C. R. Acad. Sci. Paris, Ser. I 355 (2017) 676–679

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

Existence of invariant measures for some damped stochastic dispersive equations

Existence de mesures invariantes pour des équations dispersives stochastiques

Ibrahim Ekren [a], Igor Kukavica [a], Mohammed Ziane [b]

[a] *Departement fur Mathematik, ETH Zurich, Ramistrasse 101, CH-8092, Zurich, Switzerland*
[b] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA*

A R T I C L E I N F O

Article history:
Received 13 June 2016
Accepted after revision 28 April 2017
Available online 9 May 2017

Presented by the Editorial Board

A B S T R A C T

We address the long-time behavior of solutions to damped dispersive stochastic partial differential equations, namely the KdV equation and the nonlinear Schrödinger equation on the whole space. We prove that the transition semigroup is Feller and establish the existence of an invariant measure using the asymptotic compactness property of the transition semigroup and the Aldous criterion.

R É S U M É

On étudie le comportement asymptotique des solutions d'équations dispersives stochastiques amorties, en particulier les équation de KdV et les équations de Schrödinger. Nous montrons que le semi-groupe de transition est Feller et nous établissons l'existence d'une mesure invariante grâce à la propriété de compacité asymptotique du semi-groupe de transition et au critère d'Aldous.

1. Introduction

The purpose of this Note is to report on the results in [6,7] on the long-time behavior of solutions to the stochastic damped dispersive equations, specifically the KdV equation

$$du + (\partial_x^3 u + u\partial_x u + \lambda u)\,dt = f\,dt + \Phi\,dW_t, \tag{1}$$

on $\mathbb{R}$ with a nonzero deterministic force, and the nonlinear Schrödinger equation

$$du + (\lambda u + i\Delta u + i\alpha|u|^{2\sigma}u)\,dt = \Phi\,dW_t \tag{2}$$

on $\mathbb{R}^d$, with the main purpose of providing the proof of the existence of invariant measures.

E-mail addresses: ibrahim.ekren@math.ethz.ch (I. Ekren), kukavica@usc.edu (I. Kukavica), ziane@usc.edu (M. Ziane).

http://dx.doi.org/10.1016/j.crma.2017.04.018

The existence of invariant measures for stochastic partial differential equations has been established for several important equations in mathematical physics [1,5,2,8,10,11]. In the case of dispersive equations, there are two main difficulties in carrying out the classical Krylov–Bogoliubov procedure. The first one is related to establishing the Feller property. The second and the main difficulty is the non-compactness of the domain and the lack of compactness in finite time for the solution operator necessary for establishing the tightness of the time averages. In fact, all known approaches fail due to the lack of compactness and dissipation. Thus, in order to obtain the tightness of averages, we are led to an unconventional proof. To show the tightness, we first use the existence results in [3] in order to establish uniform estimates on the solutions to the equation. These bounds give us tightness of measures on the space $L^2_{\mathrm{loc}}(\mathbb{R})$ of locally square integrable functions. To pass from tightness in $L^2_{\mathrm{loc}}(\mathbb{R})$ to tightness in $L^2(\mathbb{R})$, one intuitively needs to show that there is no mass escaping to infinity. In the stochastic framework, this means that we have convergence of the expectation of the square of the $L^2(\mathbb{R})$ norm of a sequence of solutions to the expectation of the square of the $L^2(\mathbb{R})$ norm of the limiting solution. We then use a result in [13] on the convergence in measure in Hilbert spaces to obtain the tightness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. Similar difficulties arise in the case of the Schrödinger equation. In this Note we concentrate more on the KdV equation and state the main result for the Schrödinger equation.

