Mathematical analysis/Complex analysis

Semi-continuity properties of weighted log canonical thresholds of toric plurisubharmonic functions

Propriétés de semi-continuité des seuils log canoniques à poids de fonctions plurisousharmoniques toriques

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ABSTRACT

In this note, we prove a semi-continuity theorem for certain weighted log canonical thresholds of toric plurisubharmonic functions.
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RÉSUMÉ

Dans cette note, nous démontrons un théorème de semi-continuité pour certains seuils log canoniques à poids de fonctions plurisousharmoniques toriques.
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1. Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$ and $o \in \Omega$ the origin. Let $u \in PSH(\Omega)$ and let $\mu$ be a non-negative Radon measure in $\Omega$, where $PSH(\Omega)$ denotes the set of all plurisubharmonic functions defined in $\Omega$. The weighted log canonical threshold of $u$ at $o$ is defined to be

$$c_\mu(u) := \sup \{ c \geq 0 : e^{-2cu} \text{ is } L^1(\mu) \text{ on a neighborhood of } o \}.$$ 

In [10], Hiep obtained the following semi-continuity theorem.

Theorem 1.1. Assume that $c > 0$ and $\int_\Omega e^{-2cu} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and $o \in \Omega'$. Then for $v \in PSH(\Omega')$, there exists $\delta = \delta(c, u, \Omega') > 0$ such that $\|u - v\|_{L^1(\Omega')} < \delta$ implies $c_{dV_{2n}}(v) > c$. Moreover, as $v$ converges to $u$ in $L^1(\Omega')$, the function $e^{-2cv}$ converges to $e^{-2cu}$ in $L^1$ on every relatively compact open subset $\Omega'' \subset \Omega'$.

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Therefore, it is natural to raise the following question.

**Question 1.2.** Let \( u, u_j \) be plurisubharmonic functions in \( \Omega \) such that \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega) \). Assume that \( \mu \) is a non-negative Radon measure in \( \Omega \). What are the conditions on \( \mu \) ensuring that

\[
\liminf_{j \to +\infty} c_\mu(u_j) \geq c_\mu(u). \tag{11}
\]

For the case \( \mu = dV_{2n} \), Question 1.2 is solved by Theorem 1.1. Recently, Hiep [11] showed that (11) holds when \( \mu = \|z\|^2 dV_{2n}, t \in \mathbb{R} \). He also gave an example to show that (11) is not valid when \( n = 2 \) and \( \mu = |z|^2 dV_{2n} \).

The aim of this note is to study Question 1.2. We consider here log canonical thresholds of toric plurisubharmonic functions. A function \( u \) defined on \( \Omega \) is called a toric plurisubharmonic function if \( u \) is plurisubharmonic and \( u(z) \) depends only on \( |z_1|, \ldots, |z_n| \) for any \( z \in \Omega \). In this context, we prove the following.

**Theorem 1.3.** Let \( u, u_j \) be toric plurisubharmonic functions defined on \( \Omega \) such that \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega) \). Then,

\[
\liminf_{j \to +\infty} c_{|z|^{2n} dV_{2n}}(u_j) \geq c_{|z|^{2n} dV_{2n}}(u), \quad \forall t > -n.
\]

2. **Proof of Theorem 1.3**

Some elements of pluripotential theory that are used in the following are given by [1–19].

**Lemma 2.1.** Let \( m, n \) be integer numbers with \( 0 \leq n < m \) and let \( \theta_j \in \mathbb{C}, j = 1, \ldots, n + m \) be such that \( |\theta_j| = 1 \) and \( \theta_j \neq \theta_k \), \( \forall j \neq k \).

