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Functional analysis

Relative entropy and Tsallis entropy of two accretive operators

Entropie relative et entropie de Tsallis de deux opérateurs accrétifs

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ABSTRACT

Let A and B be two accretive operators. We first introduce the weighted geometric mean of A and B together with some related properties. Afterwards, we define the relative entropy as well as the Tsallis entropy of A and B. The present definitions and their related results extend those already introduced in the literature for positive invertible operators.

RÉSUMÉ

Soient A et B deux opérateurs accrétifs. Nous introduisons d'abord une moyenne géométrique pondérée de A et de B et nous en étudions certaines propriétés. Nous définissons ensuite l'entropie relative ainsi que l'entropie de Tsallis de A et de B. Ces définitions et les résultats obtenus étendent ceux déjà énoncés dans la littérature pour les opérateurs inversibles positifs.

1. Introduction

Let $(H, \langle .,. \rangle)$ be a complex Hilbert space and let $\mathcal{B}(H)$ be the $\mathbb{C}^*$-algebra of bounded linear operators acting on H. Every $A \in \mathcal{B}(H)$ can be written in the following form

$$A = \Re A + \mathrm{i}\,\Im A, \;\text{ with } \Re A = \frac{A+A^*}{2} \text{ and } \Im A = \frac{A-A^*}{2\mathrm{i}}. \tag{1.1}$$

This is known in the literature as the so-called Cartesian decomposition of A, where the operators $\Re A$ and $\Im A$ are the real and imaginary parts of A, respectively. As usual, if A is self-adjoint (i.e. $A^* = A$), we say that A is positive if $\langle Ax, x\rangle \geq 0$ for

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all $x \in H$, and that A is strictly positive if A is positive and invertible. For $A, B \in \mathcal{B}(H)$ self-adjoint, we write $A \le B$ or $B \ge A$ to signify that $B - A$ is positive.

If $A, B \in \mathcal{B}(H)$ are strictly positive and $\lambda \in (0,1)$ is a real number, then the following quantities

$$A\nabla_\lambda B := (1-\lambda)A + \lambda B,\ A!_\lambda B := \Big((1-\lambda)A^{-1} + \lambda B^{-1}\Big)^{-1},\ A\sharp_\lambda B := A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{\lambda}A^{1/2} \tag{1.2}$$

are known, in the literature, as the λ-weighted arithmetic, λ-weighted harmonic and λ-weighted geometric operator means of A and B, respectively. If $\lambda = 1/2$, they are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. The following inequalities are well known in the literature:

$$A!_\lambda B \le A\sharp_\lambda B \le A\nabla_\lambda B. \tag{1.3}$$

For more details about the previous operator means, as well as some other weighted and generalized operator means, we refer the interested reader to the recent paper [12] and the related references cited therein. For refined and reversed inequalities of (1.3), one can consult [6] and [7] for more information.

Now, let $A \in \mathcal{B}(H)$ be as in (1.1). We say that A is accretive if its real part $\Re A$ is strictly positive. If $A, B \in \mathcal{B}(H)$ are accretive then so are A^{-1} and B^{-1}. Further, it is easy to see that the set of all accretive operators acting on H is a convex cone of $\mathcal{B}(H)$. Consequently, $A\nabla_\lambda B$ and $A!_\lambda B$ can be defined by the same formulas as previously whenever $A, B \in \mathcal{B}(H)$ are accretive. Clearly, the relationships $A\nabla_\lambda B = B\nabla_{1-\lambda}A$, $A!_\lambda B = B!_{1-\lambda}A$, $A!_\lambda B = \big(A^{-1}\nabla_\lambda B^{-1}\big)^{-1}$ are also valid for any accretive operators $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$.

