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Kimura-finiteness of quadric fibrations over smooth curves

Finitude à la Kimura de fibrations en quadriques sur des courbes lisses

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ABSTRACT

Making use of the recent theory of noncommutative mixed motives, we prove that the Voevodsky's mixed motive of a quadric fibration over a smooth curve is Kimura-finite.

RÉSUMÉ

Utilisant la théorie récente des motifs non commutatifs, nous prouvons que le motif mixte de Voevodsky d'une fibration en quadriques sur une courbe lisse est fini au sens de Kimura.

1. Introduction

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal category. Given a partition λ of an integer $n \geq 1$, consider the corresponding irreducible $\mathbb{Q}$-linear representation V_λ of the symmetric group $\mathfrak{S}_n$ and the associated idempotent $e_\lambda \in \mathbb{Q}[\mathfrak{S}_n]$. Under these notations, the Schur-functor $S_\lambda : \mathcal{C} \to \mathcal{C}$ sends an object a to the direct summand of $a^{\otimes n}$ determined by e_λ. In the particular case of the partition $\lambda = (1, \ldots, 1)$, resp. $\lambda = (n)$, the associated Schur-functor $\wedge^n := S_{(1,\ldots,1)}$, resp. $\mathrm{Sym}^n := S_{(n)}$, is called the *nth wedge product*, resp. the *nth symmetric product*. Following Kimura [11], an object $a \in \mathcal{C}$ is called *even-dimensional*, resp. *odd-dimensional*, if $\wedge^n(a)$, resp. $\mathrm{Sym}^n(a) = 0$, for some $n \gg 0$. The biggest integer $\mathrm{kim}_+(a)$, resp. $\mathrm{kim}_-(a)$, for which $\wedge^{\mathrm{kim}_+(a)} \neq 0$, resp. $\mathrm{Sym}^{\mathrm{kim}_-(a)}(a) \neq 0$, is called the *even*, resp. *odd*, *Kimura-dimension of a*. An object $a \in \mathcal{C}$ is called *Kimura-finite* if $a \simeq a_+ \oplus a_-$, with a_+ even-dimensional and a_- odd-dimensional. The integer $\mathrm{kim}(a) = \mathrm{kim}_+(a_+) + \mathrm{kim}_-(a_-)$ is called the *Kimura-dimension of a*.

Voevodsky introduced in [20] an important triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (over a perfect base field k). By construction, this category is $\mathbb{Q}$-linear, idempotent complete, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor $M(-)_{\mathbb{Q}} : \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, defined on smooth k-schemes. An important

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open problem[2] is the classification of all the Kimura-finite mixed motives and the computation of the corresponding Kimura-dimensions. On the negative side, O'Sullivan constructed a certain smooth surface S whose mixed motive $M(S)_{\mathbb{Q}}$ is *not* Kimura-finite; consult [14, §5.1] for details. On the positive side, Guletskii [7] and Mazza [14] proved, independently, that the mixed motive $M(C)_{\mathbb{Q}}$ of every smooth curve C is Kimura-finite.

The following result bootstraps Kimura-finiteness from smooth curves to quadric fibrations.

Theorem 1.1. *Let k be a field, C a smooth k-curve, and $q\colon Q \to C$ a flat quadric fibration of relative dimension $d-2$. Assume that Q is smooth and that q has only* simple degenerations, *i.e. that all the fibers of q have corank ≤ 1. Under these assumptions, the following holds:*

(i) *when d is even, the mixed motive $M(Q)_{\mathbb{Q}}$ is Kimura-finite. Moreover, we have the following equality,* $\mathrm{kim}(M(Q)_{\mathbb{Q}}) = \mathrm{kim}(M(\widetilde{C})_{\mathbb{Q}}) + (d-2)\mathrm{kim}(M(C)_{\mathbb{Q}})$, *where $D \hookrightarrow C$ stands for the finite set of critical values of q and $\widetilde{C}$ for the discriminant double cover of C (ramified over D);*

(ii) *when d is odd, k is algebraically closed, and $1/2 \in k$, the mixed motive $M(Q)_{\mathbb{Q}}$ is Kimura-finite. Moreover, we have the following equality* $\mathrm{kim}(M(Q)_{\mathbb{Q}}) = \#D + (d-1)\mathrm{kim}(M(C)_{\mathbb{Q}})$.

