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Semi-continuity properties of weighted log canonical thresholds of toric plurisubharmonic functions ☆

Propriétés de semi-continuité des seuils log canoniques à poids de fonctions plurisousharmoniques toriques

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ABSTRACT

In this note, we prove a semi-continuity theorem for certain weighted log canonical thresholds of toric plurisubharmonic functions.

RÉSUMÉ

Dans cette note, nous démontrons un théorème de semi-continuité pour certains seuils log canoniques à poids de fonctions plurisousharmoniques toriques.

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1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $o \in \Omega$ the origin. Let $u \in PSH(\Omega)$ and let μ be a non-negative Radon measure in Ω, where $PSH(\Omega)$ denotes the set of all plurisubharmonic functions defined in Ω. The weighted log canonical threshold of u at o is defined to be

$$c_\mu(u) := \sup\{c \geq 0 : \mathrm{e}^{-2cu} \text{ is } L^1(\mu) \text{ on a neighborhood of } o\}.$$

In [10], Hiep obtained the following semi-continuity theorem.

Theorem 1.1. *Assume that $c > 0$ and $\int_{\Omega'} \mathrm{e}^{-2cu} \mathrm{d}V_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and $o \in \Omega'$. Then for $v \in PSH(\Omega')$, there exists $\delta = \delta(c, u, \Omega') > 0$ such that $\|u - v\|_{L^1(\Omega')} < \delta$ implies $c_{\mathrm{d}V_{2n}}(v) > c$. Moreover, as v converges to u in $L^1(\Omega')$, the function e^{-2cv} converges to e^{-2cu} in L^1 on every relatively compact open subset $\Omega'' \Subset \Omega'$.*

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Therefore, it is natural to raise the following question.

Question 1.2. *Let u, u_j be plurisubharmonic functions in Ω such that $u_j \to u$ in $L^1_{loc}(\Omega)$. Assume that μ is a non-negative Radon measure in Ω. What are the conditions on μ ensuring that*

$$\liminf_{j\to+\infty} c_\mu(u_j) \geq c_\mu(u). \tag{1.1}$$

For the case $\mu = dV_{2n}$, Question 1.2 is solved by Theorem 1.1. Recently, Hiep [11] showed that (1.1) holds when $\mu = \|z\|^{2t} dV_{2n}$, $t \in (-n, 1]$. He also gave an example to show that (1.1) is not valid when $n = 2$ and $\mu = |z_1|^2 dV_{2n}$.

The aim of this note is to study Question 1.2. We consider here log canonical thresholds of toric plurisubharmonic functions. A function u defined on Ω is called a toric plurisubharmonic function if u is plurisubharmonic and $u(z)$ depends only on $|z_1|, \ldots, |z_n|$ for any $z \in \Omega$. In this context, we prove the following.

Theorem 1.3. *Let u, u_j be toric plurisubharmonic functions defined on Ω such that $u_j \to u$ in $L^1_{loc}(\Omega)$. Then,*

$$\liminf_{j\to+\infty} c_{\|z\|^{2t} dV_{2n}}(u_j) \geq c_{\|z\|^{2t} dV_{2n}}(u), \ \forall t > -n.$$

2. Proof of Theorem 1.3

Some elements of pluripotential theory that are used in the following are given by [1–19].

Lemma 2.1. *Let m, n be integer numbers with $0 \leq n < m$ and let $\theta_j \in \mathbb{C}$, $j = 1, \ldots, n+m$ be such that $|\theta_j| = 1$ and $\theta_j \neq \theta_k$, $\forall j \neq k$. Assume that the sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}^n} \subset \mathbb{C}$ satisfies $|a_\alpha|(2r)^{|\alpha|} \leq C$, $\forall \alpha \in \mathbb{N}^n$, for some constants $r > 0$ and $C \geq 0$ and define*

$$f(z) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \ \text{for } z \in \Delta^n(0, r).$$

Then, there exist constants $A, B > 0$ such that

$$\sum_{\alpha\in\mathbb{N}^n, |\alpha|\leq m} |a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}| \leq A \sum_{j_1,\ldots,j_n=1}^{n+m} |f(\theta_{j_1} z_1, \ldots, \theta_{j_n} z_n)| + B\|z\|^{m+1}, \ \forall z \in \Delta^n(0, r),$$

where $\Delta^n(0, r)$ denotes the polydisc of center 0 and radius r.