2. The stochastic Korteweg–de Vries equation

Fix a stochastic basis $(\Omega, \mathbb{G}, \{\mathbb{G}_t\}_{t\geq 0}, \mathbb{P})$. With $(e_i)_{i\in\mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R})$, consisting of smooth compactly supported functions and $(\beta_i)_{i\in\mathbb{N}}$, a sequence of mutually independent one-dimensional Brownian motions, denote by $W(t) = \sum_{i\in\mathbb{N}} \beta_i(t) e_i$ a cylindrical Wiener process on $L^2(\mathbb{R})$. Consider the stochastic weakly damped Korteweg–de Vries equation

$$\mathrm{d}u + (\partial_x^3 u + u\partial_x u + \lambda u)\,\mathrm{d}t = f\,\mathrm{d}t + \Phi\,\mathrm{d}W(t), \tag{3}$$

[3,14,15], where $\lambda > 0$, with the initial condition $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$.

For functions $u, v \in L^2(\mathbb{R})$, denote by $\|u\|_{L^2}$ the $L^2(\mathbb{R})$ norm of u and by (u, v) the L^2-inner product of u and v. For a Banach space B and with $T > 0$ and $p > 0$, denote by $L^p([0,T]; B)$ the space of functions from $[0,T]$ into B with integrable p-th power over $[0,T]$ and by $C([0,T]; B)$ the set of continuous functions from $[0,T]$ into B. Denote by $H^\sigma(\mathbb{R})$ the classical Sobolev spaces and by $\mathcal{B}(H^1(\mathbb{R}))$ the set of Borel measurable subsets of $H^1(\mathbb{R})$. For a Hilbert space H, we write $\mathrm{HS}(L^2; H)$ for the space of linear operators Φ from $L^2(\mathbb{R})$ into H with finite Hilbert–Schmidt norm. Similarly to functional spaces, for $p > 0$ we denote by $\mathbb{L}^p(\Omega; B)$ the space of random variables with values in B and finite p-th moment. We assume that

$$f \in H^3(\mathbb{R}), \quad \Phi \in \mathrm{HS}(L^2(\mathbb{R}); H^\sigma(\mathbb{R})), \tag{4}$$

for some $\sigma > 3$ and that

$$v \mapsto (v, f) \text{ is continuous in } L^2_{\mathrm{loc}}(\mathbb{R}). \tag{5}$$

The following statement addresses the existence and uniqueness of solutions and follows from Theorem 3.1 and Lemma 3.2 in [3].

Theorem 2.1. *Assume that $u_0 \in L^2(\Omega; H^1(\mathbb{R})) \cap L^4(\Omega; L^2(\mathbb{R}))$ is $\mathbb{G}_0$-measurable. Then there exists a unique mild solution to (3) with paths almost surely in $C([0,\infty); H^1(\mathbb{R}))$ and with $u \in L^2(\Omega; L^\infty(0,T; H^1(\mathbb{R})))$ for all $T > 0$. Additionally, if $u_0 \in L^2(\Omega; H^3(\mathbb{R}))$, then $u \in L^2(\Omega; L^\infty(0,T; H^3(\mathbb{R})))$ for all $T > 0$.*

The mild solutions of Theorem 2.1 satisfy uniform bounds in $L^2(\mathbb{R})$. More precisely, using a stopping time argument, Ito's lemma, and Burkholder–Davis–Gundy's inequality, we can show that under the assumptions of Theorem 2.1, there exists a sequence of constants $\{C_k\}_{k\geq 1}$ depending on f, Φ, and λ such that

$$\sup_{t\geq 0} \mathbb{E}\left[\|u(t)\|_{L^2}^{2k}\right] \leq C_k(\lambda, \Phi)\left(\mathbb{E}\left[\|u_0\|_{L^2}^{2k}\right] + 1\right) \tag{6}$$

holds for all $k \in \mathbb{N}$ for which $\mathbb{E}\left[\|u_0\|_{L^2}^{2k}\right] < \infty$. Furthermore, using the second invariant of the deterministic KdV equation $I(v) = \int_{\mathbb{R}} \left((\partial_x v(x))^2 - v(x)^3/3\right) \mathrm{d}x$ [15], we obtain the existence of a constant $C > 0$ such that

$$\sup_{t\geq 0} \mathbb{E}\left[\|\partial_x u(t)\|_{L^2}^{2k}\right] \leq C\left(\mathbb{E}\left[\|u_0\|_{H^1}^{2k} + \|u_0\|_{L^2}^{4k}\right] + 1\right), \qquad k = 1, 2. \tag{7}$$