To the best of the author's knowledge, Theorem 1.1 is new in the literature. It not only provides new examples of Kimura-finite mixed motives, but also computes the corresponding Kimura dimensions.

Remark 1. In the particular case where k is algebraically closed and Q, C are moreover projective, Vial proved in [19, Cor. 4.4] that the Chow motive $\mathfrak{h}(Q)_{\mathbb{Q}}$ is Kimura-finite. Since the category of Chow motives embeds fully-faithfully into $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (see [20, §4]), we then obtain in this particular case an alternative "geometric" proof of the Kimura-finiteness of $M(Q)_{\mathbb{Q}}$. Moreover, when $k=\mathbb{C}$ and d is odd, Bouali refined Vial's work by showing that $\mathfrak{h}(Q)_{\mathbb{Q}} \simeq \mathbb{Q}(-\frac{d-1}{2})^{\oplus \# D} \oplus \bigoplus_{i=0}^{d-2} \mathfrak{h}(C)_{\mathbb{Q}}(-i)$; see [4, Rk. 1.10(i)]. In this particular case, this leads to an alternative "geometric" computation of the Kimura-dimension of $M(Q)_{\mathbb{Q}}$.

2. Preliminaries

Throughout the article, k denotes a base field of arbitrary characteristic.

Dg categories. For a survey on dg categories, consult Keller's ICM talk [9]. In what follows, we write $\mathrm{dgcat}(k)$ for the category of (essentially small) dg categories and dg functors. Every (dg) k-algebra gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks, since the category of perfect complexes $\mathrm{perf}(X)$ of every k-scheme X (or, more generally, algebraic stack $\mathcal{X}$) admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; consult [9, §4.6][13] for details.

Noncommutative mixed motives. For a book, resp. survey, on noncommutative motives, consult [15], resp. [16]. Recall from [15, §8.5.1] the construction of Kontsevich's triangulated category of noncommutative mixed motives $\mathrm{NMot}(k)$; denoted by $\mathrm{NMot}^{\mathbb{A}^1}_{\mathrm{loc}}(k)$ in *loc. cit.* By construction, this category is idempotent complete, closed symmetric monoidal, and comes equipped with a symmetric monoidal functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$. In what follows, given a k-scheme X, we write $U(X)$ instead of $U(\mathrm{perf}_{\mathrm{dg}}(X))$.

Root stacks. Let X be a k-scheme, $\mathcal{L}$ a line bundle on X, $\sigma \in \Gamma(X, \mathcal{L})$ a global section, and $r > 0$ an integer. In what follows, we write $D \hookrightarrow X$ for the zero locus of σ. Recall from [5, Def. 2.2.1] (see also [1, Appendix B]) that the associated *root stack* is defined as the following fiber-product of algebraic stacks

$$\begin{array}{ccc} \sqrt[r]{(\mathcal{L},\sigma)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow{\scriptstyle \theta_r} \\ X & \xrightarrow[(\mathcal{L},\sigma)]{} & [\mathbb{A}^1/\mathbb{G}_m]\,, \end{array}$$

where θ_r stands for the morphism induced by the rth power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$.

Proposition 2.1. *We have $U(\sqrt[r]{(\mathcal{L},\sigma)/X}) \simeq U(D)^{\oplus(r-1)} \oplus U(X)$ whenever X and D are k-smooth.*

[2] Among other consequences, Kimura-finiteness implies rationality of the motivic zeta function.

Proof. By construction, the root stack comes equipped with a forgetful morphism $f: \sqrt[r]{(\mathcal{L},\sigma)/X} \to X$. As proved by Ishii–Ueda in [8, Thm. 1.6], the pull-back functor f^* is fully-faithful. Moreover, we have the following semi-orthogonal decomposition

$$\mathrm{perf}(\mathcal{X}) = \langle \mathrm{perf}(D)_{r-1}, \ldots, \mathrm{perf}(D)_1, f^*(\mathrm{perf}(X))\rangle\,,$$

where all the categories $\mathrm{perf}(D)_i$ are equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(D)$. Consequently, the proof follows from the fact that the functor U sends semi-orthogonal decomposition to direct sums (see [15, §8.4.1 and §8.4.5]). □