Assume that the sequence \( \{a_\alpha \}_{\alpha \in \mathfrak{N}} \subset \mathbb{C} \) satisfies \( |a_\alpha|(2r)^{|\alpha|} \leq C, \forall \alpha \in \mathfrak{N}^n \), for some constants \( r > 0 \) and \( C \geq 0 \) and define

\[
f(z) := \sum_{\alpha \in \mathfrak{N}^n} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \text{for } z \in \Delta^n(0, r).
\]

Then, there exist constants \( A, B > 0 \) such that

\[
\sum_{\alpha \in \mathfrak{N}^n, |\alpha| \leq m} |a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}| \leq A \sum_{j_1, \ldots, j_n=1}^{n+m} |f(\theta_{j_1} z_1, \ldots, \theta_{j_n} z_n)| + B \|z\|^{n+1}, \quad \forall z \in \Delta^n(0, r),
\]

where \( \Delta^n(0, r) \) denotes the polydisc of center 0 and radius \( r \).

**Proof.** We will prove the lemma by induction on \( n \). When \( n = 0 \), the statement is obvious. Let \( n \) be an integer number with \( n \geq 1 \). Assume that the lemma holds for \( n - 1 \). Let \( j \in \mathbb{N} \) and define

\[
p_j(z') := \sum_{\alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathfrak{N}^{n-1}, |\alpha'| \leq m-j} a_{(\alpha', j)} z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}}
\]

where \( z' = (z_1, \ldots, z_{n-1}) \in \Delta^{n-1}(0, r) \). By the hypotheses we have

\[
\det \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\theta_1 & \theta_2 & \cdots & \theta_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_1^{m+1} & \theta_2^{m+1} & \cdots & \theta_{n+m}^{m+1}
\end{bmatrix} = \prod_{1 \leq j < k \leq n+m} (\theta_k - \theta_j) \neq 0.
\]

Hence, there exists a constant \( A_n > 0 \) such that

\[
\sum_{j=0}^{n+m-1} |x_j| \leq A_n \sum_{j=1}^{n+m} |x_j| + \sum_{k=0}^{n+m-1} \theta_j^k x_k
\]

for any \( x = (x_0, \ldots, x_{n+m-1}) \in \mathbb{C}^{n+m} \). Let \( z = (z_j, z_n) \in \Delta^n(0, r) \). Applying the inequality (2.1) with \( x_j := p_j(z') z_n^j, j = 0, 1, \ldots, m \) and \( y_j = 0, j = m + 1, \ldots, n + m - 1 \), we get

\[
\sum_{j=0}^{m} |p_j(z') z_n^j| \leq A_n \sum_{j=1}^{n+m} |p_j(z') z_n^j| + \sum_{k=0}^{m} b_j \theta_j^{k} x_k.
\]

Since the lemma is assumed to hold for \( n - 1 \), there exist constants \( A', B' > 0, j_n = 0, \ldots, m \) such that
for any \( z' \in \Delta^{n-1}(0, r) \). Combining this with (2.2), we infer
\[
\sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} |a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}| = \sum_{j_n=0}^{m} \sum_{|\alpha'| \leq m-j_n} |a_{(\alpha', j_n)} z_1^{\alpha'_1} \cdots z_n^{\alpha'_n}| \\
\leq A' \sum_{j_1, \ldots, j_{n-1}=1}^{m} \sum_{j_n=0}^{m} |p_j(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})| + B' \|z\|^{m+1} \\
\leq A' A_n \sum_{j_1, \ldots, j_{n-1}, j_n=1}^{m} \sum_{k=0}^{m} |p_k(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})| z_n^{k} |z_n|^j + B' \|z\|^{m+1},
\]
for any \( z \in \Delta^n(0, r) \). We set
\[
p(z) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \text{ for } z \in \Delta^n(0, r).
\]
Then, we can find a constant \( B'' > 0 \) such that
\[
|p(z)| \leq \|f(z)\| + B'' \|z\|^{m+1}, \forall z \in \Delta^n(0, r).
\]
Moreover, since \( p(z) = \sum_{k=0}^{m} p_k(z) z_1^k \), by (2.3) we get
\[
\sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} |a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}| \leq A' A_n \sum_{j_1, \ldots, j_{n-1}=1}^{m} |p(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})| + B' \|z\|^{m+1} \\
\leq A' A_n \sum_{j_1, \ldots, j_{n-1}=1}^{m} \left[ |f(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})| + B'' \|f(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})\|^{m+1} \right] + B' \|z\|^{m+1} \\
\leq A \sum_{j_1, \ldots, j_{n-1}=1}^{m} |f(\theta_{j_1} z_1, \ldots, \theta_{j_{n-1}} z_{n-1})| + B' \|z\|^{m+1}.
\]
for all \( z \in \Delta^n(0, r) \). The proof is complete. \( \square \)