However, $A\sharp_\lambda B$ can not be defined by the same formula (1.2) when $A, B \in \mathcal{B}(H)$ are accretive, by virtue of the presence of non-integer exponents for operators in (1.2). For the particular case $\lambda = 1/2$, Drury [1] defined $A\sharp B$ via the following formula (where we continue to use the same notation)

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty \big(tA + t^{-1}B\big)^{-1}\frac{dt}{t}\right)^{-1}. \tag{1.4}$$

It is proved in [1] that $A\sharp B = B\sharp A$ and $A\sharp B = \big(A^{-1}\sharp B^{-1}\big)^{-1}$ for any accretive operator $A, B \in \mathcal{B}(H)$. It follows that (1.4) is equivalent to:

$$A\sharp B = \frac{2}{\pi}\int_0^\infty \big(tA^{-1} + t^{-1}B^{-1}\big)^{-1}\frac{dt}{t} = \frac{2}{\pi}\int_0^\infty A\big(tB + t^{-1}A\big)^{-1}B\frac{dt}{t}. \tag{1.5}$$

In this paper, we will define $A\sharp_\lambda B$ when the operators $A, B \in \mathcal{B}(H)$ are accretive. Some related operator inequalities are investigated. We also introduce the relative entropy and the Tsallis entropy for this class of operators.

2. Weighted geometric mean

We start this section by stating the following definition, which is the main tool for the present approach.

Definition 2.1. Let $A, B \in \mathcal{B}(H)$ be two accretive operators and let $\lambda \in (0,1)$. The λ-weighted geometric mean of A and B is defined by

$$A\sharp_\lambda B := \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\Big(A^{-1} + tB^{-1}\Big)^{-1}dt = \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}A\big(B + tA\big)^{-1}B\,dt. \tag{2.1}$$

In the aim to justify our previous definition we first state the following.

Proposition 2.1. *The following assertions are true:*

(i) If $A, B \in \mathcal{B}(H)$ are strictly positive then (2.1) coincides with (1.2).

(ii) If $\lambda = 1/2$ then (2.1) coincides with (1.4).

Proof. (i) Assume that $A, B \in \mathcal{B}(H)$ are strictly positive. From (2.1), it is easy to see that

$$A\sharp_\lambda B = A^{1/2}\left(\frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\Big(I + tA^{1/2}B^{-1}A^{1/2}\Big)^{-1}dt\right)A^{1/2},$$

where I denotes the identity operator on H. Since $A^{1/2}B^{-1}A^{1/2}$ is self-adjoint strictly positive then it is sufficient, by virtue of (1.2), to show that the following equality

$$a^{-\lambda} = \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\big(1+ta\big)^{-1}\mathrm{d}t$$

holds for any real number $a>0$. If we make the change of variables $u=(1+ta)^{-1}$, the previous real integral becomes after simple manipulations (here the notations B and Γ refer to the standard beta and gamma functions)

$$\frac{\sin(\lambda\pi)}{a^\lambda\pi}\int_0^1 (1-u)^{\lambda-1}u^{-\lambda}\mathrm{d}u = \frac{1}{a^\lambda}\frac{\sin(\lambda\pi)}{\pi}B\big(\lambda,1-\lambda\big) = \frac{1}{a^\lambda}\frac{\sin(\lambda\pi)}{\pi}\Gamma(\lambda)\Gamma(1-\lambda) = \frac{1}{a^\lambda}.$$

The proof of (i) is finished.

(ii) Let $A, B \in \mathcal{B}(H)$ be accretive. If $\lambda = 1/2$ then (2.1) yields

$$A\sharp B = \frac{1}{\pi}\int_0^\infty \big(A^{-1}+tB^{-1}\big)^{-1}\frac{\mathrm{d}t}{\sqrt{t}},$$

which, with the change of variables $u=\sqrt{t}$, becomes (after simple computation)

$$A\sharp B = \frac{2}{\pi}\int_0^\infty \big(A^{-1}+u^2B^{-1}\big)^{-1}\mathrm{d}u = \frac{2}{\pi}\int_0^\infty \big(u^{-1}A^{-1}+uB^{-1}\big)^{-1}\frac{\mathrm{d}u}{u}.$$

This, with (1.5) and the fact that $A\sharp B = B\sharp A$, yields the desired result. The proof of the proposition is completed. □

From a functional point of view, we are allowed to state another equivalent form of (2.1), which seems to be more convenient with a view to our objective in the sequel.