Proof. We will prove the lemma by induction on n. When $n = 0$, the statement is obvious. Let n be an integer number with $n \geq 1$. Assume that the lemma holds for $n - 1$. Let $j \in \mathbb{N}$ and define

$$p_j(z') := \sum_{\alpha'=(\alpha_1,\ldots,\alpha_{n-1})\in\mathbb{N}^{n-1}, |\alpha'|\leq m-j} a_{(\alpha',j)} z_1^{\alpha_1} \ldots z_{n-1}^{\alpha_{n-1}}$$

where $z' = (z_1, \ldots, z_{n-1}) \in \Delta^{n-1}(0, r)$. By the hypotheses we have

$$det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_1 & \theta_2 & \ldots & \theta_{n+m} \\ \ldots & \ldots & \ldots & \ldots \\ \theta_1^{n+m-1} & \theta_2^{n+m-1} & \ldots & \theta_{n+m}^{n+m-1} \end{bmatrix} = \prod_{1\leq j<k\leq n+m} (\theta_k - \theta_j) \neq 0.$$

Hence, there exists a constant $A_n > 0$ such that

$$\sum_{j=0}^{n+m-1} |x_j| \leq A_n \sum_{j=1}^{n+m} |\sum_{k=0}^{n+m-1} \theta_j^k x_k| \tag{2.1}$$

for any $x = (x_0, \ldots, x_{n+m-1}) \in \mathbb{C}^{n+m}$. Let $z = (z', z_n) \in \Delta^n(0, r)$. Applying the inequality (2.1) with $x_j := p_j(z') z_n^j$, $j = 0, 1, \ldots, m$ and $x_j = 0$, $j = m+1, \ldots, n+m-1$, we get

$$\sum_{j=0}^{m} |p_j(z') z_n^j| \leq A_n \sum_{j=1}^{n+m} |\sum_{k=0}^{m} p_k(z') z_n^k \theta_j^k|. \tag{2.2}$$

Since the lemma is assumed to hold for $n - 1$, there exist constants $A'_{j_n}, B'_{j_n} > 0$, $j_n = 0, \ldots, m$ such that

$$\sum_{\alpha'\in\mathbb{N}^{n-1},|\alpha'|\le m-j_n} |a_{(\alpha',j_n)}z_1^{\alpha_1}\dots z_{n-1}^{\alpha_{n-1}}| \le A'_{j_n}\sum_{j_1,\dots,j_{n-1}=1}^{n+m-j}|p_{j_n}(\theta_{j_1}z_1,\dots,\theta_{j_{n-1}}z_{n-1})| + B'_{j_n}\|z'\|^{m-j_n+1}$$
$$\le A'_{j_n}\sum_{j_1,\dots,j_{n-1}=1}^{n+m}|p_{j_n}(\theta_{j_1}z_1,\dots,\theta_{j_{n-1}}z_{n-1})| + B'_{j_n}\|z'\|^{m-j_n+1}$$

for any $z'\in\Delta^{n-1}(0,r)$. Combining this with (2.2), we infer

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^n,|\alpha|\le m}|a_\alpha z_1^{\alpha_1}\dots z_n^{\alpha_n}| &= \sum_{j_n=0}^{m}\sum_{\alpha'\in\mathbb{N}^{n-1},|\alpha'|\le m-j_n}|a_{(\alpha',j_n)}z_1^{\alpha_1}\dots z_{n-1}^{\alpha_{n-1}}z_n^{j_n}|\\ &\le A'\sum_{j_1,\dots,j_{n-1}=1}^{n+m}\sum_{j_n=0}^{m}|p_j(\theta_{j_1}z_1,\dots,\theta_{j_{n-1}}z_{n-1})z_n^{j_n}| + B'\|z\|^{m+1}\\ &\le A'A_n\sum_{j_1,\dots,j_{n-1},j_n=1}^{n+m}|\sum_{k=0}^{m}p_k(\theta_{j_1}z_1,\dots,\theta_{j_{n-1}}z_{n-1})z_n^k\theta_{j_n}^k| + B'\|z\|^{m+1},\end{aligned}\tag{2.3}$$

