Analytic geometry

Quot schemes and Ricci semipositivity

Schéma quot et semi-positivité de Ricci

Indranil Biswas\textsuperscript{a}, Harish Seshadri\textsuperscript{b}

\textsuperscript{a} School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
\textsuperscript{b} Indian Institute of Science, Department of Mathematics, Bangalore 560003, India

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\textbf{Abstract}

Let $X$ be a compact connected Riemann surface of genus at least two, and let $\mathcal{Q}_X(r, d)$ be the\nquot scheme that parameterizes all the torsion coherent quotients of $\mathcal{O}_X^{\mathbb{R}^r}$ of degree $d$. This\n$\mathcal{Q}_X(r, d)$ is also a moduli space of vortices on $X$. Its geometric properties have been\nextensively studied. Here we prove that the anticanonical line bundle of $\mathcal{Q}_X(r, d)$ is not nef. Equivalently, $\mathcal{Q}_X(r, d)$ does not admit any Kähler metric whose Ricci curvature is\nsemipositive.

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\textbf{Résumé}

Soit $X$ une surface de Riemann compacte et connexe de genre au moins deux, et soit\n$\mathcal{Q}_X(r, d)$ le schéma quot qui paramètre tous les quotients torsion cohérents de $\mathcal{O}_X^{\mathbb{R}^r}$\nde degré $d$. L'espace $\mathcal{Q}_X(r, d)$ est aussi un espace de modules de vortex sur $X$. Nous\ndémontrons que le fibré anticanonique de $X$ n'a pas la propriété nef. De façon équivalente, $\mathcal{Q}_X(r, d)$\nne possède aucune métrique kählérienne dont la courbure de Ricci soit semi-positive.

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1. Introduction

Take a compact connected Riemann surface $X$. The genus of $X$, which will be denoted by $g$, is assumed to be at least two. We will not distinguish between the holomorphic vector bundles on $X$ and the torsion-free coherent analytic sheaves on $X$. For a positive integer $r$, let $\mathcal{O}_X^{\mathbb{R}^r}$ be the trivial holomorphic vector bundle on $X$ of rank $r$. Fixing a positive integer $d$, let

$$\mathcal{Q} := \mathcal{Q}_X(r, d)$$  \hspace{1cm} (1.1)

be the quotient scheme that parametrizes all (torsion) coherent quotients of $\mathcal{O}_X^{\mathbb{R}^r}$ of rank zero and degree $d$ \cite{17}. Equivalently, $\mathcal{Q}$ parametrizes all coherent subsheaves of $\mathcal{O}_X^{\mathbb{R}^r}$ of rank $r$ and degree $−d$, because these are precisely the kernels of coherent

\textit{E-mail addresses:} indranil@math.tifr.res.in (I. Biswas), harish@math.iisc.ernet.in (H. Seshadri).

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quotients of $O^\text{Pr}_X$ of rank zero and degree $d$. This $Q$ is a connected smooth complex projective variety of dimension $rd$. See [6,5,4] for the properties of $Q$. It should be mentioned that $Q$ is also a moduli space of vortices on $X$, and it has been extensively studied from this point of view of mathematical physics; see [3,9,12] and references therein.

Bökstedt and Romão proved some interesting differential geometric properties of $Q$ (see [12]). In [10] and [11], we proved that $Q$ does not admit Kähler metrics with semipositive or seminegative holomorphic bisectional curvature. In this note, we continue to study the question of the existence of metrics on $Q$ whose curvature has a sign. Our aim here is to prove the following.

**Theorem 1.1.** The quotient scheme $Q$ in (1.1) does not admit any Kähler metric such that the anticanonical line bundle $K_Q^{-1}$ is Hermitian semipositive.

Since semipositive holomorphic bisectional curvature implies semipositive Ricci curvature for a Kähler metric, Theorem 1.1 generalizes the main result of [11].

Recall that a holomorphic line bundle $L$ on a compact complex manifold $M$ is said to be Hermitian semipositive if $L$ admits a smooth Hermitian structure such that the corresponding Hermitian connection has the property that its curvature form is semipositive. The anticanonical line bundle on $M$ will be denoted by $K_M^{-1}$. Note that if $M$ admits a Kähler metric such that the corresponding Ricci curvature is semipositive, then $K_M^{-1}$ is Hermitian semipositive. Indeed, in that case, the Hermitian connection on $K_M^{-1}$ for the Hermitian structure induced by such a Kähler metric has semipositive curvature. The converse statement, that the Hermitian semipositivity of $K_M^{-1}$ implies the existence of Kähler metrics with semipositive Ricci curvature, is also true by Yau’s solution to the Calabi conjecture [1,2,19].

The proof of Theorem 1.1 is based on a recent work of Demailly, Campana, and Peternell on the classification of compact Kähler manifolds $M$ with semipositive $K_M^{-1}$ [15,14]. This classification implies that if $K_M^{-1}$ is semipositive, then there is a nontrivial Abelian ideal in the Lie algebra of holomorphic vector fields on $M$, provided $b_1(M) > 0$. On the other hand, for $M = Q$, this Lie algebra is isomorphic to $sl(r, \mathbb{C})$, which does not have any nontrivial Abelian ideal.