Lemma 2.2. *For any accretive $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$, there holds*

$$A\sharp_\lambda B = \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\big(A!_t B\big)\mathrm{d}t. \tag{2.2}$$

Proof. If in (2.1) we make the change of variables $t=u/(1-u)$, $u\in[0,1)$, we obtain the desired result. The details are simple and are therefore omitted here. □

Using the previous lemma, it is not hard to verify that the following formula

$$A\sharp_\lambda B = B\sharp_{1-\lambda}A$$

persists for any accretive $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$. Moreover, it is clear that $\Re\big(A\nabla_\lambda B\big) = (\Re A)\nabla_\lambda(\Re B)$. About $A!_\lambda B$, we state the following lemma, which will also be needed in the sequel.

Lemma 2.3. *For any accretive $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$, it holds that*

$$\Re\big(A!_\lambda B\big) \geq \big(\Re A\big)!_\lambda\big(\Re B\big). \tag{2.3}$$

Proof. Let $f(A) = \big(\Re(A^{-1})\big)^{-1}$ be defined on the convex cone of accretive operators $A \in \mathcal{B}(H)$. In [10], Mathias proved that f is operator convex, i.e.

$$f\Big((1-\lambda)A+\lambda B\Big) \leq (1-\lambda)f(A)+\lambda f(B).$$

This means that

$$\Big(\Re\big((1-\lambda)A+\lambda B\big)^{-1}\Big)^{-1} \leq (1-\lambda)\big(\Re(A^{-1})\big)^{-1}+\lambda\big(\Re(B^{-1})\big)^{-1}.$$

Replacing in the latter inequality A and B by the accretive operators A^{-1} and B^{-1}, respectively, and using the fact that the map $X \longmapsto X^{-1}$ is operator monotone increasing for $X \in \mathcal{B}(H)$ strictly positive, we then deduce (2.3). □

We now are in a position to state our first main result (which extends Theorem 1.1 of [9]).

Theorem 2.4. *Let $A, B \in \mathcal{B}(H)$ be accretive and $\lambda \in (0,1)$. Then*

$$\Re\Big(A\sharp_\lambda B\Big) \geq (\Re A)\sharp_\lambda(\Re B). \tag{2.4}$$

Proof. By (2.2) with (2.3) we can write

$$\Re\Big(A\sharp_\lambda B\Big) = \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\Re\big(A!_t B\big)\mathrm{d}t \geq \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\big(\Re A\big)!_t\big(\Re B\big)\mathrm{d}t,$$

which, when combined with Proposition 2.1, implies the desired result. □

3. Relative/Tsallis operator entropy

Let $A, B \in \mathcal{B}(H)$ be strictly positive and $\lambda \in (0,1)$. The relative operator entropy $\mathcal{S}(A|B)$ and the Tsallis relative operator entropy $\mathcal{T}_\lambda(A|B)$ are defined by

$$\mathcal{S}(A|B) := A^{1/2}\log\Big(A^{-1/2}BA^{-1/2}\Big)A^{1/2}, \tag{3.1}$$

$$\mathcal{T}_\lambda(A|B) := \frac{A\sharp_\lambda B - A}{\lambda}, \tag{3.2}$$

see [2–4] for instance. The Tsallis relative operator entropy is a parametric extension in the sense that

$$\lim_{\lambda\to 0}\mathcal{T}_\lambda(A|B) = \mathcal{S}(A|B). \tag{3.3}$$

For more details about these operator entropies, we refer the reader to [5] and [11] and the related references cited therein.

Our aim in this section is to extend $\mathcal{S}(A|B)$ and $\mathcal{T}_\lambda(A|B)$ for accretive $A, B \in \mathcal{B}(H)$. Following the previous study, we suggest that $\mathcal{T}_\lambda(A|B)$ can be defined by the same formula (3.2) whenever $A, B \in \mathcal{B}(H)$ are accretive and so $A\sharp_\lambda B$ is given by (2.2). Precisely, we have the following.

