(M)$ [7,12]. The formality theorem is highly non trivial and has many applications, one of which is the deformation quantization of Poisson manifolds.

In this Note, we study the Gerstenhaber algebra structures associated with a \mathfrak{g} -manifold and we establish a formality theorem for \mathfrak{g} -manifolds. By a \mathfrak{g} -manifold, we mean a smooth manifold equipped with an infinitesimal action of a Lie algebra \mathfrak{g} . In this situation, the analogues of $T_{\text{poly}}^\bullet(M)$ and $D_{\text{poly}}^\bullet(M)$ are the Chevalley–Eilenberg's complexes $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ and $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$, respectively – they are briefly mentioned in Dolgushev's work [4, concluding remarks]. Both of them are naturally dglas (see Lemma 3.1 and Lemma 3.2) and their cohomologies are Gerstenhaber algebras.

In order to state the formality theorem and the precise relation between these two Gerstenhaber algebras, one must take into consideration the obstruction to the existence of a \mathfrak{g} -invariant affine connection on M , the Atiyah cocycle $R_{1,1}^\vee \in \mathfrak{g}^\vee \otimes \Gamma(T_M^\vee \otimes \text{End } T_M)$, which is a Chevalley–Eilenberg 1-cocycle of the \mathfrak{g} -module $\Gamma(T_M^\vee \otimes \text{End } T_M)$. More precisely, we must call upon its cohomology class, the Atiyah class $\alpha_{M/\mathfrak{g}} \in H_{\text{CE}}^1(\mathfrak{g}, \Gamma(T_M^\vee \otimes \text{End } T_M))$, which we introduce in Proposition 4.1.

The Todd cocycle $\text{td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M)$ of a \mathfrak{g} -manifold M is defined in terms of the Atiyah cocycle in Eq. (5). The corresponding class in Chevalley–Eilenberg cohomology is the Todd class $\text{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H_{\text{CE}}^k(\mathfrak{g}, \Omega^k(M))$. See Eq. (6).

The main results of this Note are a formality theorem for \mathfrak{g} -manifolds (Theorem 5.1) and its consequence: a Kontsevich–Duflo type theorem for \mathfrak{g} -manifolds (Theorem 5.2).

The theorem above is parallel in spirit to an analogue of Duflo's Theorem – a classical result of Lie theory – discovered by Kontsevich in complex geometry [7]. Kontsevich observed that, for a complex manifold X , the composition $\text{hkr} \circ (\text{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$ is an isomorphism of associative algebras. Here Td_X denotes the Todd class of the tangent bundle T_X and $HH^\bullet(X)$ denotes the Hochschild cohomology groups of the complex manifold X , i.e. the groups $\text{Ext}_{\mathcal{O}_{X \times X}}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. The multiplications on $H^\bullet(X, \Lambda^\bullet T_X)$ and $HH^\bullet(X)$ are the wedge product and the Yoneda product respectively. A detailed proof of Kontsevich's result appeared in [2]. It is worth mentioning that the map $\text{hkr} \circ (\text{Td}_X)^{\frac{1}{2}}$ actually respects the Gerstenhaber algebra structures on both cohomologies; this was brought to light in [2].

2. Preliminary: Chevalley–Eilenberg cohomology

Let \mathfrak{g} be a Lie algebra over \mathbb{k} (\mathbb{k} is \mathbb{R} or \mathbb{C}). Given a \mathfrak{g} -module E , one may consider the Chevalley–Eilenberg cochain complex

$$\cdots \longrightarrow \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E \xrightarrow{d_{\text{CE}}} \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} E \xrightarrow{d_{\text{CE}}} \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E \longrightarrow \cdots,$$

where d_{CE} is the Chevalley–Eilenberg differential. More generally, given a bounded below complex of left \mathfrak{g} -modules

$$\cdots \longrightarrow E^{p-1} \xrightarrow{d_E} E^p \xrightarrow{d_E} E^{p+1} \longrightarrow \cdots,$$

