C. R. Acad. Sci. Paris, Ser. I 355 (2017) 553–558

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional analysis/Mathematical physics

Eigenvalues behaviours for self-adjoint Pauli operators with unsigned perturbations and admissible magnetic fields ☆

Asymptotiques de valeurs propres pour des opérateurs de Pauli autoadjoints perturbés par des potentiels de signe non fixé en présence d'un champ magnétique non constant

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A R T I C L E I N F O

Article history:
Received 20 October 2016
Accepted after revision 19 April 2017
Available online 4 May 2017

Presented by the Editorial Board

A B S T R A C T

We investigate the discrete spectrum behaviour for the 2d Pauli operator with nonconstant magnetic field, perturbed by a *sign-indefinite* self-adjoint electric potential that decays polynomially at infinity. A localisation of the eigenvalues and new asymptotics are established.

R É S U M É

Cette note est consacrée à l'étude du comportement des valeurs propres (discrètes) associées à l'opérateur de Pauli 2d en présence d'un champ magnétique non constant et d'un potentiel électrique autoadjoint de signe non fixé qui décroît polynomialement à l'infini. De nouvelles asymptotiques sur les valeurs propres sont obtenues en plus de leur localisation sur le spectre.

1. Introduction and results

We consider a quantum spin-$\frac{1}{2}$ non-relativistic particle submitted to an electromagnetic field and described by the Pauli operator

☆ The two authors have been supported by the Chilean Program Núcleo Milenio de Física Matemática RC120002. D. Sambou is supported by the Chilean Fondecyt Grant 3170411. The authors are grateful to J.-F. Bony for his suggestion in the use of the reduction (2.2), G. Raikov for his helpful suggestions during the revision of this note, and the anonymous referee for his helpful remarks, suggestions and comments.

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http://dx.doi.org/10.1016/j.crma.2017.04.007
1631-073X/

$$H(b,V) := \begin{pmatrix} (-\mathrm{i}\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-\mathrm{i}\nabla - \mathbf{A})^2 + b \end{pmatrix} + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathbb{C}^2), \tag{1.1}$$

where $V = V(x)$, $x \in \mathbb{R}^2$, is a 2×2 Hermitian matrix-valued potential, and $\mathbf{A}$ is a vector potential generating the magnetic field $b = \nabla \wedge \mathbf{A}$. We assume $b = b(x)$ to be an admissible magnetic field in the sense that there exists a constant $b_0 > 0$ such that

$$b(x) = b_0 + \tilde{b}(x), \tag{1.2}$$

with the Poisson equation $\Delta\tilde{\varphi} = \tilde{b}$ admitting a solution $\tilde{\varphi} \in \mathrm{C}^2(\mathbb{R}^2)$ satisfying $\sup_{x\in\mathbb{R}^2} |D^\alpha \tilde{\varphi}(x)| < \infty$ for $\alpha \in \mathbb{N}$, $|\alpha| \le 2$. We refer for instance to [4] for examples of admissible magnetic fields.

In the unperturbed case where $V = 0$, the spectrum of $H(b,0)$ belongs to $\{0\} \cup [\zeta, +\infty)$ with $\zeta = 2b_0 \mathrm{e}^{-2\mathrm{osc}(\tilde{\varphi})}$ and $\mathrm{osc}(\tilde{\varphi}) := \sup_{x\in\mathbb{R}^2} \tilde{\varphi}(x) - \inf_{x\in\mathbb{R}^2} \tilde{\varphi}(x)$. Furthermore, 0 is an eigenvalue of infinite multiplicity (see, e.g., [4]). Notice that in the constant magnetic field case $b = b_0$, we have $\zeta = 2b_0$ the first Landau level of the shifted Schrödinger operator $(-\mathrm{i}\nabla - \mathbf{A})^2 + b$. The case where V is of definite sign has been already studied in [4]. In the present note, we are interested in the sign-indefinite potentials V of the form

$$V(x) := \begin{pmatrix} 0 & \overline{U(x)} \\ U(x) & 0 \end{pmatrix}, \quad \text{for} \quad x \in \mathbb{R}^2, \tag{1.3}$$

where the function $U(x) \in \mathbb{C}$ satisfies

$$|U(x)| = \mathcal{O}(\langle x\rangle^{-m}), \quad \langle x\rangle := \sqrt{1 + |x|^2}, \quad \text{for some} \quad m > 0. \tag{1.4}$$

Remark. The potentials V of the form (1.3) are *sign-indefinite* since their eigenvalues are given by $\pm|U(x)|$.