Definition 3.1. Let $A, B \in \mathcal{B}(H)$ be accretive and let $\lambda \in (0,1)$. The Tsallis relative operator entropy of A and B is defined by

$$\mathcal{T}_\lambda(A|B) = \frac{\sin\lambda\pi}{\lambda\pi}\int_0^1\left(\frac{t}{1-t}\right)^\lambda\left(\frac{A!_t B - A}{t}\right)\mathrm{d}t. \tag{3.4}$$

This, with (3.2) and (2.4), immediately yields

$$\Re\Big(\mathcal{T}_\lambda(A|B)\Big) \geq \mathcal{T}_\lambda\big(\Re A|\Re B\big)$$

for any accretive $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$.

In view of (3.4), the extension of $\mathcal{S}(A|B)$ can be introduced via the following definition (where we always conserve the same notation, for the sake of simplicity).

Definition 3.2. Let $A, B \in \mathcal{B}(H)$ be accretive. The relative operator entropy of A and B is defined by

$$\mathcal{S}(A|B) = \int_0^1 \frac{A!_t B - A}{t}\mathrm{d}t. \tag{3.5}$$

The following proposition gives a justification as regards the previous definition.

Proposition 3.1. *If $A, B \in \mathcal{B}(H)$ are strictly positive then (3.5) coincides with (3.1).*

Proof. Assume that $A, B \in \mathcal{B}(H)$ are strictly positive. By (3.5), with the definition of $A!_t B$, it is easy to see that

$$\mathcal{S}(A|B) = A^{1/2}\left(\int_0^1 \frac{\Big((1-t)I + tA^{1/2}B^{-1}A^{1/2}\Big)^{-1} - I}{t}\mathrm{d}t\right)A^{1/2}.$$

By similar arguments as those for the proof of Proposition 2.1, it is sufficient to show that

$$\log a = \int_0^1 \frac{\left(1-t+ta^{-1}\right)^{-1}-1}{t}\mathrm{d}t$$

is valid for any $a>0$. This follows from a simple computation of this latter real integral, so completing the proof. □

Theorem 3.2. *Let $A, B \in \mathcal{B}(H)$ be accretive. Then*

$$\Re\Big(\mathcal{S}(A|B)\Big) \geq \mathcal{S}\big(\Re A|\Re B\big). \tag{3.6}$$

Proof. By (3.5) with Lemma 2.3 we have

$$\Re\big(\mathcal{S}(A|B)\big) = \int_0^1 \frac{\Re(A!_t B)-\Re A}{t}\mathrm{d}t \geq \int_0^1 \frac{(\Re A)!_t\big(\Re B\big)-\Re A}{t}\mathrm{d}t.$$

This, with Proposition 3.1, immediately yields (3.6). □

Proposition 3.3. *If $A, B \in \mathcal{B}(H)$ are strictly positive then (3.4) coincides with (3.2).*

Proof. Putting $1-t=\frac{1}{s+1}$, (3.4) is calculated as

$$\mathcal{T}_\lambda(A|B) = \frac{\sin\lambda\pi}{\lambda\pi}\int_0^\infty s^{\lambda-1}\left\{\left(A^{-1}+sB^{-1}\right)^{-1}-(1+s)^{-1}A\right\}\mathrm{d}s.$$

For $A, B > 0$, we have

$$\frac{\sin\lambda\pi}{\pi}\int_0^\infty s^{\lambda-1}\left(A^{-1}+sB^{-1}\right)^{-1}\mathrm{d}s = A\sharp_\lambda B$$

and

$$\frac{\sin\lambda\pi}{\pi}\int_0^\infty s^{\lambda-1}(1+s)^{-1}\mathrm{d}s = 1,$$

which imply the assertion. □

We note that Proposition 3.3 is a generalization of Proposition 3.1. We end this section by stating the following remark.