we may consider the double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^{q+1} & \xrightarrow{d_{CE}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^{q+1} & \xrightarrow{d_{CE}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^{q+1} \longrightarrow \cdots \\
 & & (-1)^{p-1} \text{id} \otimes d_E \uparrow & & (-1)^p \text{id} \otimes d_E \uparrow & & (-1)^{p+1} \text{id} \otimes d_E \uparrow \\
 \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^q & \xrightarrow{d_{CE}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^q & \xrightarrow{d_{CE}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^q \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

where d_{CE} is the Chevalley–Eilenberg differential corresponding to the \mathfrak{g} -module structure on E^\bullet . By definition, the Chevalley–Eilenberg cohomology of \mathfrak{g} with coefficients in the complex of \mathfrak{g} -modules (E^\bullet, d_E) is the total cohomology of the double complex above:

$$H_{CE}^k(\mathfrak{g}, E^\bullet \xrightarrow{d_E} E^{\bullet+1}) = H^k(\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} E^\bullet))$$

3. Hochschild–Kostant–Rosenberg theorem for \mathfrak{g} -manifolds

3.1. Polyvector fields

Let M be a \mathfrak{g} -manifold with infinitesimal action given by a Lie algebra morphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$. It is well known that the space of polyvector fields $T_{\text{poly}}^\bullet(M) = \bigoplus_{k=-1}^\infty \Gamma(\Lambda^{k+1} T_M)$ on M , together with the wedge product and the Schouten bracket $[\cdot, \cdot]$, forms a Gerstenhaber algebra. Moreover, the \mathfrak{g} -action on M and the Schouten bracket together determine a \mathfrak{g} -module structure on $T_{\text{poly}}^k(M)$ for each $k \geq -1$:

$$a \cdot \gamma = [\varphi(a), \gamma] \quad \forall a \in \mathfrak{g}, \gamma \in T_{\text{poly}}^k(M).$$

Therefore $\cdots \rightarrow T_{\text{poly}}^k(M) \xrightarrow{0} T_{\text{poly}}^{k+1}(M) \rightarrow \cdots$ is a complex of \mathfrak{g} -modules. Its Chevalley–Eilenberg cohomology

$$H_{CE}^k(\mathfrak{g}, T_{\text{poly}}^\bullet(M) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)) = H^k(\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M)))$$

is the total cohomology of the double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & 0 \uparrow & & 0 \uparrow & & 0 \uparrow \\
 \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^{q+1}(M) & \xrightarrow{d_{CE}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^{q+1}(M) & \xrightarrow{d_{CE}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^{q+1}(M) \longrightarrow \cdots \\
 & & 0 \uparrow & & 0 \uparrow & & 0 \uparrow \\
 \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^q(M) & \xrightarrow{d_{CE}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^q(M) & \xrightarrow{d_{CE}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^q(M) \longrightarrow \cdots \\
 & & 0 \uparrow & & 0 \uparrow & & 0 \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Extend the Schouten bracket $[\cdot, \cdot]$ on $T_{\text{poly}}^\bullet(M)$ to $\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M)$ as follows:

$$[\alpha \otimes \mathcal{X}, \beta \otimes \mathcal{Y}] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [\mathcal{X}, \mathcal{Y}] \tag{1}$$

for any $\alpha \otimes \mathcal{X} \in \Lambda^{p_1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^{q_1}(M)$ and $\beta \otimes \mathcal{Y} \in \Lambda^{p_2} \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^{q_2}(M)$.

The following lemma can be easily verified.

Lemma 3.1. *The graded \mathbb{k} -vector space $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$, together with the Chevalley–Eilenberg differential d_{CE} , the wedge product \wedge and the bracket defined by Eq. (1) is a differential Gerstenhaber algebra. As a consequence, $H_{CE}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M))$ is a Gerstenhaber algebra.*

3.2. Polydifferential operators

On a smooth manifold M , one also has the dgla of polydifferential operators, $D_{\text{poly}}^\bullet(M)$.