for any $z\in\Delta^n(0,r)$. We set

$$p(z) := \sum_{\alpha\in\mathbb{N}^n,|\alpha|\le m} a_\alpha z_1^{\alpha_1}\dots z_n^{\alpha_n},\ \text{for } z\in\Delta^n(0,r).$$

Then, we can find a constant $B''>0$ such that

$$|p(z)|\le|f(z)|+B''\|z\|^{m+1},\ \forall z\in\Delta^n(0,r).$$

Moreover, since $p(z)=\sum_{k=0}^m p_k(z')z_n^k$, by (2.3) we get

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^n,|\alpha|\le m}|a_\alpha z_1^{\alpha_1}\dots z_n^{\alpha_n}| &\le A'A_n\sum_{j_1,\dots,j_n=1}^{n+m}|p(\theta_{j_1}z_1,\dots,\theta_{j_n}z_n)| + B'\|z\|^{m+1}\\ &\le A'A_n\sum_{j_1,\dots,j_n=1}^{n+m}\Big[|f(\theta_{j_1}z_1,\dots,\theta_{j_n}z_n)| + B''\|(\theta_{j_1}z_1,\dots,\theta_{j_n}z_n)\|^{m+1}\Big] + B'\|z\|^{m+1}\\ &\le A\sum_{j_1,\dots,j_n=1}^{n+m}|f(\theta_{j_1}z_1,\dots,\theta_{j_n}z_n)| + B\|z\|^{m+1},\end{aligned}$$

for all $z\in\Delta^n(0,r)$. The proof is complete. □

Lemma 2.2. *Let k be an integer number with $1\le k\le n$ and let $u,u_j\in PSH^-(\Delta^n(0,3r))$ such that $u_j\to u$ in $L^1_{\text{loc}}(\Delta^n(0,3r))$ and*

$$\int_{\Delta^n(0,3r)}|z_k|^{2t}e^{-u}dV_{2n}<+\infty$$

for some $t\in\mathbb{N}$. Then, for every $\varepsilon>0$, there exist a positive integer number j_0 and a complex number $\lambda\in\mathbb{C}\setminus\{0\}$ such that, for each integer number $j\ge j_0$, we can find holomorphic functions $F_{j,k}$ and $G_{j,k}$ defined on $\Delta^n(0,r)$ satisfying

(i) $\int_{\Delta^n(0,r)}|F_{j,k}|^2e^{-u_j}dV_{2n}<+\infty$;
(ii) $F_{j,k}(z)=z_k^t+(z_k-\lambda)G_{j,k}(z)$;
(iii) $G_{j,k}(z)=\sum_{\alpha\in\mathbb{N}^n}a_{j,k,\alpha}z^\alpha$ *with* $|\lambda a_{j,k,\alpha}|\le\varepsilon r^{-|\alpha|}$, $\forall\alpha\in\mathbb{N}^n$.

Proof. The proof is almost the same as the one given in [10]. For the convenience of the reader, we sketch the proof of the lemma. Without loss of generality, we can assume that $k=n$. By Fubini's theorem, we have:

$$\int_{\Delta(0,3r)}\left[\int_{\Delta^{n-1}(0,3r)}e^{-u(z',z_n)}dV_{2n-2}(z')\right]|z_n|^{2t}dV_2(z_n)$$

$$\leq \int_{\Delta^n(0,3r)} |z_n|^{2t} e^{-u} dV_{2n} < +\infty.$$

Let $\delta > 0$. Since $u_j \to u$ in $L^1_{loc}(\Delta^n(0,3r))$, we can find $\lambda \in \Delta(0, \frac{\varepsilon r^{-t+1}}{2+4\varepsilon r^{-t}})\backslash\{0\}$ such that $u_j(\bullet,\lambda) \to u(\bullet,\lambda)$ in $L^1_{loc}(\Delta^{n-1}(0,3r))$ and

$$\int_{\Delta^{n-1}(0,3r)} e^{-u(z',\lambda)} dV_{2n-2}(z') < \frac{\varepsilon^2\delta}{|\lambda|^{2t+2}}.$$