Under condition (1.4), V is relatively compact with respect to $H(b,0)$ so that $\sigma_{\mathrm{ess}}\big(H(b,V)\big) = \sigma_{\mathrm{ess}}\big(H(b,0)\big)$, where σ_{ess} denotes the essential spectrum. However, $H(b,V)$ may have a discrete spectrum $\sigma_{\mathrm{disc}}\big(H(b,V)\big)$ that can accumulate at 0. The aim of this note is to study this discrete spectrum near the low-ground energy 0. The novelty of this work arises from sign-indefinite perturbations we consider and the behaviours we obtain. This is probably one of the first works dealing with sign-indefinite perturbations in a magnetic framework, see also the recent work [6] where the case of 3d Pauli operators are studied in a resonance point of view. We denote

$$H_\pm := (-\mathrm{i}\nabla - \mathbf{A})^2 \pm b \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2) := \mathrm{L}^2(\mathbb{R}^2, \mathbb{C}), \tag{1.5}$$

the component operators of the Pauli operator (1.1). Let $p := p(b)$ be the orthogonal projection of $\mathrm{L}^2(\mathbb{R}^2)$ onto the (infinite dimensional) kernel of H_-. The corresponding projection in the constant magnetic field case will be denoted $p_0 := p(b_0)$. For a bounded operator $B \in \mathscr{L}\big(\mathrm{L}^2(\mathbb{R}^2)\big)$, we introduce the operator $\mathbf{W}(B)$ defined by

$$\big(\mathbf{W}(B) f\big)(x) := \overline{U}(x) B (Uf)(x). \tag{1.6}$$

If I denotes the identity operator on $\mathrm{L}^2(\mathbb{R}^2)$, then $\mathbf{W}(I)$ is the multiplication operator by the function $x \longmapsto |U(x)|^2$. This function will be denoted $\mathbf{W}(I)$ again. Our results are strongly related to the operator $\mathbf{W}(B)$ through the Toepliz operator

$$p\mathbf{W}(B)p, \qquad B = I \quad \text{or} \quad H_+^{-1}. \tag{1.7}$$

Since the spectrum of the invertible operator H_+ belongs to $[\zeta, +\infty)$ and U fulfils (1.4), then it follows from [4, Lemma 3.5] that the positive self-adjoint operators $p\mathbf{W}(I)p$ and $p\mathbf{W}\big(H_+^{-1}\big)p$ are compact on $\mathrm{L}^2(\mathbb{R}^2)$. For further use, let us introduce the following:

Assumption (A). The function $U \in C^1(\mathbb{R}^2)$ satisfies

$$0 \le U(x) \le C\langle x\rangle^{-m}, \; |\nabla U(x)| \le C\langle x\rangle^{-m-1}, \quad x \in \mathbb{R}^2, \tag{1.8}$$

for some constants $C > 0$, $m > 0$, and $U(x) = U_0\big(\frac{x}{|x|}\big)|x|^{-m}(1 + o(1))$, $|x| \to +\infty$, with $0 \not\equiv U_0 \in C^0(\mathbb{S}^1)$.

Integrated density of states (IDS): For $x \in \mathbb{R}^2$, let $\chi_{T,x}$ be the characteristic function of the square $x + (T/2, T/2)^2$ with $T > 0$. Denote $\mathbf{1}_I(H_-)$ the spectral projection of H_- on the interval $I \subset \mathbb{R}$. A non-increasing function $g : \mathbb{R} \longrightarrow [0, +\infty)$ is called an IDS for the operator H_- if it satisfies for any $x \in \mathbb{R}^2$

$$g(t) = \lim_{T\to\infty} T^{-2}\,\mathrm{Tr}\big[\chi_{T,x} \mathbf{1}_{(-\infty,t)}(H_-) \chi_{T,x}\big],$$

for each point t of continuity of g (see, e.g., [4]).