Let $u_0 \in H^1(\mathbb{R})$ be a deterministic initial condition, and let u be the corresponding solution to (3). For all $B \in \mathcal{B}(H^1(\mathbb{R}))$, we define the transition probabilities of the equation by $P_t(u_0, B) = \mathbb{P}(u_t \in B)$. Also, for any function $\xi \in C_b(H^1; \mathbb{R})$ and $t \geq 0$, denote

$$P_t\xi(u_0) = \mathbb{E}[\xi(u_t)] = \int_{H^1} \xi(u) P_t(u_0, \mathrm{d}u). \tag{8}$$

The following statement is our main result on the stochastic damped KdV equation.

Theorem 2.2. *Suppose $\lambda > 0$, and assume that f and Φ verify (4) and (5). Then there exists an invariant measure for the damped stochastic KdV equation (3).*

Here we outline the main steps; for complete details, cf. [6]. First, we establish that the transition semigroup satisfies the Feller property, i.e. the following statement holds.

Lemma 2.3 *(Feller Property). Under the assumptions of Theorem 2.1, the semigroup P_t is Feller on $H^1(\mathbb{R})$. Namely, for $\xi \in C_b(H^1, \mathbb{R})$ and with $u_0^1, u_0^2, \ldots \in H^1(\mathbb{R})$ satisfying $\|u_0^n - u_0\|_{H^1} \to 0$ as $n \to \infty$, where $u_0 \in H^1(\mathbb{R})$, the convergence $P_t\xi(u_0^n) \to P_t\xi(u_0)$ as $n \to \infty$ holds for all $t \geq 0$.*

The next goal is to prove the tightness of averages originating from the initial datum $u_0 = 0$. More precisely, we have the following statement.

Lemma 2.4 *(Tightness). Under the assumptions of Theorem 2.1, the family of measures μ_n is tight on $H^1(\mathbb{R})$, where μ_n is given by*

$$\mu_n(\cdot) = \frac{1}{n}\int_0^n P_t(0, \cdot)\,\mathrm{d}t, \qquad n = 1, 2, \ldots \tag{9}$$

The main tool in establishing the tightness is the asymptotic compactness property of the semi-group; a precise statement of this property is given by the next statement.

Lemma 2.5. *For any sequence of deterministic initial conditions u_0^n satisfying $R = \sup_n \left\{\|u_0^n\|_{H^1}^2\right\} < \infty$ and a sequence of nonnegative numbers $t_1, t_2, \ldots$ such that $\lim_{n\to\infty} t_n = \infty$, the set of probabilities $\{P_{t_n}(u_0^n, \cdot) : n \in \mathbb{N}\}$ is tight in H^1.*

Proof of Lemma 2.5. We assume without loss of generality that $t_1 < t_2 < \ldots$. The proof is divided into several steps. In Step 1, we let $\{u_0^n\}_{n=1}^\infty$ be a sequence of initial conditions as above, and we denote by $\{u^n(t)\}_{n=1}^\infty$ the respective solutions of (3). We intend to show that there exists a subsequence of $\{u_{t_n}^n\}$ that converges in distribution in H^1. By the uniform bounds stated above, we have the estimate $\sup_n \mathbb{E}[\|u^n(t_n)\|_{H^1}^2] \leq C(R)$. Using this bound, the fact that bounded sets in $H^1(\mathbb{R})$ are relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R})$, and Prokhorov's theorem in $L^2_{\mathrm{loc}}(\mathbb{R})$, we conclude that there exists an $L^2_{\mathrm{loc}}(\mathbb{R})$ valued random variable ξ (possibly defined on another probability space) and a subsequence of $\{u_{t_n}^n\}$ such that $u_{t_n}^n \to \xi$ in distribution in $L^2_{\mathrm{loc}}(\mathbb{R})$ as $n \to \infty$. Now using the Monotone Convergence theorem, we show that ξ is $H^1(\mathbb{R})$-valued.