**Lemma 2.2.** Let \( k \) be an integer number with \( 1 \leq k \leq n \) and let \( u, u_j \in PSH^-(\Delta^n(0, 3r)) \) such that \( u_j \to u \) in \( L^1_{loc}(\Delta^n(0, 3r)) \) and
\[
\int_{\Delta^n(0, 3r)} |z_k|^{2t} e^{u} dV_{2n} < +\infty
\]
for some \( t \in \mathbb{N} \). Then, for every \( \varepsilon > 0 \), there exist a positive integer number \( j_0 \) and a complex number \( \lambda \in \mathbb{C} \setminus \{0\} \) such that, for each integer number \( j \geq j_0 \), we can find holomorphic functions \( F_{j,k} \) and \( G_{j,k} \) defined on \( \Delta^n(0, r) \) satisfying
\begin{align*}
(i) \quad & \int_{\Delta^n(0, r)} |F_{j,k}|^2 e^{-u} dV_{2n} < +\infty; \\
(ii) \quad & F_{j,k}(z) = z_k^l + (z_k - \lambda) G_{j,k}(z); \\
(iii) \quad & G_{j,k}(z) = \sum_{\alpha \in \mathbb{N}^n} a_{j,k,\alpha} z^\alpha \text{ with } |\alpha a_{j,k,\alpha}| \leq \varepsilon r^{-|\alpha|}, \forall \alpha \in \mathbb{N}^n.
\end{align*}

**Proof.** The proof is almost the same as the one given in [10]. For the convenience of the reader, we sketch the proof of the lemma. Without loss of generality, we can assume that \( k = n \). By Fubini’s theorem, we have:
\[
\int_{\Delta^n(0, 3r)} \left[ \int_{\Delta^{n-1}(0, 3r)} e^{-u(z',z_n)} dV_{2n-2}(z') \right] |z_n|^2 dV_2(z_n)
\]
\[
\leq \int_{\Delta^{n+1}(0, 3r)} |z|^{2r} e^{-u} dV_{2n} < +\infty.
\]

Let \( \delta > 0 \). Since \( u_j \to u \) in \( L^1_{\text{loc}}(\Delta^n(0, 3r)) \), we can find \( \lambda \in \Delta(0, \frac{\varepsilon^2 - 4\delta}{2r^2 + 4\varepsilon^2 - 4r}) \setminus \{0\} \) such that \( u_j(\bullet, \lambda) \to u(\bullet, \lambda) \) in \( L^1_{\text{loc}}(\Delta^n(0, 3r)) \) and
\[
\int_{\Delta^{n+1}(0, 3r)} e^{-u(z', \lambda)} dV_{2n-2}(z') < \frac{\varepsilon^2 \delta}{|\lambda|^{2r+2}}.
\]

Theorem 1.1 implies that there exists a positive integer number \( j_0 \) such that
\[
\int_{\Delta^{n+1}(0, 2r)} e^{-u_j(z', \lambda)} dV_{2n-2}(z') \leq \frac{\varepsilon^2 \delta}{|\lambda|^{2r+2}}, \quad \forall j \geq j_0.
\]