Let M be a manifold, let \mathcal{R} denote its algebra of smooth functions $C^\infty(M)$, and let $D_{\text{poly}}^0(M)$ denote the algebra of differential operators on M . Denote by $D_{\text{poly}}^k(M)$, $k \geq 0$, the space of $(k + 1)$ -differential operators on M , i.e. the tensor product $D_{\text{poly}}^0(M) \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} D_{\text{poly}}^0(M)$ of $(k + 1)$ copies of the left \mathcal{R} -module $D_{\text{poly}}^0(M)$. Denote also by $D_{\text{poly}}^{-1}(M)$ the space of smooth functions $\mathcal{R} = C^\infty(M)$. It is well known that endowing $D_{\text{poly}}^\bullet(M) = \bigoplus_{k=-1}^\infty D_{\text{poly}}^k(M)$ with the Hochschild differential d_H , the cup product $D_{\text{poly}}^k(M) \otimes D_{\text{poly}}^l(M) \xrightarrow{\smile} D_{\text{poly}}^{k+l+1}(M)$, and the Gerstenhaber bracket $[[\cdot, \cdot]]$ makes it a Gerstenhaber algebra up to homotopy [5].

Following our earlier notations, now assume that M is a \mathfrak{g} -manifold with infinitesimal action $\varphi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$. Analogously to the polyvector field case, the Lie algebra \mathfrak{g} acts on $D_{\text{poly}}^\bullet(M)$ by:

$$a \cdot \mu = [[\varphi(a), \mu]] \quad \forall a \in \mathfrak{g}, \mu \in D_{\text{poly}}^\bullet(M).$$

Since the Gerstenhaber bracket satisfies the graded Jacobi identity, this infinitesimal \mathfrak{g} -action on $D_{\text{poly}}^\bullet(M)$ is compatible with the Hochschild differential. Consequently $\cdots \rightarrow D_{\text{poly}}^k(M) \xrightarrow{d_H} D_{\text{poly}}^{k+1}(M) \rightarrow \cdots$ is a complex of \mathfrak{g} -modules, and therefore we have the Chevalley–Eilenberg cohomology

$$H_{\text{CE}}^k(\mathfrak{g}, D_{\text{poly}}^\bullet(M) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)) = H^k(\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))),$$

which is, by definition, the total cohomology of the double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^{q+1}(M) & \xrightarrow{d_{\text{CE}}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^{q+1}(M) & \xrightarrow{d_{\text{CE}}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^{q+1}(M) \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & \Lambda^{p-1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^q(M) & \xrightarrow{d_{\text{CE}}} & \Lambda^p \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^q(M) & \xrightarrow{d_{\text{CE}}} & \Lambda^{p+1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^q(M) \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

Extend the cup product \smile and the Gerstenhaber bracket $[[\cdot, \cdot]]$ to $\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M)$ as follows:

$$(\alpha \otimes \xi) \smile (\beta \otimes \eta) = (-1)^{q_1 p_2} (\alpha \wedge \beta) \otimes (\xi \smile \eta) \tag{2}$$

$$[[\alpha \otimes \xi, \beta \otimes \eta]] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [[\xi, \eta]] \tag{3}$$

for any $\alpha \otimes \xi \in \Lambda^{p_1} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^{q_1}(M)$ and $\beta \otimes \eta \in \Lambda^{p_2} \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^{q_2}(M)$.

Again the following lemma is immediate.

Lemma 3.2.

- (i) The graded \mathbb{k} -vector space $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$, together with the differential $d_{\text{CE}} + \text{id} \otimes d_H$ and the Gerstenhaber bracket $[[\cdot, \cdot]]$ defined by Eq. (3), is a dgla.
- (ii) The graded \mathbb{k} -vector space $H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M))$, together with the cup product and the Gerstenhaber bracket defined by Eqs. (2) and (3), is a Gerstenhaber algebra.

3.3. Hochschild–Kostant–Rosenberg theorem

Given a smooth manifold M , there is a natural embedding $\text{hkr} : T_{\text{poly}}^\bullet(M) \hookrightarrow D_{\text{poly}}^\bullet(M)$, called Hochschild–Kostant–Rosenberg map, and defined by

$$\text{hkr}(X_1 \wedge \cdots \wedge X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}, \quad \forall X_i \in \mathfrak{X}(M), \tag{4}$$

where S_k is the symmetric group on k objects. The Hochschild–Kostant–Rosenberg theorem for smooth manifolds states that hkr is a quasi-isomorphism, i.e. the induced morphism in cohomology $\text{hkr} : T_{\text{poly}}^\bullet(M) \xrightarrow{\cong} H^\bullet(D_{\text{poly}}^\bullet(M), d_H)$ is an isomorphism of vector spaces [6,7].