Orbit categories. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be an $\mathbb{Q}$-linear symmetric monoidal additive category and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object. Following [6,10], the *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a,b) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b\otimes \mathcal{O}^{\otimes n})$. Given objects a, b, c and morphisms $\mathrm{f} = \{f_n\}_{n\in\mathbb{Z}}$ and $\mathrm{g} = \{g_n\}_{n\in\mathbb{Z}}$, the ith-component of $\mathrm{g}\circ\mathrm{f}$ is defined as $\sum_n (g_{i-n}\otimes \mathcal{O}^{\otimes n})\circ f_n$. The functor $\pi: \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$, given by $a \mapsto a$ and $f \mapsto \mathrm{f} = \{f_n\}_{n\in\mathbb{Z}}$, where $f_0 := f$ and $f_n := 0$ if $n \neq 0$, is endowed with an isomorphism $\pi\circ(-\otimes\mathcal{O}) \Rightarrow \pi$ and is 2-universal among all such functors. Finally, the category $\mathcal{C}/_{-\otimes\mathcal{O}}$ is $\mathbb{Q}$-linear, additive, and inherits from $\mathcal{C}$ a symmetric monoidal structure making π symmetric monoidal.

3. Proof of Theorem 1.1

Following Kuznetsov [12, §3] (see also Auel–Bernardara–Bolognesi [3, §1.2]), let E be a vector bundle of rank d on C, $p: \mathbb{P}(E) \to C$ the projectivization of E on C, $\mathcal{O}_{\mathbb{P}(E)}(1)$ the Grothendieck line bundle on $\mathbb{P}(E)$, $\mathcal{L}$ a line bundle on C, and finally $\rho \in \Gamma(C, S^2(E^\vee)\otimes\mathcal{L}^\vee) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2)\otimes\mathcal{L}^\vee)$ a global section. Given this data, $Q \subset \mathbb{P}(E)$ is defined as the zero locus of ρ on $\mathbb{P}(E)$ and $q: Q \to C$ as the restriction of p to Q; the relative dimension of q is equal to $d-2$. Consider also the discriminant global section $\mathrm{disc}(q) \in \Gamma(C, \det(E^\vee)^{\otimes 2}\otimes(\mathcal{L}^\vee)^{\otimes d})$ and the associated zero locus $D \hookrightarrow C$. Note that D agrees with the finite set of critical values of q. Recall from [12, §3.5] (see also [3, §1.6]) that when d is even, we can consider the discriminant double cover $\widetilde{C}$ of C. By construction, $\widetilde{C}$ is ramified over D. Since by hypothesis q has only simple degenerations, $\widetilde{C}$ is, moreover, k-smooth. Finally, recall from [12, §3.6] (see also [3, §1.7]) that when d is odd and $1/2 \in k$, we can consider the root stack $\mathcal{X} := \sqrt[2]{(\det(E^\vee)^{\otimes 2}\otimes(\mathcal{L}^\vee)^{\otimes d}, \mathrm{disc}(q))/C}$.

Under the above notations, we have the following computation.

Proposition 3.1. *Let $q: Q \to C$ be a flat quadric fibration as above.*

(a) *When d is even, we have an isomorphism $U(Q)_{\mathbb{Z}[1/2]} \simeq U(\widetilde{C})_{\mathbb{Z}[1/2]} \oplus U(C)^{\oplus(d-2)}_{\mathbb{Z}[1/2]}$.*

(b) *When d is odd, k is algebraically closed, and $1/2 \in k$, we have $U(Q) \simeq U(D) \oplus U(C)^{\oplus(d-1)}$.*

Proof. Recall from [12, §3] (see also [3, §1.5]) the construction of the sheaf $\mathcal{C}_0$ of even parts of the Clifford algebra associated with q. As proved in [12, Thm. 4.2] (see also [3, Thm. 2.2.1]), we have the following semi-orthogonal decomposition

$$\mathrm{perf}(Q) = \langle \mathrm{perf}(C;\mathcal{C}_0), \mathrm{perf}(C)_1, \ldots, \mathrm{perf}(C)_{d-2}\rangle\,,$$

where $\mathrm{perf}(C;\mathcal{C}_0)$ stands for the category of perfect $\mathcal{C}_0$-modules and $\mathrm{perf}(C)_i := q^*(\mathrm{perf}(C))\otimes\mathcal{O}_{Q/C}(i)$. Note that all the categories $\mathrm{perf}(C)_i$ are equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(C)$. Since U sends semi-orthogonal decompositions to direct sums, we then obtain a direct sum decomposition