2. Proof of Theorem 1.1

2.1. Semipositive Ricci curvature

Let $J^d(X) = \text{Pic}^d(X)$ be the connected component of the Picard group of $X$ that parameterizes the isomorphism classes of holomorphic line bundles on $X$ of degree $d$. Let $S^d(X)$ denote the space of all effective divisors on $X$ of degree $d$, so $S^d(X) = X^d/P_d$ is the symmetric product, with $P_d$ being the group of permutations of $\{1, \ldots, d\}$. Let

$$p : S^d(X) \longrightarrow \text{Pic}^d(X)$$

be the natural morphism that sends a divisor on $X$ to the holomorphic line bundle on $X$ defined by it.

Take any coherent subsheaf $F \subset O^\text{Pr}_X$ of rank $r$ and degree $-d$. Let

$$s_F : O^\text{Pr}_X = (O^\text{Pr}_X)^* \longrightarrow F^*$$

be the dual of the inclusion of $F$ in $O^\text{Pr}_X$. Its exterior product

$$\bigwedge^r s_F : O_X = \bigwedge^r O^\text{Pr}_X \longrightarrow \bigwedge^r F^*$$

is a holomorphic section of the holomorphic line bundle $\bigwedge^r F^*$ of degree $d$. Therefore, the divisor $\text{div}(\bigwedge^r s_F)$ is an element of $S^d(X)$. Consequently, we have a morphism

$$\varphi : Q \longrightarrow S^d(X), \quad F \mapsto \text{div}(\bigwedge^r s_F),$$

where $Q$ is defined in (1.1). We note that when $r = 1$, then $\varphi$ is an isomorphism.

Assume that $Q$ admits a Kähler metric $\omega$ such that $K_Q^{-1}$ is Hermitian semipositive. Then there is a connected finite étale Galois covering

$$f : \tilde{Q} \longrightarrow Q$$

such that $(\tilde{Q}, f^* \omega)$ is holomorphically isometric to a product

$$\gamma : \tilde{Q} \longrightarrow A \times C \times H \times F,$$

where

- $A$ is an Abelian variety,
- $C$ is a simply connected Calabi–Yau manifold (holonomy is $SU(c)$, where $c = \dim C$),
• $H$ is a simply connected hyper-Kähler manifold (holonomy is $\text{Sp}(h/2)$, where $h = \dim H$), and
• $F$ is a rationally connected smooth projective variety such that $K_F^{-1}$ is Hermitian semipositive.

(See [15, Theorem 3.1].) Henceforth, we will identify $\tilde{Q}$ with $A \times C \times H \times F$ using $\gamma$ in (2.4). We note that $F$ is simply connected because it is rationally connected [13, p. 545, Theorem 3.5], [18, p. 362, Proposition 2.3].

2.2. A lower bound of $d$

We know that $b_1(Q) = 2g$, and the induced homomorphism

$$(p \circ \varphi)_*: H_1(Q, \mathbb{Q}) \to H_1(\text{Pic}^d(X), \mathbb{Q}),$$

where $p$ and $\varphi$ are constructed in (2.1) and (2.2), respectively, is an isomorphism [5], [6, p. 649, Remark]. Since $f$ in (2.2) is a finite étale covering, the induced homomorphism

$$f_* : H_1(\tilde{Q}, \mathbb{Q}) \to H_1(Q, \mathbb{Q})$$

is surjective. Therefore, the homomorphism

$$(p \circ \varphi \circ f)_*: H_1(\tilde{Q}, \mathbb{Q}) \to H_1(\text{Pic}^d(X), \mathbb{Q})$$

is surjective. Thus, there is no nonconstant holomorphic map from a compact simply connected Kähler manifold to an Abelian variety. In particular, there are no nonconstant holomorphic maps from $C$, $H$ and $F$ in (2.4) to $\text{Pic}^d(X)$. Hence, the map $p \circ \varphi \circ f$ factors through a map

$$\beta : A \to \text{Pic}^d(X).$$

In other words, there is a commutative diagram

$$\begin{array}{ccc}
\tilde{Q} = A \times C \times H \times F & \xrightarrow{p \circ \varphi \circ f} & \text{Pic}^d(X) \\
q \downarrow & & \| \text{Id} \\
A & \xrightarrow{\beta} & \text{Pic}^d(X)
\end{array}$$

(2.6)

where $q$ is the projection of $A \times C \times H \times F$ to the first factor. Since $H_1(A \times C \times H \times F, \mathbb{Z}) = H_1(A, \mathbb{Z})$ (as $C$, $H$ and $F$ are simply connected), and $(p \circ \varphi \circ f)_*$ in (2.5) is surjective, it follows that the homomorphism

$$\beta_* : H_1(A, \mathbb{Q}) \to H_1(\text{Pic}^d(X), \mathbb{Q})$$

induced by $\beta$ is surjective. This immediately implies that the map $\beta$ is surjective. Since $\beta$ is surjective, from the commutativity of (2.6) we know that the map $p$ is surjective. This implies that

$$d = \dim \text{Pic}^d(X) \geq \dim \text{Pic}^d(X) = g \geq 2.$$ 

(2.7)