Theorem 1.1 implies that there exists a positive integer number j_0 such that

$$\int_{\Delta^{n-1}(0,2r)} e^{-u_j(z',\lambda)} dV_{2n-2}(z') \leq \frac{\varepsilon^2\delta}{|\lambda|^{2t+2}}, \ \forall j \geq j_0.$$

By the L^2-extension theorem of Ohsawa and Takegoshi, we can find holomorphic functions $F_{j,n}$ on $\Delta^n(0,2r)$ such that $F_{j,n}(\bullet,\lambda) = \lambda^t$ in $\Delta^{n-1}(0,2r)$ and

$$\begin{aligned}\int_{\Delta^n(0,2r)} |F_{j,n}|^2 e^{-u_j} dV_{2n} &\leq A \int_{\Delta^{n-1}(0,2r)} |\lambda|^{2t} e^{-u_j(z',\lambda)} dV_{2n-2}(z') \\ &\leq \frac{A\varepsilon^2\delta}{|\lambda|^2} < +\infty,\end{aligned} \tag{2.4}$$

where A is a positive constant which only depends on r and n. Let $a \in \Delta^n(0,r)$. Since $|F_{j,n}|^2$ are plurisubharmonic functions in $\Delta^n(0,2r)$, from (2.4), we get

$$\begin{aligned}|F_{j,n}(a)|^2 &\leq \frac{1}{\pi^n(2r-|a_1|)^2\ldots(2r-a_n)^2} \int_{\Delta^n(0,2r)} |F_{j,n}|^2 dV_{2n} \\ &\leq \frac{1}{\pi^n r^{2n}} \int_{\Delta^n(0,2r)} |F_{j,n}|^2 e^{-u_j} dV_{2n} \\ &\leq \frac{A\varepsilon^2\delta}{\pi^n r^{2n}|\lambda|^2}.\end{aligned}$$

It follows that

$$\|F_{j,n}\|^2_{\Delta^n(0,r)} \leq \frac{A\varepsilon^2\delta}{\pi^n r^{2n}|\lambda|^2}. \tag{2.5}$$

Since $F_{j,n}(z',\lambda) - \lambda^t = 0$ for all $z' \in \Delta^{n-1}(0,2r)$, there exist holomorphic functions $G_{j,n}$ on $\Delta^n(0,2r)$ such that

$$F_{j,n}(z) = z_n^t + (z_n - \lambda)G_{j,n}(z), \ \forall z \in \Delta^n(0,2r).$$

Now, by the maximum principle for plurisubharmonic functions, we infer that

$$\begin{aligned}\|\lambda G_{j,n}\|_{\Delta^n(0,r)} &= \|\lambda G_{j,n}\|_{\Delta^{n-1}(0,r)\times\partial\Delta(0,r)} \\ &\leq \frac{|\lambda|}{r-|\lambda|}\|F_{j,n} - z_n^t\|_{\Delta^{n-1}(0,r)\times\partial\Delta(0,r)} \\ &\leq \frac{|\lambda|}{r-|\lambda|}\|F_{j,n}\|_{\Delta^n(0,r)} + \frac{|\lambda| r^t}{r-|\lambda|}.\end{aligned} \tag{2.6}$$