Remark. If $b = b_0$ is constant, then there exists naturally an IDS for the operator H_- given by

$$g(t) = \frac{b_0}{2\pi} \sum_{q=0}^{\infty} \chi_{\mathbb{R}_+}(t - 2b_0 q), \quad t \in \mathbb{R}, \quad \chi_{\mathbb{R}_+}(t) = \begin{cases} 1 & \text{if } t \in \mathbb{R}^+, \\ 0 & \text{otherwise.} \end{cases}$$

In the next results, the discrete eigenvalues of the operator $H(b, eV)$ are counted according to their multiplicity defined by (2.6), for which we conjecture that it coincides with the geometric multiplicity.

Theorem 1.1. *Assume that V and U fulfil (1.3) and (1.4), respectively. Then, there exists a discrete set $\mathcal{E} \subset \mathbb{R}$ such that for any $e \in \mathbb{R} \setminus \mathcal{E}$ and any $0 < r_0 \ll 1$, the following holds:*

(i) *localization: if z is a discrete eigenvalue of $H(b, eV)$ near zero, then $z \le 0$;*
(ii) *asymptotic: Suppose that $\#\{z \in \sigma(p\mathbf{W}(H_+^{-1})p) : z \ge r\} \to +\infty$ as $r \searrow 0$. Then, there exists a positive sequence $(r_\ell)_\ell$ tending to 0 such that as $\ell \longrightarrow \infty$,*

$$\#\{z \in \sigma_{\mathrm{disc}}(H(b, eV)) : -r_0 e^2 \le z < -r_\ell e^2\} = \#\{z \in \sigma(p\mathbf{W}(H_+^{-1})p) : z \ge r_\ell\}\big(1 + o(1)\big); \tag{1.9}$$

(iii) *upper bound: assume that there exists an IDS g for the operator H_-. If $\mathbf{W}(I)$ satisfies Assumption (A) and $\mathrm{Tr}\, \mathbf{1}_{[r(1+\nu), r(1-\nu)]}\big(p\mathbf{W}(H_+^{-1})p\big) = \mathrm{Tr}\, \mathbf{1}_{[r,1]}\big(p\mathbf{W}(H_+^{-1})p\big)\big(o(1) + \mathcal{O}(\nu)\big)$ for any $0 < \nu \ll 1$, $r \searrow 0$, then,*

$$\limsup_{r \searrow 0} \frac{\#\{z \in \sigma_{\mathrm{disc}}(H(b, eV)) : -r_0 e^2 \le z < -re^2\}}{\#\{z \in \sigma\left(\zeta^{-1} p\mathbf{W}(I)p\right) : z \ge r\}} \le 1. \tag{1.10}$$

Furthermore, if the magnetic field is constant (*i.e.* $b = b_0$), we obtain the following theorem.

Theorem 1.2.

(i) *Assume that $\#\{z \in \sigma\left((p_0 U p_0)^* p_0 U p_0\right) : z \ge 2rb_0\} = \phi(r)\big(1 + o(1)\big)$, $r \searrow 0$, where $\phi\big(r(1 \pm \nu)\big) = \phi(r)\big(1 + o(1) + \mathcal{O}(\nu)\big)$ for any $0 < \nu \ll 1$. Suppose, moreover, that $\mathrm{Tr}\, \mathbf{1}_{[r,1]}\big(p_0\mathbf{W}(H_+^{-1})p_0\big) = \phi(r)\big(1 + o(1)\big)$, $\phi(r) \longrightarrow +\infty$, $r \searrow 0$. Then, as $r \searrow 0$,*

$$\#\{z \in \sigma_{\mathrm{disc}}(H(b_0, eV)) : -r_0 e^2 \le z < -re^2\} = \#\{z \in \sigma\left((p_0 U p_0)^* p_0 U p_0\right) : z \ge 2rb_0\}\big(1 + o(1)\big). \tag{1.11}$$

(ii) *Assume that U satisfies Assumption (A). Then, as $r \searrow 0$,*

$$\#\{z \in \sigma_{\mathrm{disc}}(H(b_0, eV)) : -r_0 e^2 \le z < -re^2\} = \#\{z \in \sigma(p_0 U p_0) : z \ge (2rb_0)^{\frac{1}{2}}\}\big(1 + o(1)\big). \tag{1.12}$$

Remarks.