2.3. Albanese for $\tilde{Q}$

The homomorphism of fundamental groups

$$\varphi_* : \pi_1(Q) \to \pi_1(\text{Pic}^d(X))$$

induced by $\varphi$ in (2.2) is an isomorphism [8, Proposition 4.1]. Since $d \geq 2$ (see (2.7)), the homomorphism of fundamental groups

$$p_* : \pi_1(\text{Pic}^d(X)) \to \pi_1(\text{Pic}^d(X))$$

induced by $p$ in (2.1) is an isomorphism. Indeed, $\pi_1(\text{Pic}^d(X))$ is the Abelianization

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] = H_1(X, \mathbb{Z})$$

of $\pi_1(X)$ [16]. Combining these we conclude that the homomorphism of fundamental groups

$$(p \circ \varphi)_* : \pi_1(Q) \to \pi_1(\text{Pic}^d(X))$$

(2.8)

induced by $p \circ \varphi$ is an isomorphism.

Since the homomorphism in (2.8) is an isomorphism, the covering $f$ in (2.3) is induced by a covering of $\text{Pic}^d(X)$. In other words, there is a finite étale Galois covering...
$$\mu : J \rightarrow \text{Pic}^d(X)$$

and a morphism $$\lambda : \tilde{Q} \rightarrow J$$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\tilde{Q} & \xrightarrow{f} & Q \\
\downarrow{\lambda} & & \downarrow{p \circ \varphi} \\
J & \xrightarrow{\mu} & \text{Pic}^d(X)
\end{array}$$

(2.9)

(2.10)

where $$f$$ is the covering map in (2.3). The projection $$q$$ in (2.6) is clearly the Albanese morphism for $$\tilde{Q}$$, because $$C, H$$ and $$F$$ are all simply connected. On the other hand, $$p \circ \varphi$$ is the Albanese morphism for $$Q$$ [11, Corollary 2.2]. Therefore, its pullback, namely, $$\lambda$$, is the Albanese morphism for $$\tilde{Q}$$. Consequently, we have $$A = J$$ with $$\lambda$$ coinciding with the projection $$q$$ in (2.6). Henceforth, we will identify $$A$$ and $$q$$ with $$J$$ and $$\lambda$$ respectively.

2.4. Vector fields

The differential $$df$$ of $$f$$ identifies $$T\tilde{Q}$$ with $$f^*TQ$$, because $$f$$ is étale. Using the trace homomorphism $$t : f_*O_{\tilde{Q}} \rightarrow O_Q$$, we have

$$f_*T\tilde{Q} = f_*f^*TQ \xrightarrow{p f} (f_*O_{\tilde{Q}}) \otimes T\tilde{Q} \xrightarrow{\iota} O_Q \otimes T\tilde{Q} = T\tilde{Q},$$

where $$p f$$ is given by the projection formula. This produces a homomorphism

$$\Phi : H^0(\tilde{Q}, T\tilde{Q}) = H^0(Q, f_*T\tilde{Q}) \rightarrow H^0(Q, TQ)$$

(2.11)

(see (2.12)) is an ideal in this Lie algebra. Since $$A = J$$ is a covering of Pic$$^d(X)$$, we have

$$\dim H^0(A, TA) = \dim \text{Pic}^d(X) = g > 1.$$  

Since $$H^0(A, TA)$$ is an ideal in $$H^0(\tilde{Q}, T\tilde{Q})$$, it follows immediately that

$$\Phi(H^0(A, TA)) \subset \Phi(H^0(\tilde{Q}, T\tilde{Q})) = H^0(Q, TQ)$$

is an ideal, where $$\Phi$$ is constructed in (2.11). Note that $$H^0(A, TA)$$ is an Abelian Lie algebra, so the Lie algebra $$\Phi(H^0(A, TA))$$ is also Abelian.

Since $$\mu : J = A \rightarrow \text{Pic}^d(X)$$ in (2.9) is a covering map between Abelian varieties, the trace map $$H^0(A, TA) \rightarrow H^0(\text{Pic}^d(X), TPic^d(X))$$ is an isomorphism. In view of this, from the commutativity of the diagram in (2.10), it follows that the restriction

$$\Phi_{|H^0(A, TA)} : H^0(A, TA) \rightarrow H^0(Q, TQ)$$

is injective (see (2.12) and (2.11)). But $$H^0(Q, TQ) = sl(r, \mathbb{C})$$ [7, p. 1446, Theorem 1.1]. Hence the Lie algebra $$H^0(Q, TQ)$$ does not contain any nonzero Abelian ideal. This is in contradiction with the earlier result that $$\Phi(H^0(A, TA))$$ is a nonzero Abelian ideal in $$H^0(Q, TQ)$$ of dimension $$g$$ (see (2.13)). This completes the proof of Theorem 1.1.

References