Suppose we are given a \mathfrak{g} -manifold M . Then the map $\text{id} \otimes \text{hkr} : \Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M) \rightarrow \Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M)$ is a morphism of double complexes. Abusing notations, the induced morphism on Chevalley–Eilenberg cohomologies will also be denoted by hkr .

Proposition 3.1 ([8]). *Let M be a \mathfrak{g} -manifold. The Hochschild–Kostant–Rosenberg map*

$$\text{hkr} : H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M) \xrightarrow{\cong} H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$$

is an isomorphism of vector spaces.

The proof is a straightforward spectral sequence computation relying on the classical Hochschild–Kostant–Rosenberg theorem for smooth manifolds.

4. Atiyah class of a \mathfrak{g} -manifold

The Atiyah class was originally introduced by Atiyah for holomorphic vector bundles [1]. Atiyah classes can also be defined for Lie algebroid pairs [3] and dg vector bundles [10]. In this section, we introduce the notions of Atiyah class and Todd class of a \mathfrak{g} -manifold.

Let M be a \mathfrak{g} -manifold with infinitesimal action $\mathfrak{g} \ni a \mapsto \hat{a} \in \mathfrak{X}(M)$. Given an affine connection ∇ on M , the Atiyah 1-cocycle associated with ∇ is defined as the map $R_{1,1}^\nabla : \mathfrak{g} \times \mathfrak{X}(M) \rightarrow \text{End}_{\mathcal{R}} \mathfrak{X}(M)$ given by

$$R_{1,1}^\nabla(a, X) = \nabla_{\hat{a}} \nabla_X - \nabla_X \nabla_{\hat{a}} - \nabla_{[\hat{a}, X]},$$

where $a \in \mathfrak{g}$, $X \in \mathfrak{X}(M)$, and $\mathcal{R} = C^\infty(M)$.

Following [3], we prove the following

Proposition 4.1.

- (i) *The Atiyah cocycle $R_{1,1}^\nabla \in \mathfrak{g}^\vee \otimes \Gamma(T_M^\vee \otimes \text{End } T_M)$ is a Chevalley–Eilenberg 1-cocycle of the \mathfrak{g} -module $\Gamma(T_M^\vee \otimes \text{End } T_M)$.*
- (ii) *The cohomology class $\alpha_{M/\mathfrak{g}} \in H_{\text{CE}}^1(\mathfrak{g}, \Gamma(T_M^\vee \otimes \text{End } T_M))$ of the 1-cocycle $R_{1,1}^\nabla$ does not depend on the choice of connection ∇ .*

The cohomology class $\alpha_{M/\mathfrak{g}}$ is called the Atiyah class of the \mathfrak{g} -manifold M . It is the obstruction class to the existence of a \mathfrak{g} -invariant connection on M , i.e. an affine connection ∇ on M satisfying

$$[\hat{a}, \nabla_X Y] = \nabla_{[\hat{a}, X]} Y + \nabla_X [\hat{a}, Y]$$

for all $a \in \mathfrak{g}$ and $X, Y \in \mathfrak{X}(M)$.

Proposition 4.2. *Let M be a \mathfrak{g} -manifold. The Atiyah class $\alpha_{M/\mathfrak{g}}$ of M vanishes if and only if there exists a \mathfrak{g} -invariant connection on M .*

Note that if \mathfrak{g} is a compact Lie algebra, $\alpha_{M/\mathfrak{g}}$ vanishes since \mathfrak{g} -invariant connections always exist.

The Todd class of complex vector bundles plays an important role in the Riemann–Roch theorem. In our context, the Todd cocycle of a \mathfrak{g} -manifold M is the Chevalley–Eilenberg cocycle

$$\text{td}_{M/\mathfrak{g}} = \det \left(\frac{R_{1,1}^\nabla}{1 - e^{-R_{1,1}^\nabla}} \right) \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M), \tag{5}$$

with $\Omega^k(M)$, $k \geq 0$, being the natural \mathfrak{g} -module. Its corresponding Chevalley–Eilenberg cohomology class is the Todd class $\text{Td}_{M/\mathfrak{g}}$. Alternatively

$$\text{Td}_{M/\mathfrak{g}} = \det \left(\frac{\alpha_{M/\mathfrak{g}}}{1 - e^{-\alpha_{M/\mathfrak{g}}}} \right) \in \bigoplus_{k=0} H_{\text{CE}}^k(\mathfrak{g}, \Omega^k(M)). \tag{6}$$

Since the Lie algebra \mathfrak{g} is finite dimensional, the above expression for the Todd class $\text{Td}_{M/\mathfrak{g}}$ reduces to a finite sum.