In Step 2, we establish convergence in distribution in L^2. In order to prove this, we use

$$\lim_n \mathbb{E}[\|u_{t_n}^n\|_{L^2}^2] = \mathbb{E}[\|\xi\|_{L^2}^2]. \tag{10}$$

The proof of this fact is technical and relies in particular on the energy method [14]. For details on the proof of (10), see [6,7].

By Prokhorov's theorem and the uniform bounds on the solutions, which imply the uniform integrability of $\|u_{t_n}^n\|_{L^2}^2$, we obtain the tightness in distribution in L^2 of measures of $\{u_{t_n}^n\}$. Note that any limiting measure can only be the measure of ξ. Thus $u_{t_n}^n \to \xi$ in distribution in L^2. With similar arguments, due to the fundamental fact that $\mathbb{E}[I(\xi)] = \lim_n \mathbb{E}[I(u_{t_n}^n)]$ (see [6] for the proof), we have $u_{t_n}^n \to \xi$ in distribution in H^1. $\square$

Note that if K is a compact subset of $H^1(\mathbb{R})$, the set of measures on $H^1(\mathbb{R})$ given by $\{P_s(v, \cdot) : s \in [0, 1],\ v \in K\}$ is tight (cf. [6]). We are now ready to provide a sketch of the proof of Lemma 2.4.

Proof of Lemma 2.4. Fix $\epsilon > 0$. The asymptotic compactness of the equation implies that the set of probabilities $\{P_n(0, \cdot) : n \geq 0\}$ on $H^1(\mathbb{R})$ is tight. We choose a compact set $K_\epsilon \subseteq H^1(\mathbb{R})$ such that $\sup_n P_n(0, K_\epsilon^c) \leq \epsilon/2$. Additionally, since the set of probabilities $\{P_s(v, \cdot) : s \in [0, 1],\ v \in K_\epsilon\}$ defined on $H^1(\mathbb{R})$ is tight, we may pick another compact $A_\epsilon \subseteq H^1(\mathbb{R})$ such that $\sup_{s\in[0,1],\, v\in K_\epsilon} P_s(v, A_\epsilon^c) \leq \epsilon/2$. By a direct computation, we obtain $\mu_n(A_\epsilon^c) \leq \epsilon$ (cf. [6,7] for details). $\square$

3. The stochastic Schrödinger equation

In this section, we state the result establishing the existence of invariant measures for the stochastic damped Schrödinger equation

$$\mathrm{d}u + (\lambda u + \mathrm{i}\Delta u + \mathrm{i}\alpha|u|^{2\sigma} u)\,\mathrm{d}t = \Phi\,\mathrm{d}W_t \tag{11}$$

where $\lambda > 0, \alpha = 1$, or $\alpha = -1$ and $\Phi \in HS(L^2(\mathbb{R}^d); H^1(\mathbb{R}^d))$. We assume $0 \le \sigma < 2/(d-2)$ if $d \ge 3$ and $\sigma \ge 0$ if $d = 1, 2$. Under these assumptions, we have by [4] the following: for every $\mathbb{G}_0$ measurable, $H^1(\mathbb{R}^n)$-valued random variable u_0, there exists an $H^1(\mathbb{R}^d)$-valued and continuous solution $\{u_t\}_{t\ge 0}$ of (11) with the initial condition u_0. The following statement is the main result addressing the damped nonlinear Schrödinger equation.

