Mathematical analysis

# Circumventing the lack of dissipation in certain Oldroyd models 

# Comment contourner le manque de dissipation de certains modèles de Oldroyd 

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#### Abstract

We modify an argument of Renardy proving existence and regularity for a subset of a class of models of non-Newtonian fluids suggested by Oldroyd, including the upper-convected and lower-convected Maxwellian models. We suggest an effective method for solving these models, which can provide a variational formulation suitable for finite element computation.


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## R É S U M É

Nous modifions le raisonnement utilisé par Renardy pour prouver l'existence et la régularité de solutions d'une sous-classe de modèles de fluides non newtoniens introduits par Oldroyd, comme les modèles maxwelliens de sur-convection et sous-convection. Nous proposons une méthode itérative variationnelle de calcul de solutions qui s'adapte aux éléments finis.
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## 1. Introduction

We summarize here results obtained in [7] regarding models for non-Newtonian fluids that are a subset of the Oldroyd models [9], including the upper-convected and lower-convected Maxwellian models. The subset we study involves three parameters, the fluid kinematic viscosity $\eta$ and two rheological parameters $\lambda_{1}$ and $\mu_{1}$. We refer to this subset as the "three-parameter" subset. We modify the existence proof of Renardy [10] and show that it can be the basis for an effective solution algorithm.

[^0]Well-posedness has also been established [4] for a "five-parameter" subset of the Oldroyd models [9] involving two additional rheological parameters $\lambda_{2}$ and $\mu_{2}$. The techniques used for these models are quite different from the ones used by Renardy [10] and revisited here. For some reasons explained in [7], we are forced to limit our approach to the threeparameter case. The approaches are complementary, and this potentially reflects significant differences in these models. In [4], $\lambda_{2} \neq 0$ is explicitly required, and (as far as we are aware) the bounds obtained would degenerate as $\lambda_{2} \rightarrow 0$. The condition $\lambda_{2}>0$ leads to an explicit dissipation term that is used in obtaining bounds. When $\lambda_{2}=0$, such explicit dissipation is missing. Thus there is an open question regarding bounds, when $\lambda_{2}>0$, that hold uniformly for $\lambda_{2}$ small.

### 1.1. Notation

We assume that the fluid domain $\mathcal{D} \subset \mathbb{R}^{d}$ is connected and has a Lipschitz boundary $\partial \mathcal{D}$. For simplicity, we assume that the boundary conditions on the fluid velocity are Dirichlet: $\mathbf{u}=\mathbf{0}$ on $\partial \mathcal{D}$. We utilize standard Sobolev spaces $W_{q}^{s}(\mathcal{D})$ for nonnegative integers $s$ and $1 \leq q \leq \infty$, consisting of functions whose derivatives of order $s$ or less are in the Lebesgue space $L_{q}(\mathcal{D})$ [5,1,3]. For vector-valued functions $\mathbf{v}$ and matrix-valued functions $\mathbf{T}$, we will write $\mathbf{v} \in W_{q}^{s}(\mathcal{D})^{d}$ or $\mathbf{T} \in W_{q}^{s}(\mathcal{D})^{d^{2}}$ to indicate that each component of $\mathbf{v}$ or $\mathbf{T}$ is $W_{q}^{S}(\mathcal{D})$. We will also write the corresponding norms for vector-valued and tensor-valued functions via

$$
\|\mathbf{T}\|_{W_{q}^{s}(\mathcal{D})}=\sum_{m=0}^{s}\left\|\left|\nabla^{m} \mathbf{T}\right|\right\|_{L_{q}(\mathcal{D})}
$$

where for tensor quantities $\mathbf{T}$ of any order $r \geq 1$, we denote by $|\mathbf{T}|$ the Euclidean norm of $\mathbf{T}$ when viewed as a vector of dimension $d^{r}$.

Regarding the regularity of the domain boundary, we make the following assumptions. Consider the elliptic equations

$$
\begin{equation*}
v-\Delta v=f \text { in } \mathcal{D}, \quad \nabla v \cdot \mathbf{n}=0 \text { on } \partial \mathcal{D} \tag{1.1}
\end{equation*}
$$

where $\mathbf{n}$ is the unit outer normal to $\partial \mathcal{D}$, and

$$
\begin{equation*}
-\Delta v=f \text { in } \mathcal{D}, \quad v=0 \text { on } \partial \mathcal{D} \tag{1.2}
\end{equation*}
$$