Example 1. Consider the case of the 1-dimensional abelian Lie algebra $\mathfrak{g} = \mathbb{R}$ acting on the real line $M = \mathbb{R}$. The infinitesimal action is uniquely determined by a vector field $Q = q(x) \frac{d}{dx} \in \mathfrak{X}(\mathbb{R})$. The Chevalley–Eilenberg complex $(\Lambda^\bullet \mathfrak{g}^\vee \otimes \Gamma(T_M^\vee \otimes \text{End } T_M), d_{\text{CE}})$ is then isomorphic to the 2-term complex

$$0 \longrightarrow C^\infty(\mathbb{R}) \xrightarrow{d_Q} C^\infty(\mathbb{R}) \longrightarrow 0,$$

where the map d_Q is given by

$$d_Q(f) = \frac{d(fq)}{dx} = f'q + fq',$$

for $f \in C^\infty(\mathbb{R})$. Let ∇ be the trivial affine connection on the manifold $M = \mathbb{R}$, i.e. $\nabla_{\frac{d}{dx}} \frac{d}{dx} = 0$. Under the above isomorphism, the Atiyah 1-cocycle $R_{1,1}^\nabla$ is simply the second order derivative of q :

$$R_{1,1}^\nabla = q'' \in C^\infty(\mathbb{R}) \cong \mathfrak{g}^\vee \otimes \Gamma(T_M \otimes \text{End } T_M).$$

As a consequence, the Atiyah class vanishes if and only if there exists a smooth function y defined on the whole real line and satisfying the differential equation $qy' + q'y = q''$. For instance, if $Q = x^2 \frac{d}{dx}$, the Atiyah class is non-trivial since no function $y \in C^\infty(\mathbb{R})$ satisfies $x^2 \frac{dy}{dx} + 2xy = 2$ and therefore there exists no Q -invariant connection on \mathbb{R} .

5. Formality theorem and Kontsevich–Duflo theorem for \mathfrak{g} -manifolds

The main results of this Note are a formality theorem for \mathfrak{g} -manifolds and its consequence: a Kontsevich–Duflo type theorem for \mathfrak{g} -manifolds.

Theorem 5.1 (Formality theorem for \mathfrak{g} -manifolds). *Given a \mathfrak{g} -manifold M and an affine torsionfree connection ∇ on M , there exists an L_∞ quasi-isomorphism Φ from the dgla $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ to the dgla $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$ whose first Taylor coefficient Φ_1 satisfies the following two properties:*

- (i) Φ_1 is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the cohomologies);
- (ii) Φ_1 is equal to the composition $\text{hkr} \circ \text{td}_{M/\mathfrak{g}}^{\frac{1}{2}}$ of the HKR map and the action of the square root of the Todd cocycle $\text{td}_{M/\mathfrak{g}}^{\frac{1}{2}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M)$ on $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ by contraction.

As an immediate consequence, we have the following theorem.

Theorem 5.2 (Kontsevich–Duflo type theorem for \mathfrak{g} -manifolds). *Given a \mathfrak{g} -manifold M , the map*

$$\text{hkr} \circ \text{Td}_{M/\mathfrak{g}}^{\frac{1}{2}} : H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M) \xrightarrow{\cong} H_{\text{CE}}^\bullet(\mathfrak{g}, D_{\text{poly}}^\bullet(M)) \xrightarrow{d_H} D_{\text{poly}}^{\bullet+1}(M)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\text{Td}_{M/\mathfrak{g}}^{\frac{1}{2}}$ of the Todd class $\text{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H_{\text{CE}}^k(\mathfrak{g}, \Omega^k(M))$ acts on $H_{\text{CE}}^\bullet(\mathfrak{g}, T_{\text{poly}}^\bullet(M)) \xrightarrow{0} T_{\text{poly}}^{\bullet+1}(M)$ by contraction.