Remark 3.1. Analog of (1.3), for accretive $A, B \in \mathcal{B}(H)$, does not persist, i.e.

$$\Re\Big(A!_\lambda B\Big) \leq \Re\Big(A\sharp_\lambda B\Big) \leq \Re\Big(A\nabla_\lambda B\Big)$$

fails for some accretive $A, B \in \mathcal{B}(H)$. For $\lambda = 1/2$, this was pointed out in [9] and the same arguments may be used for general $\lambda \in (0,1)$.

However, the following remark is worth to be mentioned.

Remark 3.2. In [8] (see Section 3, Theorem 3), M. Lin presented an extension of the geometric mean-arithmetic mean inequality $A\sharp B \leq A\nabla B$ from positive matrices to accretive matrices (called there sector matrices). By similar arguments, we can obtain an analogue inequality between the λ-weighted geometric mean $A\sharp_\lambda B$ and the λ-weighted arithmetic mean $A\nabla_\lambda B$, when A and B are sector matrices. We omit the details about this latter point to the reader.

4. More about $A\sharp_\lambda B$

We preserve the same notation as previously. The operator mean $A\sharp_\lambda B$ enjoys more other properties which we will discuss in this section. For any real numbers $\alpha, \beta > 0$, we set $\alpha\sharp_\lambda\beta = \alpha^{1-\lambda}\beta^\lambda$ the real λ-weighted geometric mean of α and β.

Now, the following proposition may be stated.

Proposition 4.1. *For any accretive $A, B \in \mathcal{B}(H)$ and $\lambda \in (0,1)$ the following equality*

$$(\alpha A)\sharp_\lambda(\beta B) = (\alpha\sharp_\lambda\beta)\big(A\sharp_\lambda B\big) \tag{4.1}$$

holds for every real numbers $\alpha, \beta > 0$.

Proof. Since $A\sharp_\lambda B = B\sharp_{1-\lambda}A$ it is then sufficient to prove that $(\alpha A)\sharp_\lambda B = \alpha^{1-\lambda}(A\sharp_\lambda B)$. By equation (2.1), we have

$$(\alpha A)\sharp_\lambda B = \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\Big(\frac{A^{-1}}{\alpha} + tB^{-1}\Big)^{-1}\mathrm{d}t = \alpha\frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\big(A^{-1} + \alpha tB^{-1}\big)^{-1}\mathrm{d}t.$$

If we make the change of variables $u = \alpha t$, and we use again (2.1), we immediately obtain the desired equality after simple manipulations. □

We now state the following result, which is also of interest.

Theorem 4.2. *Let $A, B \in \mathcal{B}(H)$ be accretive and $\lambda \in (0,1)$. Then the following inequality*

$$\sum_{k=1}^n\Big\langle\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1}x_k, x_k\Big\rangle \le \left(\sum_{k=1}^n\big\langle(\Re A)^{-1}x_k, x_k\big\rangle\right)\sharp_\lambda\left(\sum_{k=1}^n\big\langle(\Re B)^{-1}x_k, x_k\big\rangle\right) \tag{4.2}$$

holds true, for any family of vectors $(x_k)_{k=1}^n \in H$.

Proof. By (2.4), with the left-hand side of (1.3), we have

$$\Re\big(A\sharp_\lambda B\big) \ge (\Re A)\sharp_\lambda(\Re B) \ge (\Re A)!_\lambda(\Re B),$$

from which we deduce

$$\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1} \le (1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1}.$$

Replacing in this latter inequality A by tA, with $t > 0$ a real number, and using Proposition 4.1, we obtain (after a simple manipulation)

$$t^\lambda\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1} \le (1-\lambda)(\Re A)^{-1} + t\lambda(\Re B)^{-1}.$$