By the \( L^2 \)-extension theorem of Ohsawa and Takegoshi, we can find holomorphic functions \( F_{j,n} \) on \( \Delta^n(0, 2r) \) such that \( F_{j,n}(\bullet, \lambda) = \lambda^j \) in \( \Delta^{n+1}(0, 2r) \) and
\[
\int_{\Delta^n(0, 2r)} |F_{j,n}|^2 e^{-u_j} dV_{2n} \leq A \int_{\Delta^{n+1}(0, 2r)} |\lambda|^{2r} e^{-u_j(z', \lambda)} dV_{2n-2}(z')
\]
\[
\leq \frac{A \varepsilon^2 \delta}{|\lambda|^{2r}} < +\infty.
\]

(2.4)

where \( A \) is a positive constant which only depends on \( r \) and \( n \). Let \( a \in \Delta^n(0, r) \). Since \( |F_{j,n}|^2 \) are plurisubharmonic functions in \( \Delta^n(0, 2r) \), from (2.4), we get
\[
|F_{j,n}(a)|^2 \leq \frac{1}{\pi^n (2r - |a_1|)^2 \ldots (2r - |a_n|)^2} \int_{\Delta^n(0, 2r)} |F_{j,n}|^2 dV_{2n}
\]
\[
\leq \frac{1}{\pi^n r^{2n}} \int_{\Delta^n(0, 2r)} |F_{j,n}|^2 e^{-u_j} dV_{2n}
\]
\[
\leq \frac{A \varepsilon^2 \delta}{\pi^n r^{2n} |\lambda|^2}.
\]

(2.5)

It follows that
\[
\|F_{j,n}\|_{\Delta^n(0, r)}^2 \leq \frac{A \varepsilon^2 \delta}{\pi^n r^{2n} |\lambda|^2}.
\]

(2.6)

Since \( F_{j,n}(z', \lambda) - \lambda^j = 0 \) for all \( z' \in \Delta^n(0, 2r) \), there exist holomorphic functions \( G_{j,n} \) on \( \Delta^n(0, 2r) \) such that
\[
F_{j,n}(z) = z^n + (z_n - \lambda)G_{j,n}(z), \quad \forall z \in \Delta^n(0, 2r).
\]

Now, by the maximum principle for plurisubharmonic functions, we infer that
\[
\|\lambda G_{j,n}\|_{\Delta^n(0, r)} = \|\lambda G_{j,n}\|_{\Delta^{n-1}(0, r) \times \partial \Delta(0, r)}
\]
\[
\leq \frac{|\lambda|}{r - |\lambda|} \|F_{j,n} - z^n\|_{\Delta^{n-1}(0, r) \times \partial \Delta(0, r)}
\]
\[
\leq \frac{|\lambda|}{r - |\lambda|} \|F_{j,n}\|_{\Delta^n(0, r)} + \frac{|\lambda|^2}{r - |\lambda|}.
\]

(2.6)

Since \( \lambda \in \Delta(0, \frac{\varepsilon^2 - 4\delta}{2r^2 + 4\varepsilon^2 - 4r}) \setminus \{0\} \), we have \( \frac{|\lambda|}{r - |\lambda|} \leq \frac{\varepsilon^2 - 4\delta}{2r^2 + 4\varepsilon^2 - 4r} \) and \( \frac{1}{r - |\lambda|} \leq \frac{2}{r} \). Combining this with (2.5) and (2.6) we get
\[
\|\lambda G_{j,n}\|_{\Delta^n(0, r)} \leq \frac{A^{1/2} \delta^{1/2} \varepsilon^{1/2}}{\pi^{n/2} r^n (r - |\lambda|)} + \frac{|\lambda|^2}{r - |\lambda|} \leq \frac{2A^{1/2} \delta^{1/2} \varepsilon^{1/2}}{\pi^{n/2} r^{n+1}} + \frac{\varepsilon}{2}.
\]

If we choose \( \delta \) such that \( \frac{4A^{1/2} \delta^{1/2}}{\pi^{n/2} r^{n+1}} < 1 \) then
\[
\|\lambda G_{j,n}\|_{\Delta^n(0, r)} \leq \varepsilon .
\]