(i) The proof of Theorem 1.2, (ii) shows that $\#\{z \in \sigma(p_0\mathbf{W}(H_+^{-1})p_0) : z \ge r\} \to +\infty$ as $r \searrow 0$. Then, by Theorem 1.1, (ii), the asymptotic (1.9) holds with $p = p_0$.
(ii) Notice that thanks to the asymptotics of [5, Lemma 3.3], (1.12) implies that the number of eigenvalues of the operator $H(b_0, eV)$ near 0 satisfies

$$\#\{z \in \sigma_{\mathrm{disc}}\big(H(b_0, eV)\big) : -r_0 e^2 \le z < -re^2\} = C_m \left(\frac{1}{2b_0}\right)^{1/m} r^{-1/m}\big(1 + o(1)\big), \tag{1.13}$$

as $r \searrow 0$, where

$$C_m := \frac{b_0}{4\pi} \int_{\mathbb{S}^1} U_0(t)^{2/m} \mathrm{d}t. \tag{1.14}$$

In particular, it holds from (1.13) that the eigenvalues of $H(b_0, eV)$ less than $-re^2$ accumulate at zero with an accumulation rate of order $r^{-1/m}$, whereas it was of order $r^{-2/m}$ for V of definite sign in [4].
(iii) Otherwise, we can expect that this kind of accumulation also occurs near all the Landau levels $2b_0 q$, $q \in \mathbb{N}$, of the operator $H(b_0, V)$. However, the spectral analysis is more difficult due to the contribution of the half-Pauli operators $H_\pm$ near each Landau level $2b_0 q$, $q \in \mathbb{N}^*$.

2. Strategy of the proofs

We explain the main ideas of the proofs and the relationship between the initial operator and the new quantities we are going to introduce. First, let us introduce some useful notations. For $\mathscr{H}$ a separable Hilbert space, we denote $S_\infty(\mathscr{H})$ (resp. $GL(\mathscr{H})$) the set of compact (resp. invertible) linear operators in $\mathscr{H}$. Let $D \subseteq \mathbb{C}$ be a connected open set, $Z \subset D$ be a discrete and closed subset, $A : \overline{D} \backslash Z \longrightarrow GL(\mathscr{H})$ be a finite meromorphic operator-valued function (see, e.g., [2] and [3, Section 4]) and Fredholm at each point of Z. For an operator A that does not vanish on γ a positive oriented contour, the index of A with respect to γ is defined by

$$\mathrm{Ind}_\gamma\, A := \frac{1}{2i\pi}\mathrm{tr}\int_\gamma A'(z)A(z)^{-1}\mathrm{d}z = \frac{1}{2i\pi}\mathrm{tr}\int_\gamma A(z)^{-1}A'(z)\mathrm{d}z. \tag{2.1}$$

2.1. Reduction of the problem

Let us consider the punctured disk $D(0,\epsilon)^* := \{z \in \mathbb{C} : 0 < |z| < \epsilon\}$ for $0 < \epsilon < \zeta$. For $z \in D(0,\epsilon)^*$ small enough, we have

$$\big(H(b,V) - z\big)\begin{pmatrix} I & 0 \\ -(H_+ - z)^{-1}U & (H_+ - z)^{-1}\end{pmatrix} = \begin{pmatrix} H_- - z - \overline{U}(H_+ - z)^{-1}U & \overline{U}(H_+ - z)^{-1} \\ 0 & I\end{pmatrix},$$

so that the following characterisation holds:

$$H(b,V) - z \quad \text{is invertible} \;\Leftrightarrow\; H_- - z - \overline{U}(H_+ - z)^{-1}U \quad \text{is invertible.} \tag{2.2}$$

Thus, we reduce the study of the discrete eigenvalues of $H(b,V)$ near $z = 0$ to the analysis of the non-invertibility of the operator $H_- - z - \overline{U}(H_+ - z)^{-1}U$. It is not difficult to prove the following lemma, which gives a new representation of the operator $\overline{U}(H_+ - z)^{-1}U$.