Theorem 5.1 follows from a more general result of ours, a formality theorem for Lie pairs, whose detailed proof will appear in a forthcoming revision of [8]. A pair of Lie algebroids (or Lie pair in short) consists of a Lie algebroid L and a Lie subalgebroid A of L . Given any Lie pair, our formality theorem for Lie pairs establishes an L_∞ quasi-isomorphism Φ from the polyvector fields ‘on the pair’ to the polydifferential operators ‘on the pair.’ The first ‘Taylor coefficient’ Φ_1 of the L_∞ quasi-isomorphism Φ preserves the associative algebra structures up to homotopy and admits an explicit description in terms of the Hochschild–Kostant–Rosenberg map and the Todd cocycle of the Lie pair. Now every \mathfrak{g} -manifold M determines in a canonical way a matched pair: $(\mathfrak{g} \times M, T_M)$ [11, Example 5.5] [9]. The notation $\mathfrak{g} \times M$ refers to the transformation Lie algebroid arising from the infinitesimal \mathfrak{g} -action on M . Therefore, we can form a Lie pair (L, A) , where $L = (\mathfrak{g} \times M) \bowtie T_M$ and $A = \mathfrak{g} \times M$. For this particular pair, the polyvector fields and polydifferential operators reduce to $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ and $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$, respectively. **Theorem 5.1** then follows from our formality theorem for Lie pairs [8].

To the best of our knowledge, the first construction of an L_∞ quasi-isomorphism from the dgla $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} T_{\text{poly}}^\bullet(M))$ to the dgla $\text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{k}} D_{\text{poly}}^\bullet(M))$ can be credited to Dolgushev [4, concluding remarks].

Applications of **Theorem 5.1** to the deformation quantization of \mathfrak{g} -manifolds will be considered elsewhere.

Acknowledgements

The authors thank Ruggero Bandiera, Martin Bordemann, Vasily Dolgushev, Olivier Elchinger, Marco Manetti, Boris Shoikhet, Jim Stasheff, and Dima Tamarkin for inspiring discussions and useful comments. Mathieu Stiénon would like to express his gratitude to the Université Paris-Diderot, for their hospitality while work on this project was underway.

References

- [1] M.F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85 (1957) 181–207, MR 0086359.
- [2] D. Calaque, M. Van den Bergh, Hochschild cohomology and Atiyah classes, *Adv. Math.* 224 (5) (2010) 1839–1889, MR 2646112.
- [3] Z. Chen, M. Stiénon, P. Xu, From Atiyah classes to homotopy Leibniz algebras, *Commun. Math. Phys.* 341 (1) (2016) 309–349, MR 3439229.
- [4] V. Dolgushev, Covariant and equivariant formality theorems, *Adv. Math.* 191 (1) (2005) 147–177, MR 2102846 (2006c:53101).
- [5] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math. (2)* 78 (1963) 267–288, MR 0161898.
- [6] G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, *Trans. Amer. Math. Soc.* 102 (1962) 383–408, MR 0142598.
- [7] M. Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* 66 (3) (2003) 157–216, MR 2062626 (2005i:53122).
- [8] H. Liao, M. Stiénon, P. Xu, Formality theorem and Kontsevich–Duflo type theorem for Lie pairs, *ArXiv e-prints*, arXiv:1605.09722, 2016.
- [9] K.C.H. Mackenzie, Drinfeld doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids, *Electron. Res. Announc. Amer. Math. Soc.* 4 (1998) 74–87, MR 1650045.
- [10] R.A. Mehta, M. Stiénon, P. Xu, The Atiyah class of a dg-vector bundle, *C. R. Acad. Sci. Paris, Ser. I* 353 (4) (2015) 357–362, MR 3319134.
- [11] T. Mokri, Matched pairs of Lie algebroids, *Glasg. Math. J.* 39 (2) (1997) 167–181, MR 1460632.
- [12] D.E. Tamarkin, Operadic Proof of M. Kontsevich’s Formality Theorem, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.), The Pennsylvania State University, PA, USA, 1999, MR 2699544.