Since $\lambda \in \Delta(0, \frac{\varepsilon r^{-t+1}}{2+4\varepsilon r^{-t}})\backslash\{0\}$, we have $\frac{|\lambda|}{r-|\lambda|} \leq \frac{\varepsilon r^{-t}}{2}$ and $\frac{1}{r-|\lambda|} \leq \frac{2}{r}$. Combining this with (2.5) and (2.6) we get

$$\|\lambda G_{j,n}\|_{\Delta^n(0,r)} \leq \frac{A^{1/2}\delta^{1/2}\varepsilon}{\pi^{n/2}r^n(r-|\lambda|)} + \frac{|\lambda| r^t}{r-|\lambda|} \leq \frac{2A^{1/2}\delta^{1/2}\varepsilon}{\pi^{n/2}r^{n+1}} + \frac{\varepsilon}{2}.$$

If we choose δ such that $\frac{4A^{1/2}\delta^{1/2}}{\pi^{n/2}r^{n+1}} < 1$ then

$$\|\lambda G_{j,n}\|_{\Delta^n(0,r)} \leq \varepsilon.$$

Now, since $G_{j,n}$ is a holomorphic function in $\Delta^n(0,r)$, we can write

$$G_{j,n}(z)=\sum_{\alpha\in\mathbb{N}^n} a_{j,n,\alpha}z^\alpha,\ z\in\Delta^n(0,r).$$

Finally, the Cauchy integral formula gives

$$|\lambda a_{j,n,\alpha}|\le\frac{\|\lambda G_{j,n}\|_{\Delta^n(0,r)}}{r^{|\alpha|}}\le\frac{\varepsilon}{r^{|\alpha|}},\ \forall\alpha\in\mathbb{N}^n.$$

The proof is complete. □

We are now able to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality, we can assume that $u, u_j\in PSH^-(\Omega)$. Since $\log\|z\|^2$ is a plurisubharmonic function in $\mathbb{C}^n$, Theorem 1.1 easily shows that we only need to prove Theorem 1.3 in the case $t\in\mathbb{N}^*$. Let $c<c_{\|z\|^{2t}\mathrm{d}V_{2n}}(u)$ and let $r>0$ such that $\Delta^n(0,3r)\subset\Omega$ and

$$\int\limits_{\Delta^n(0,3r)}\|z\|^{2t}\mathrm{e}^{-2cu}\mathrm{d}V_{2n}<+\infty.$$

It remains to prove that there exists $j_0\in\mathbb{N}^*$ such that for every $j\ge j_0$, we can find $\delta_j\in(0,r)$ with

$$\int\limits_{\Delta^n(0,\delta_j)}\|z\|^{2t}\mathrm{e}^{-2cu_j}\mathrm{d}V_{2n}<+\infty.$$

Let k be an integer number with $1\le k\le n$. By Lemma 2.2, there exist a positive integer number j_k and a complex number $\lambda_k\in\mathbb{C}\backslash\{0\}$ such that, for every $j\ge j_k$, we can find the holomorphic functions $F_{j,k}$ and $G_{j,k}$ defined on $\Delta^n(0,r)$ that satisfy:

(i) $\int_{\Delta^n(0,r)}|F_{j,k}|^2\mathrm{e}^{-2cu_j}\mathrm{d}V_{2n}<+\infty$;
(ii) $F_{j,k}(z)=z_k^t+(z_k-\lambda_k)G_{j,k}(z)$;
(iii) $G_{j,k}(z)=\sum_{\alpha\in\mathbb{N}^n}a_{j,k,\alpha}z^\alpha$ with $2|\lambda_k a_{j,k,\alpha}|r^{|\alpha|-t}\le 1$, $\forall\alpha\in\mathbb{N}^n$.

We now write

$$F_{j,k,l}(z)=\sum_{\alpha\in\mathbb{N}^n} b_{j,k,\alpha}z^\alpha,\ z\in\Delta^n(0,r).$$