Lemma 2.1. *For z small enough, the operator $\overline{U}(H_+ - z)^{-1}U$ admits the representation*

$$\overline{U}(H_+ - z)^{-1}U = \mathbf{w}^*\,(I + M(z))\,\mathbf{w}, \tag{2.3}$$

where $\mathbf{w} := H_+^{-1/2}U$ and

$$z \longmapsto M(z) := z\sum_{k\geq 0} z^k H_+^{-k-1} \tag{2.4}$$

is analytic near $z = 0$.

Let $R_-(z)$ denote the resolvent of the operator H_-. We have the following:

Lemma 2.2. *For z small enough, the operator-valued function*

$$D(0,\epsilon)^* \ni z \longmapsto \mathcal{T}_V(z) := \big(I + M(z)\big)\mathbf{w}R_-(z)\mathbf{w}^*,$$

is analytic with values in $S_\infty\left(L^2(\mathbb{R}^2)\right)$.

Proof. Since $M(z)$ and $R_-(z)$ are well defined and analytic for z in $D(0,\epsilon)^*$, then the analyticity of $\mathcal{T}_V(z)$ follows. The compactness holds from that of $UR_-(z)\overline{U}$, by combining the diamagnetic inequality and [7, Theorem 2.13]. □

We have the following Birman–Schwinger principle:

Proposition 2.1. *For z_0 near zero, the following assertions are equivalent:*

(i) *z_0 is a discrete eigenvalue of $H(b,V)$,*
(ii) *$I - \mathcal{T}_V(z_0)$ is not invertible.*

Proof. Set $\mathscr{R}(z) := \left(H_- - z - \overline{U}(H_+ - z)^{-1}U\right)^{-1}$. Then, the proof follows directly from (2.2), the fact that $\mathscr{R}(z)$ and $\big(I + M(z)\big)\mathbf{w}\mathscr{R}(z)\mathbf{w}^*$ have the same poles (the discrete eigenvalues z) near 0, together with the identity

$$\Big(I - \big(I + M(z)\big)\mathbf{w}R_-(z)\mathbf{w}^*\Big)\Big(I + \big(I + M(z)\big)\mathbf{w}\mathscr{R}(z)\mathbf{w}^*\Big) = I. \quad \square \tag{2.5}$$

In Proposition 2.1, (ii), z_0 is said to be a characteristic value of the operator-valued function $I - \mathcal{T}_V(\cdot)$. Sometimes, by abuse of language, we will say that z_0 is a characteristic value of the operator $I - \mathcal{T}_V(z)$. The multiplicity of a discrete eigenvalue z_0 is defined by

$$\mathrm{mult}(z_0) := \mathrm{Ind}_\gamma \left(I - \mathcal{T}_V(\cdot) \right), \tag{2.6}$$

where γ is a small positively oriented contour containing z_0 as the unique discrete eigenvalue of $H(b, V)$ (see (2.1)). We will denote $\mathcal{Z}\big(I - \mathcal{T}_V(\cdot)\big)$ the set of characteristic values of $I - \mathcal{T}_V(\cdot)$.