This means that, for any $x \in H$ and $t > 0$, we have

$$t^\lambda\Big\langle\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1}x, x\Big\rangle \le (1-\lambda)\big\langle(\Re A)^{-1}x, x\big\rangle + t\lambda\big\langle(\Re B)^{-1}x, x\big\rangle,$$

and so

$$t^\lambda\sum_{k=1}^n\Big\langle\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1}x_k, x_k\Big\rangle \le (1-\lambda)\sum_{k=1}^n\big\langle(\Re A)^{-1}x_k, x_k\big\rangle + t\lambda\sum_{k=1}^n\big\langle(\Re B)^{-1}x_k, x_k\big\rangle, \tag{4.3}$$

holds for any $(x_k)_{k=1}^n \in H$ and $t > 0$. If $x_k = 0$ for each $k = 1, 2, ..., n$, then (4.2) is an equality. Assume that $x_k \ne 0$ for some $k = 1, 2, ..., n$. If we take

$$t = \left(\frac{\sum_{k=1}^n\Big\langle\Big(\Re\big(A\sharp_\lambda B\big)\Big)^{-1}x_k, x_k\Big\rangle}{\sum_{k=1}^n\big\langle(\Re B)^{-1}x_k, x_k\big\rangle}\right)^{1/(1-\lambda)} > 0$$

in (4.3) and then compute and reduce, we immediately obtain the desired inequality. The details are very simple and therefore are omitted here. □

As a consequence of the previous theorem, we obtain the following.

Corollary 4.3. *Let A, B and λ be as above. Then*

$$\left\|\left(\Re\left(A\sharp_\lambda B\right)\right)^{-1}\right\| \le \left\|\left(\Re A\right)^{-1}\right\|^{1-\lambda}\left\|\left(\Re B\right)^{-1}\right\|^{\lambda}, \tag{4.4}$$

where, for any $T \in \mathcal{B}(H)$, $\|T\| := \sup_{\|x\|=1}\|Tx\|$ is the usual norm of $\mathcal{B}(H)$.

Proof. It follows from (4.2) with $n = 1$ and the fact that

$$\|T\| := \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|=1}\langle Tx, x\rangle$$

whenever $T \in \mathcal{B}(H)$ is a positive operator. □

It is interesting to see whether (4.4) holds for any unitarily invariant norm. Theorem 4.2 gives an inequality about $\left(\Re(A\sharp_\lambda B)\right)^{-1}$. The following result gives another inequality but involving $\Re(A\sharp_\lambda B)$.

Theorem 4.4. *Let A, B and λ be as in Theorem 4.2. Then*

$$\left(\Re e\langle x^*, x\rangle\right)^2 \le \left(\langle\Re\left(A\sharp_\lambda B\right)x^*, x^*\rangle\right)\left(\langle\left(\Re A\right)^{-1}x, x\rangle\sharp_\lambda\langle\left(\Re B\right)^{-1}x, x\rangle\right) \tag{4.5}$$

for all $x, x^* \in H$.

Proof. Following [13], for any $T \in \mathcal{B}(H)$ strictly positive, the following equality

$$\langle T^{-1}x^*, x^*\rangle = \sup_{x\in H}\left\{2\Re e\langle x^*, x\rangle - \langle Tx, x\rangle\right\}$$

is valid for all $x^* \in H$. This, with (4.2) for $n = 1$, immediately implies that

$$2\Re e\langle x^*, x\rangle \le \langle\Re\left(A\sharp_\lambda B\right)x^*, x^*\rangle + \langle\left(\Re A\right)^{-1}x, x\rangle\sharp_\lambda\langle\left(\Re B\right)^{-1}x, x\rangle$$

holds for all $x^*, x \in H$. In the latter inequality, we can, of course, replace x^* by tx^* for any real number t, for obtaining

$$2t\Re e\langle x^*, x\rangle \le t^2\langle\Re\left(A\sharp_\lambda B\right)x^*, x^*\rangle + \langle\left(\Re A\right)^{-1}x, x\rangle\sharp_\lambda\langle\left(\Re B\right)^{-1}x, x\rangle. \tag{4.6}$$

If $x^* = 0$ the inequality (4.5) is obviously an equality. We then assume that $x^* \ne 0$. If in inequality (4.6) we take

$$t = \frac{\Re e\langle x^*, x\rangle}{\langle\Re(A\sharp_\lambda B)x^*, x^*\rangle}$$

then we obtain, after all reduction, the desired inequality (4.5), so completes the proof. □

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