Let $\beta\in\mathbb{N}^n$ such that $\beta_k=t$ and $\beta_l=0$, $\forall l\ne k$. Then, by (ii) we have

$$b_{j,k,\alpha}=\begin{cases}-\lambda_k a_{j,k,\alpha} & \text{if } \alpha_k=0, \alpha\ne\beta,\\ 1-\lambda_k a_{j,k,\alpha} & \text{if } \alpha_k=0, \alpha=\beta,\\ -\lambda_k a_{j,k,\alpha}+a_{j,k,(\alpha_1,\dots,\alpha_{k-1},\alpha_k-1,\alpha_{k+1},\dots,\alpha_n)} & \text{if } \alpha_k>0, \alpha\ne\beta,\\ 1-\lambda_k a_{j,k,\alpha}+a_{j,k,(\alpha_1,\dots,\alpha_{k-1},\alpha_k-1,\alpha_{k+1},\dots,\alpha_n)} & \text{if } \alpha_k>0, \alpha=\beta.\end{cases}\tag{2.7}$$

First we claim that there exists $\theta>0$ such that

$$\theta|z_k|^t\le\sum_{\alpha\in\mathbb{N}^n,|\alpha|\le t}|b_{j,k,\alpha}z^\alpha|,\ \forall z\in\Delta^n(0,r).\tag{2.8}$$

Indeed, let $\{\gamma^s\}\subset\mathbb{N}^n$ be such that $\gamma_k^s=s$ and $\gamma_l^s=0$, $\forall l\ne k$. Put

$$s_0:=\begin{cases}\inf\{s\in\mathbb{N}:a_{j,k,\gamma^s}\ne 0\} & \text{if } a_{j,k,\gamma^{t-1}}\ne 0,\\ t & \text{if } a_{j,k,\gamma^{t-1}}=0.\end{cases}$$

Then, $0\le s_0\le t$ and $a_{j,k,\gamma^{s_0}}\ne 0$ if $s_0<t$. From (2.7), we have

$$b_{j,k,\gamma^{s_0}}=\begin{cases}1-\lambda_k a_{j,k,\beta} & \text{if } s_0=t,\\ -\lambda_k a_{j,k,\gamma^{s_0}} & \text{if } 0\le s_0<t.\end{cases}$$

Therefore, by (iii) we get $\theta:=|b_{j,k,\gamma^{s_0}}|r^{t-s_0}>0$, and hence

$$\theta|z_k|^t\le\sum_{\alpha\in\mathbb{N}^n,|\alpha|\le t}|b_{j,k,\alpha}z^\alpha|,\ \forall z\in\Delta^n(0,r).$$

This proves the claim. Now, by (iii) and (2.7), we get

$$|b_{j,k,\alpha}|r^{|\alpha|} \le \frac{r^t}{2} + \frac{r^{t+1}}{2|\lambda_k|}, \ \forall \alpha \in \mathbb{N}^n, |\alpha| > t+1.$$

Hence

$$|b_{j,k,\alpha}|r^{|\alpha|} \le C_{j,k}, \ \forall \alpha \in \mathbb{N}^n,$$

for some constant $C_{j,k}$. Let $\theta_s \in \mathbb{C}$, $s = 1, \ldots, n+t$ such that $|\theta_s| = 1$ and $\theta_s \ne \theta_h$, $\forall s \ne h$. Lemma 2.1 implies that there exist constants $A_{j,k}, B_{j,k} > 0$ such that

$$\sum_{\alpha \in \mathbb{N}^n, |\alpha| \le t} |b_{j,k,\alpha} z^\alpha| \le A_{j,k} \sum_{s_1,\ldots,s_n=1}^{n+t} |F_{j,k}(\theta_{s_1} z_1, \ldots, \theta_{s_n} z_n)| + B_{j,k}\|z\|^{t+1}, \ \forall z \in \Delta^n(0, \frac{r}{2}).$$

Combining this with (2.8), we infer that

$$2\|z\|^{2t} \le A \sum_{k=1}^{n} \sum_{s_1,\ldots,s_n=1}^{n+t} |F_{j,k}(\theta_{s_1} z_1, \ldots, \theta_{s_n} z_n)|^2 + B\|z\|^{2t+1}, \ \forall z \in \Delta^n(0, \frac{r}{2}), \forall j \ge j_0 := \max_{1 \le k \le n} j_k,$$

where A, B are positive constants. Choose $\delta_j > 0$ such that $\delta_j(\frac{2}{r} + B) < 1$. Then

$$\|z\|^{2t} \le A \sum_{k=1}^{n} \sum_{s_1,\ldots,s_n=1}^{n+t} |F_{j,k}(\theta_{s_1} z_1, \ldots, \theta_{s_n} z_n)|^2, \ \forall z \in \Delta^n(0, \delta_j), \forall j \ge j_0. \tag{2.9}$$