2.2. Sketch of proof of Theorem 1.1

As preparation, we point out some facts. Since p is the orthogonal projection onto $\ker H_-$ and $p^\perp := 1 - p$, then we have

$$R_-(z) = (H_- - z)^{-1} p + (H_- - z)^{-1} p^\perp = -z^{-1} p + (H_- - z)^{-1} p^\perp.$$

In particular, this implies that

$$\mathcal{T}_V(z) = -\frac{1}{z}\, \mathbf{w} p \mathbf{w}^* - z^{-1} M(z) \mathbf{w} p \mathbf{w}^* + \big(I + M(z)\big) \mathbf{w} R_-(z) p^\perp \mathbf{w}^*. \tag{2.7}$$

In the first term of the r.h.s. of (2.7), write the operator $\mathbf{w} p \mathbf{w}^*$ as $\mathbf{w} p \mathbf{w}^* = (p\mathbf{w}^*)^*(p\mathbf{w}^*)$. By the definition of $\mathbf{w}$ in Lemma 2.1, we have $\mathbf{w}^*\mathbf{w} = \overline{U} H_+^{-1} U$. Since $\sigma(H_+) \subset [\zeta, \infty)$, then we have

$$(p\mathbf{w}^*)(p\mathbf{w}^*)^* = p\mathbf{w}^*\mathbf{w}p = p\mathbf{W}(H_+^{-1})p \le \zeta^{-1}\, p\mathbf{W}(I)p, \tag{2.8}$$

where $\mathbf{W}(\bullet)$ is the operator defined by (1.6). According to Proposition 2.1, the discrete eigenvalues z of the operator $H(b, eV)$ near 0 are the characteristic values of the operator $I - \mathcal{T}_{eV}(z)$. Let us set $K_V(z) := \mathbf{w} p \mathbf{w}^* - z A(z)$, where

$$A(z) := -z^{-1} M(z) \mathbf{w} p \mathbf{w}^* + \big(I + M(z)\big) \mathbf{w} R_-(z) p^\perp \mathbf{w}^*. \tag{2.9}$$

Thus, we have

$$I - \mathcal{T}_{eV}(z) = I + \frac{\mathrm{e}^2}{z} K_V(z) = I - \frac{K_V^{(e)}(\lambda)}{\lambda}, \tag{2.10}$$

with the rescaling $\lambda = -z/\mathrm{e}^2$ and the operator $K_V^{(e)}(\lambda)$ defined by $K_V^{(e)}(\lambda) := K_V(-\lambda \mathrm{e}^2)$, so that $K_V^{(e)}(0) = K_V(0) = \mathbf{w} p \mathbf{w}^*$. Moreover, $\big(K_V^{(e)}\big)'(\lambda) = -\mathrm{e}^2 K_V'(-\lambda \mathrm{e}^2)$ so that $\big(K_V^{(e)}\big)'(0) = -\mathrm{e}^2 K_V'(0)$. Let Π_0 be the orthogonal projection onto $\ker K_V(0)$. The compactness of the operator $K_V'(0)\Pi_0$ implies the existence of a discrete set $\{e_n\} \subset \mathbb{R}$ finite or infinite such that the operator $I + \mathrm{e}^2 K_V'(0)\Pi_0$ is invertible for each $e \in \mathcal{E} := \mathbb{R} \setminus \{e_n\}$. For $L \in S_\infty\left(L^2(\mathbb{R}^2)\right)$, we set

$$n_+(r, L) := \mathrm{rank}\, \mathbf{1}_{[r,\infty)}(L), \quad r > 0, \tag{2.11}$$

where $\mathbf{1}_{[r,\infty)}(L)$ is the spectral projection of L on the interval $[r, \infty)$. We have

$$n_+\big(r, \mathbf{w} p \mathbf{w}^*\big) = n_+\big(r, p\mathbf{w}^*\mathbf{w}p\big) = n_+\Big(r, p\mathbf{W}(H_+^{-1})p\Big), \quad r > 0. \tag{2.12}$$

Then, (2.8) implies that

$$n_+\big(r, \mathbf{w} p \mathbf{w}^*\big) \le n_+\left(r, \zeta^{-1} p\mathbf{W}(I)p\right), \quad r > 0. \tag{2.13}$$

We return now to the proof of Theorem 1.1.