Now, since \( G_{j,n} \) is a holomorphic function in \( \Delta^n(0, r) \), we can write
Finally, the Cauchy integral formula gives

$$|\lambda a_{j,n,a}| \leq \frac{||\lambda G_{j,n}||_{\Delta^n(0,r)}}{r^{|\alpha|}}, \forall \alpha \in \mathbb{N}^n.$$  

The proof is complete. \(\square\)

We are now able to give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Without loss of generality, we can assume that \(u, u_j \in PSH^-(\Omega)\). Since \(\log ||z||^2\) is a plurisubharmonic function in \(\mathbb{C}^n\), Theorem 1.1 easily shows that we only need to prove Theorem 1.3 in the case \(t \in \mathbb{N}^*\). Let \(c < c_{|z|^2}dv_{2n}(u)\) and let \(r > 0\) such that \(\Delta^n(0, 3r) \subset \Omega\) and

$$\int_{\Delta^n(0,3r)} ||z||^2 e^{-2cu} dv_{2n} < +\infty.$$  

It remains to prove that there exists \(j_0 \in \mathbb{N}^*\) such that for every \(j \geq j_0\), we can find \(\delta_j \subset (0, r)\) with

$$\int_{\Delta^n(0,\delta_j)} ||z||^2 e^{-2cu} dv_{2n} < +\infty.$$  

Let \(k\) be an integer number with \(1 \leq k \leq n\). By Lemma 2.2, there exist a positive integer number \(j_k\) and a complex number \(\lambda_k \in \mathbb{C}\setminus\{0\}\) such that, for every \(j \geq j_k\), we can find the holomorphic functions \(F_{j,k}\) and \(G_{j,k}\) defined on \(\Delta^n(0, r)\) that satisfy:

(i) \(\int_{\Delta^n(0,r)} |F_{j,k}|^2 e^{-2c_{j_k}u} dv_{2n} < +\infty;\)
(ii) \(F_{j,k}(z) = z_k^r + (z_k - \lambda_k)G_{j,k}(z);\)
(iii) \(G_{j,k}(z) = \sum_{\alpha \in \mathbb{N}^n} a_{j,k,\alpha} z^\alpha\) with \(2|\lambda_k a_{j,k,\alpha} r^{\alpha} - 1|^2 \leq 1, \forall \alpha \in \mathbb{N}^n.\)

We now write

$$F_{j,k}(z) = \sum_{\alpha \in \mathbb{N}^n} b_{j,k,\alpha} z^\alpha, \ z \in \Delta^n(0, r).$$

Let \(\beta \in \mathbb{N}^n\) such that \(\beta_k = t\) and \(\beta_l = 0, \forall l \neq k\). Then, by (ii) we have

$$b_{j,k,\alpha} = \begin{cases} -\lambda_k a_{j,k,\alpha} & \text{if } \alpha_k = 0, \alpha \neq \beta, \\ 1 - \lambda_k a_{j,k,\alpha} & \text{if } \alpha_k = 0, \alpha = \beta, \\ -\lambda_k a_{j,k,\alpha} + a_{j,k,\alpha_1,\ldots,\alpha_k-1,\alpha_{k+1},\ldots,\alpha_n} & \text{if } \alpha_k > 0, \alpha \neq \beta, \\ 1 - \lambda_k a_{j,k,\alpha} + a_{j,k,\alpha_1,\ldots,\alpha_k-1,\alpha_{k+1},\ldots,\alpha_n} & \text{if } \alpha_k > 0, \alpha = \beta. \end{cases} \tag{2.7}$$

First we claim that there exists \(\theta > 0\) such that

$$\theta |z_k|^t \leq \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq t} |b_{j,k,\alpha} z^\alpha|, \forall z \in \Delta^n(0, r). \tag{2.8}$$