Now, since the u_j's are toric plurisubharmonic functions, we conclude by (i) that

$$\int_{\Delta^n(0,\delta_j)} |F_{j,k}(\theta_{s_1} z_1, \ldots, \theta_{s_n} z_n)|^2 \mathrm{e}^{-2cu_j(z)} \mathrm{d}V_{2n}(z) < +\infty$$

for all $s_1, \ldots, s_n = 1, \ldots, n+t$. Hence, by (2.9) this implies that

$$\int_{\Delta^n(0,\delta_j)} \|z\|^{2t} \mathrm{e}^{-2cu_j(z)} \mathrm{d}V_{2n}(z) < +\infty, \ \forall j \ge j_0.$$

The proof is complete. □

References

[1] B. Berndtsson, The extension theorem of Ohsawa–Takegoshi and the theorem of Donnelly–Fefferman, Ann. Inst. Fourier 46 (1996) 1083–1094.
[2] J.-P. Demailly, J. Kollár, Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds, Ann. Sci. Éc. Norm. Supér. (4) 34 (2001) 525–556.
[3] J.-P. Demailly, P.H. Hiep, A sharp lower bound for the log canonical threshold, Acta Math. 212 (2014) 1–9.
[4] Q. Guan, Z. Li, Multiplier ideal sheaves associated with weights of log canonical threshold one, Adv. Math. 302 (2016) 40–47.
[5] Q. Guan, X. Zhou, A proof of Demailly's strong openness conjecture, Ann. of Math. (2) 182 (2015) 605–616.
[6] Q. Guan, X. Zhou, Effectiveness of Demailly's strong openness conjecture and related problems, Invent. Math. 202 (2) (2015) 635–676.
[7] Q. Guan, X. Zhou, Characterization of multiplier ideal sheaves with weights of Lelong number one, Adv. Math. 285 (2015) 1688–1705.
[8] L.M. Hai, N.X. Hong, V.V. Hung, A result on the comparison principle for the log canonical threshold of plurisubharmonic functions, Ann. Pol. Math. 112 (2014) 109–114.
[9] P.H. Hiep, A comparison principle for the log canonical threshold, C. R. Acad. Sci. Paris, Ser. I 351 (2013) 441–443.
[10] P.H. Hiep, The weighted log canonical threshold, C. R. Acad. Sci. Paris, Ser. I 352 (2014) 283–288.
[11] P.H. Hiep, Continuity properties of certain weighted log canonical thresholds, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 34–39.
[12] P.H. Hiep, Log canonical thresholds and Monge–Ampère masses, Math. Ann. (2017), http://dx.doi.org/10.1007/s00208-017-1518-2.
[13] P.H. Hiep, T. Tung, The weighted log canonical thresholds of toric plurisubharmonic functions, C. R. Acad. Sci. Paris, Ser. I 353 (2) (2015) 127–131.
[14] N.X. Hong, H. Viet, Local property of maximal plurifinely plurisubharmonic functions, J. Math. Anal. Appl. 441 (2016) 586–592.
[15] N.X. Hong, N.V. Trao, T.V. Thuy, Convergence in capacity of plurisubharmonic functions with given boundary values, Int. J. Math. 28 (3) (2017), article ID: 1750018, 14 p.
[16] M. Klimek, Pluripotential Theory, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991.
[17] T. Ohsawa, K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. 195 (1987) 197–204.
[18] A. Rashkovskii, Extremal cases for the log canonical threshold, C. R. Acad. Sci. Paris, Ser. I 353 (1) (2015) 21–24.
[19] N.V. Trao, H. Viet, N.X. Hong, Approximation of plurifinely plurisubharmonic functions, J. Math. Anal. Appl. 450 (2017) 1062–1075.