(i)–(ii): The claim (i) follows immediately from [1, Corollary 3.4] with z replaced by $\lambda = -z/\mathrm{e}^2$, thanks to (2.10). To deal with the claim (ii), introduce the sector $\mathcal{C}_\alpha(a, a') := \{x + \mathrm{i}y \in \mathbb{C} : a \le x \le a', -\alpha x \le y \le \alpha x\}$, with $a > 0$ tending to 0, $a' > 0$ fixed, and $\alpha > 0$. Proposition 2.1 together with (2.10) show that z is a discrete eigenvalue near zero if and only if λ is a characteristic value of $I - \mathcal{T}_{eV}(-\lambda \mathrm{e}^2)$. Moreover, the proof of (i) shows that for $-r_0\mathrm{e}^2 \le z < -r\mathrm{e}^2$, $0 < r_0 \ll 1$, the characteristic values $\lambda = -z/\mathrm{e}^2$ are concentrated in the sector $\lambda \in \mathcal{C}_\alpha(r, r_0) \cap \mathbb{R}$, for any $\alpha > 0$. Hence, by setting $\mathcal{N}\big(\mathcal{C}_\alpha(r, r_0) \cap \mathbb{R}\big) := \#\big\{\lambda \in \mathcal{Z}\big(I - \mathcal{T}_{eV}(-\lambda \mathrm{e}^2)\big) : \lambda \in \mathcal{C}_\alpha(r, r_0) \cap \mathbb{R}\big\}$, one has

$$\#\big\{z \in \sigma_{\mathrm{disc}}\big(H(b, eV)\big) : -r_0\mathrm{e}^2 \le z < -r\mathrm{e}^2\big\} = \mathcal{N}\big(\mathcal{C}_\alpha(r, r_0) \cap \mathbb{R}\big) + \mathcal{O}(1), \quad 0 < r_0 \ll 1. \tag{2.14}$$

For an interval $\Lambda \subset \mathbb{R}^*$, let

$$n(\Lambda) := \mathrm{Tr}\, \mathbf{1}_\Lambda\big(K_V(0)\big), \tag{2.15}$$

be the number of eigenvalues of the operator $K_V(0)$ lying in Λ and counted according to their multiplicity. In view of (2.13), we have $n\big([r, r_0]\big) \le n_+\left(r, \zeta^{-1} p\mathbf{W}(I)p\right)$, so that (ii) follows from (2.14) together with [1, Corollary 3.9] and (2.12).

(iii): Concerning (iii), if there exists an IDS for the operator H_- and if the function $\mathbf{W}(I)$ satisfies Assumption (A), then by [5, Lemma 3.3] we have $n_+\left(r, \zeta^{-1} p\mathbf{W}(I)p\right) = \widetilde{C}_m(\zeta r)^{-1/m}\left(1+o(1)\right)$, $r \searrow 0$, for some constant $\widetilde{C}_m > 0$. Otherwise, [1, Theorem 3.7] implies that for any $\nu > 0$ small enough, there exists $r(\nu) > 0$ such that for all $0 < r < r(\nu)$, we have

$$\mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) = n\big([r,1]\big)\big(1+\mathcal{O}\big(\nu|\ln\nu|^2\big)\big) + \mathcal{O}\big(|\ln\nu|^2\big)n\big([r(1-\nu), r(1+\nu)]\big) + \mathcal{O}_\nu(1), \tag{2.16}$$

where the $\mathcal{O}$'s are uniform with respect to r, ν but the $\mathcal{O}_\nu$ may depend on ν. Since we have $n\big([r,1]\big) \le n_+\left(r, \zeta^{-1}p\mathbf{W}(I)p\right)$, then if $\mathrm{Tr}\,\mathbf{1}_{[r(1+\nu),r(1-\nu)]}\big(p\mathbf{W}(H_+^{-1})p\big) = \mathrm{Tr}\,\mathbf{1}_{[r,1]}\big(p\mathbf{W}(H_+^{-1})p\big)\big(o(1)+O(\nu)\big)$, we deduce from (2.16) that

$$\begin{aligned}\mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) &\le n_+\left(r, \zeta^{-1}p\mathbf{W}(I)p\right)\left(1+\mathcal{O}\big(\nu|\ln\nu|^2\big)\right)\\ &\quad + \mathcal{O}\big(|\ln\nu|^2\big)n_+\left(r, \zeta^{-1}p\mathbf{W}(I)p\right)\big(o(1)+\mathcal{O}(\nu)\big) + \mathcal{O}_\nu(1).\end{aligned} \tag{2.17}$$