$$U(Q) \simeq U(\mathrm{perf}^{\mathrm{dg}}(C;\mathcal{C}_0)) \oplus U(C)^{\oplus(d-2)}\,, \tag{1}$$

where $\mathrm{perf}^{\mathrm{dg}}(C;\mathcal{C}_0)$ stands for the dg enhancement of $\mathrm{perf}(C;\mathcal{C}_0)$ induced from $\mathrm{perf}_{\mathrm{dg}}(Q)$. As explained in [12, Prop. 4.9] (see also [3, §2.2]), the inclusion of categories $\mathrm{perf}(C;\mathcal{C}_0) \hookrightarrow \mathrm{perf}(Q)$ is of Fourier–Mukai type. Therefore, the associated kernel leads to a Fourier–Mukai Morita equivalence between $\mathrm{perf}^{\mathrm{dg}}(C;\mathcal{C}_0)$ and $\mathrm{perf}_{\mathrm{dg}}(C;\mathcal{C}_0)$. Consequently, we can replace $\mathrm{perf}^{\mathrm{dg}}(C,\mathcal{C}_0)$ by $\mathrm{perf}_{\mathrm{dg}}(C;\mathcal{C}_0)$ in decomposition (1).

Item (a). As explained in [12, §3.5] (see also [3, §1.6]), the category $\mathrm{perf}(C;\mathcal{C}_0)$ is equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(\widetilde{C};\mathcal{B}_0)$, where $\mathcal{B}_0$ is a certain sheaf of Azumaya algebras over $\widetilde{C}$ of rank $2^{(d/2)-1}$. Therefore, the associated kernel leads to a Fourier–Mukai Morita equivalence between $\mathrm{perf}_{\mathrm{dg}}(C;\mathcal{C}_0)$ and $\mathrm{perf}_{\mathrm{dg}}(\widetilde{C};\mathcal{B}_0)$. As proved in [18, Thm. 2.1], since $\mathcal{B}_0$ is a sheaf of Azumaya algebras of rank $2^{(d/2)-1}$, the noncommutative mixed motive $U(\mathrm{perf}_{\mathrm{dg}}(\widetilde{C};\mathcal{B}_0))_{\mathbb{Z}[1/2]}$ is isomorphic to $U(\widetilde{C})_{\mathbb{Z}[1/2]}$. Consequently, the $\mathbb{Z}[1/2]$-linearization of the right-hand side of (1) reduces to $U(\widetilde{C})_{\mathbb{Z}[1/2]} \oplus U(C)^{\oplus(d-2)}_{\mathbb{Z}[1/2]}$.

Item (b). As explained in [12, Cor. 3.16] (see also [3, §1.7]), since k is algebraically closed and $1/2 \in k$, the category $\mathrm{perf}(C;\mathcal{C}_0)$ is equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(\mathcal{X})$. This implies that the dg category $\mathrm{perf}_{\mathrm{dg}}(C;\mathcal{C}_0)$ is Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X})$. Consequently, since C and D are k-smooth, we conclude from Proposition 2.1 that the right-hand side of (1) reduces to $U(D) \oplus U(C)^{\oplus(d-1)}$. □

We now have all the ingredients necessary to conclude the proof of Theorem 1.1.

Item (i). As proved in [17, Thm. 2.8], there exists a $\mathbb{Q}$-linear, fully-faithful, symmetric monoidal functor Φ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{X \mapsto \mathrm{perf}_{\mathrm{dg}(X)}} & \mathrm{dgcat}(k) \\ \downarrow{\scriptstyle M(-)_{\mathbb{Q}}} & & \downarrow{\scriptstyle U(-)_{\mathbb{Q}}} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} & & \mathrm{NMot}(k)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]} & \xrightarrow[\Phi]{} & \mathrm{NMot}(k)_{\mathbb{Q}}\,, \end{array} \tag{2}$$

where $\underline{\mathrm{Hom}}(-,-)$ stands for the internal Hom of the closed symmetric monoidal structure on $\mathrm{NMot}(k)_{\mathbb{Q}}$ and $\mathbb{Q}(1)[2]$ for the Tate mixed motive. Since the functor π, resp. Φ, is additive, resp. fully-faithful and additive, we conclude from the combination of Proposition 3.1 with the commutative diagram (2) that

$$\pi(M(Q)_{\mathbb{Q}}) \simeq \pi(M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-2)})\,. \tag{3}$$