Indeed, let \(\{\gamma^s\} \subset \mathbb{N}^n\) be such that \(\gamma_k^s = s\) and \(\gamma_l^s = 0, \forall l \neq k\). Put

$$s_0 := \begin{cases} \inf\{s \in \mathbb{N} : a_{j,k,\gamma^s} \neq 0\} & \text{if } a_{j,k,\gamma^{s-1}} \neq 0, \\ t & \text{if } a_{j,k,\gamma^{s-1}} = 0. \end{cases}$$

Then, \(0 \leq s_0 \leq t\) and \(a_{j,k,\gamma^0} \neq 0\) if \(s_0 < t\). From (2.7), we have

$$b_{j,k,\gamma^0} = \begin{cases} 1 - \lambda_k a_{j,k,\beta} & \text{if } s_0 = t, \\ -\lambda_k a_{j,k,\gamma^0} & \text{if } 0 \leq s_0 < t. \end{cases}$$

Therefore, by (iii) we get \(\theta := |b_{j,k,\gamma^0} r^{t-\gamma^0} > 0\), and hence

$$\theta |z_k|^t \leq \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq t} |b_{j,k,\alpha} z^\alpha|, \forall z \in \Delta^n(0, r).$$
This proves the claim. Now, by (iii) and (2.7), we get
\[ |b_{j,k,\alpha}|r^{r|\alpha|} \leq \frac{r^{t} + r^{t+1}}{2|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n, |\alpha| > t + 1. \]
Hence
\[ |b_{j,k,\alpha}|r^{r|\alpha|} \leq C_{j,k}, \quad \forall \alpha \in \mathbb{N}^n, \]
for some constant $C_{j,k}$. Let $\theta_i \in \mathbb{C}$, $s = 1, \ldots, n + t$ such that $|\theta_s| = 1$ and $\theta_s \neq \theta_h$, $\forall s \neq h$. Lemma 2.1 implies that there exist constants $A_{j,k}, B_{j,k} > 0$ such that
\[ \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq t} |b_{j,k,\alpha}|z^\alpha \leq A_{j,k} \sum_{s_1, \ldots, s_n=1}^{n+t} |F_{j,k}(\theta_s z_1, \ldots, \theta_s z_n)| + B_{j,k}z^{t+1}, \quad \forall z \in \Delta^n(0, \frac{r}{2}). \]

Combining this with (2.8), we infer that
\[ 2\|z\|^{2t} \leq A \sum_{k=1}^{n} \sum_{s_1, \ldots, s_n=1}^{n+t} |F_{j,k}(\theta_s z_1, \ldots, \theta_z z_n)|^2 + B\|z\|^{2t+1}, \quad \forall z \in \Delta^n(0, \frac{r}{2}), \forall j \geq j_0 := \max_{1 \leq k \leq n} j_k. \]
where $A, B$ are positive constants. Choose $\delta_j > 0$ such that $\delta_j(\frac{1}{2} + B) < 1$. Then
\[ \|z\|^{2t} \leq A \sum_{k=1}^{n} \sum_{s_1, \ldots, s_n=1}^{n+t} |F_{j,k}(\theta_s z_1, \ldots, \theta_z z_n)|^2, \quad \forall z \in \Delta^n(0, \delta_j), \forall j \geq j_0. \] (2.9)

Now, since the $u_j$’s are toric plurisubharmonic functions, we conclude by (i) that
\[ \int_{\Delta^n(0, \delta_j)} |F_{j,k}(\theta_s z_1, \ldots, \theta_z z_n)|^2 e^{-2cu_j(z)} dV_{2n}(z) < +\infty \]
for all $s_1, \ldots, s_n = 1, \ldots, n + t$. Hence, by (2.9) this implies that
\[ \int_{\Delta^n(0, \delta_j)} \|z\|^{2t} e^{-2cu_j(z)} dV_{2n}(z) < +\infty, \quad \forall j \geq j_0. \]

The proof is complete. \(\square\)

References