Since $\mathcal{N}\big(\mathcal{C}_\alpha(r,r_0)\cap\mathbb{R}\big) = \mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) + \mathcal{O}(1)$, then putting this together with (2.14) and (2.17), we get

$$\limsup_{r\searrow 0} \frac{\#\left\{z \in \sigma_{\mathrm{disc}}(H(b,eV)) : -r_0 e^2 \le z < -re^2\right\}}{n_+\left(r, \zeta^{-1}p\mathbf{W}(I)p\right)} \le 1 + \mathcal{O}\big(\nu|\ln\nu|^2\big) + \mathcal{O}\big(|\ln\nu|^2\big)\mathcal{O}(\nu).$$

Now, letting ν tend to 0, the claim (iii) follows immediately.

2.3. Sketch of the proof of Theorem 1.2

If the magnetic field is constant, then $(2b_0)^{-1}p_0 = H_+^{-1}p_0 \le H_+^{-1}$. This implies that

$$(2b_0)^{-1}\big(p_0Up_0\big)^*\big(p_0Up_0\big) \le p_0\mathbf{W}\big(H_+^{-1}\big)p_0. \tag{2.18}$$

(i): If the assumptions of item (i) are satisfied, then we have

$$n\big(\big[r(1-\nu), r(1+\nu)\big]\big) = n\big([r,1]\big)\big(o(1)+\mathcal{O}(\nu)\big),$$

$0 < \nu \ll 1$. Since $\phi(r) \longrightarrow \infty$, then it follows easily from (2.16) that

$$\mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) = n\big([r,1]\big)\big(1+o(1)\big) = \phi(r)\big(1+o(1)\big), \quad r \searrow 0. \tag{2.19}$$

Now, (2.14) together with the identities $\mathcal{N}\big(\mathcal{C}_\alpha(r,r_0)\cap\mathbb{R}\big) = \mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) + \mathcal{O}(1)$ and (2.19) give (i).

(ii): If the magnetic field is constant, remember that we have $\zeta = 2b_0$. Thus, if the function U satisfies $U \ge 0$, then (2.13) together with (2.18) imply that

$$n_+\Big((2rb_0)^{\frac{1}{2}}, p_0Up_0\Big) \le n_+\big(r, K_V(0)\big) \le n_+\big(2rb_0, p_0\mathbf{W}(I)p_0\big), \quad r > 0. \tag{2.20}$$

Recall that $\mathbf{W}(I) = |U|^2$ as function. Therefore, if $U \ge 0$ satisfies Assumption (A), then [5, Lemma 3.3] implies that the l.h.s. and the r.h.s. quantities of (2.20) have the same first asymptotic term as $r \searrow 0$. Namely as $r \searrow 0$, $n_+\big((2rb_0)^{\frac{1}{2}}, p_0Up_0\big) = C_m(2b_0)^{-1/m}r^{-1/m}\big(1+o(1)\big)$ and $n_+\big(2rb_0, p_0\mathbf{W}(I)p_0\big) = C_m(2b_0)^{-1/m}r^{-1/m}\big(1+o(1)\big)$, the constant $C_m > 0$ being defined by (1.14). This implies that

$$n_+\big(r, K_V(0)\big) = C_m(2b_0)^{-1/m}r^{-1/m}\big(1+o(1)\big), \quad r \searrow 0. \tag{2.21}$$

Then, (ii) follows from (2.14) together with [1, Corollary 3.11] and the identity $\mathcal{N}\big(\mathcal{C}_\alpha(r,r_0)\cap\mathbb{R}\big) = \mathcal{N}\big(\mathcal{C}_\alpha(r,1)\cap\mathbb{R}\big) + \mathcal{O}(1)$.

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