By definition of the orbit category, there exist then morphisms

$$\mathrm{f} = \{f_n\}_{n\in\mathbb{Z}} \in \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(M(Q)_{\mathbb{Q}}, (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-1)})(n)[2n]),$$
$$\mathrm{g} = \{g_n\}_{n\in\mathbb{Z}} \in \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-1)}, M(Q)_{\mathbb{Q}}(n)[2n])$$

verifying the equalities $\mathrm{g}\circ\mathrm{f} = \mathrm{id} = \mathrm{f}\circ\mathrm{g}$; in order to simplify the exposition, we write $-(n)[2n]$ instead of $-\otimes\mathbb{Q}(1)[2]^{\otimes n}$. Moreover, only finitely many of these morphisms are non-zero. Let us choose an integer $N \gg 0$ such that $f_n = g_n = 0$ for every $|n| > N$. The sets $\{f_n \,|\, -N \leq n \leq N\}$ and $\{g_{-n}(n) \,|\, -N \leq n \leq N\}$ give then rise to the following morphisms between mixed motives:

$$\alpha : M(Q)_{\mathbb{Q}} \longrightarrow \bigoplus_{n=-N}^{N} (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-1)})(n)[2n],$$
$$\beta : \bigoplus_{n=-N}^{N} (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-1)})(n)[2n] \longrightarrow M(Q)_{\mathbb{Q}}\,.$$

The composition $\beta\circ\alpha$ agrees with the 0th component of $\mathrm{g}\circ\mathrm{f} = \mathrm{id}$, *i.e.* with the identity of $M(Q)_{\mathbb{Q}}$. Consequently, $M(Q)_{\mathbb{Q}}$ is a direct summand of the direct sum $\bigoplus_{n=-N}^{N}(M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus(d-1)})(n)[2n]$. Using the fact that $M(\widetilde{C})_{\mathbb{Q}}$ and $M(C)_{\mathbb{Q}}$ are both Kimura-finite, that $\wedge^2(\mathbb{Q}(1)[2]) = 0$, and that Kimura-finiteness is stable under direct sums, direct summands, and tensor products, we hence conclude that the mixed motive $M(Q)_{\mathbb{Q}}$ is also Kimura-finite. This finishes the proof of the first claim.

Let us now prove the second claim. Let X be a smooth k-scheme such that $M(X)_{\mathbb{Q}}$ is Kimura-finite. Note that since the functor π is symmetric monoidal and additive, the object $\pi(M(X)_{\mathbb{Q}})$ of the orbit category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]}$ is also Kimura-finite. As explained in [2, §3], we have the following equality $\mathrm{kim}(M(X)_{\mathbb{Q}}) = \chi(M(X)_{\mathbb{Q},+}) - \chi(M(X)_{\mathbb{Q},-})$, where χ stands for the Euler characteristic computed in the rigid symmetric monoidal category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Therefore, since the functor π is moreover faithful, we observe that $\mathrm{kim}(M(X)_{\mathbb{Q}}) = \mathrm{kim}(\pi(M(X)_{\mathbb{Q}}))$. This leads to the following equalities:

$$\mathrm{kim}(M(?)_{\mathbb{Q}}) = \mathrm{kim}(\pi(M(?)_{\mathbb{Q}})) \qquad ? \in \{Q, \widetilde{C}, C\}\,. \tag{4}$$

The Kimura-dimension of a direct sum of Kimura-finite objects is equal to the sum of the Kimura-dimension of each one of the objects. Hence, using the above computation (3) and the fact that the functor π is additive, we conclude that

$$\mathrm{kim}(\pi(M(Q)_{\mathbb{Q}})) = \mathrm{kim}(\pi(M(\widetilde{C})_{\mathbb{Q}})) + (d-1)\mathrm{kim}(\pi(M(C)_{\mathbb{Q}}))\,. \tag{5}$$

The proof of the second claim follows now from the combination of the above equalities (4)–(5).

Item (ii). The proof is similar to the one of item (i): simply replace $\widetilde{C}$ by D, $(d-1)$ by $(d-2)$, and use the fact that $\mathrm{kim}(M(D)_{\mathbb{Q}}) = \#D$.

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