Theorem 3.1. *Under the assumptions above, there exists an invariant measure for the stochastic damped nonlinear Schrödinger equation.*

The main difficulties in the proof are the lack of smoothing and compactness properties of the solution operator in finite time. Kim [12] obtained the existence of an invariant measure for the defocussing ($\alpha = 1$) Schrödinger equation in L^2 for a restricted range of exponents $\sigma < 2/d$. His proof is based on the existence of two invariants that give uniform bounds in both L^2 and H^1. Therefore, using the Feller property, the existence of invariant measures follows. In the case $\sigma > 2/d$, one needs to work in the phase space H^1 where solutions exist globally; however, the lack of an invariant in H^2 makes the proof of existence of invariant measure difficult.

In order to overcome these difficulties, we follow a similar approach to that of the KdV equation presented above, and we establish an asymptotic compactness property of the solution operator. Namely, we prove that for every sequence of solutions resulting from H^1-bounded initial conditions and for every sequence of times diverging to ∞, there exists a subsequence of solutions and a sequence of times such that the marginals of these solutions at these times converge in distribution in H^1. For this purpose, we employ the conserved quantities used classically for the deterministic analog of the equations [9,14]. We also use the energy equation approach. For details, cf. [7].

Acknowledgements

I.E. was partly supported by the Swiss National Science Foundation Grant SNF 200021-153555, I.K. was supported in part by the NSF grants DMS-1311943 and DMS-1615239, while M.Z. was supported in part by the NSF grant DMS-1109562.

References

[1] P. Constantin, N. Glatt-Holtz, V. Vicol, Unique ergodicity for fractionally dissipated, stochastically forced 2D Euler equations, Commun. Math. Phys. 330 (2) (2014) 819–857.
[2] G. Da Prato, J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, UK, 1996.
[3] A. de Bouard, A. Debussche, On the stochastic Korteweg–de Vries equation, J. Funct. Anal. 154 (1) (1998) 215–251.
[4] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equation in H^1, Stoch. Anal. Appl. 21 (1) (2003) 97–126.
[5] A. Debussche, C. Odasso, Ergodicity for a weakly damped stochastic non-linear Schrödinger equation, J. Evol. Equ. 5 (3) (2005) 317–356.
[6] I. Ekren, I. Kukavica, M. Ziane, Existence of invariant measures for the stochastic damped KdV equation, Indiana Univ. Math. J. (2017), in press.
[7] I. Ekren, I. Kukavica, M. Ziane, Existence of invariant measures for the stochastic damped Schrödinger equation, Stoch. Partial Differ. Equ., Anal. Computat. (2017), in press.
[8] F. Flandoli, Dissipativity and invariant measures for stochastic Navier–Stokes equations, Nonlinear Differ. Equ. Appl. 1 (4) (1994) 403–423.
[9] O. Goubet, Regularity of the attractor for a weakly damped nonlinear Schrödinger equation, Appl. Anal. 60 (1–2) (1996) 99–119.
[10] N. Glatt-Holtz, J. Mattingly, G. Richards, On unique ergodicity in nonlinear stochastic partial differential equations, J. Stat. Phys. (2017), in press.
[11] M. Hairer, J.C. Mattingly, Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing, Ann. of Math. (2) 164 (3) (2006) 993–1032.
[12] J.U. Kim, Invariant measures for a stochastic nonlinear Schrödinger equation, Indiana Univ. Math. J. 55 (2) (2006) 687–717.
[13] Y. Prokhorov, Convergence of random processes and limit theorems in probability theory, Theory Probab. Appl. 2 (2) (1956) 157–214.
[14] R. Rosa, The global attractor of a weakly damped, forced Korteweg–de Vries equation in $H^1(\mathbb{R})$, in: VIth Workshop on Partial Differential Equations, Part II, Rio de Janeiro, 1999, in: Mat. Contemp., vol. 19, 2000, pp. 129–152.
[15] R. Temam, Sur un problème non linéaire, J. Math. Pures Appl. (9) 48 (1969) 159–172.