We introduce the following condition: suppose that the domain $\mathcal{D}$ has the property that there is a constant $C$ such that each problem (1.1) and (1.2) has a unique solution $v \in H^{2}(\mathcal{D})$ for all $f \in L_{2}(\mathcal{D})$ satisfying

$$
\begin{equation*}
\|v\|_{H^{2}(\mathcal{D})} \leq C\|f\|_{L_{2}(\mathcal{D})} \tag{1.3}
\end{equation*}
$$

Similarly, we consider a Stokes system,

$$
\begin{equation*}
-\Delta \mathbf{v}+\nabla p=\mathbf{f} \text { in } \mathcal{D}, \quad \nabla \cdot \mathbf{v}=0 \text { in } \mathcal{D}, \quad \mathbf{v}=\mathbf{0} \text { on } \partial \mathcal{D} \tag{1.4}
\end{equation*}
$$

We introduce the following condition: suppose that, for some $q>1$, the domain $\mathcal{D}$ has the property that there is a constant $C_{q, \mathcal{D}}$ such that, for all $\mathbf{f} \in L_{q}(\mathcal{D})^{d}$, there is a unique pair $\mathbf{v} \in W_{q}^{2}(\mathcal{D})^{d}$ and $p \in W_{q}^{1}(\mathcal{D}) / \mathbb{R}$ solving (1.4) such that

$$
\begin{equation*}
\|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D}) / \mathbb{R}} \leq C_{q, \mathcal{D}}\|\mathbf{f}\|_{L_{q}(\mathcal{D})} \text { for all } \mathbf{f} \in L_{q}(\mathcal{D})^{d} \tag{1.5}
\end{equation*}
$$

We assume this holds for all $q \leq q_{0}$ where $q_{0}>1$. Ultimately, many of the results will be restricted to the case $q_{0}>d$, where $d$ is the dimension of $\mathcal{D}$.

## 2. Rheology models

In all (time-independent) models of fluids, the basic equation can be written as

$$
\begin{equation*}
\mathbf{u} \cdot \nabla \mathbf{u}+\nabla p=\nabla \cdot \mathbf{T}+\mathbf{f} \tag{2.6}
\end{equation*}
$$

where $\mathbf{T}$ is called the extra (or deviatoric) stress and $\mathbf{f}$ represents externally given data. The models differ only according to the dependence of the stress on the velocity $\mathbf{u}$.

A three parameter subset of the eight-parameter model of Oldroyd [9] for the extra stress takes the form

$$
\mathbf{T}+\lambda_{1}\left(\mathbf{u} \cdot \nabla \mathbf{T}+\mathbf{R T}+\mathbf{T R}^{t}\right)-\mu_{1}(\mathbf{E T}+\mathbf{T E})=2 \eta \mathbf{E}
$$

where the five parameters $\lambda_{2}, \mu_{2}, \mu_{0}, v_{0}$, and $\nu_{1}$ in [9] are set to zero, and $\mathbf{R}=\frac{1}{2}\left(\nabla \mathbf{u}^{t}-\nabla \mathbf{u}\right)$ and $\mathbf{E}=\frac{1}{2}\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{t}\right)$. This can be written equivalently as

$$
\mathbf{T}+\lambda_{1}\left(\mathbf{u} \cdot \nabla \mathbf{T}-(\nabla \mathbf{u}) \mathbf{T}-\mathbf{T}\left(\nabla \mathbf{u}^{t}\right)\right)+\left(\lambda_{1}-\mu_{1}\right)(\mathbf{E T}+\mathbf{T E})=2 \eta \mathbf{E} .
$$

We can write the full model in the steady case as

$$
\begin{align*}
& \mathbf{u} \cdot \nabla \mathbf{u}+\nabla p=\nabla \cdot \mathbf{T}+\mathbf{f} \text { in } \mathcal{D} \\
& \nabla \cdot \mathbf{u}=0 \text { in } \mathcal{D}, \quad \mathbf{u}=\mathbf{0} \text { on } \partial \mathcal{D}  \tag{2.7}\\
& \mathbf{T}+\lambda_{1}\left(\mathbf{u} \cdot \nabla \mathbf{T}-(\nabla \mathbf{u}) \mathbf{T}-\mathbf{T}\left(\nabla \mathbf{u}^{t}\right)\right)+\left(\lambda_{1}-\mu_{1}\right)(\mathbf{E T}+\mathbf{T E})=2 \eta \mathbf{E} \text { in } \mathcal{D} . \tag{2.8}
\end{align*}
$$

When $\lambda_{1}=\mu_{1}$, (2.8) is the upper-convected Maxwellian model [10]. When $\lambda_{1}=-\mu_{1}$, (2.8) is the lower-convected Maxwellian model.

The first mathematical results on solutions for visco-elastic fluid models were presented by Renardy [10,11]. The first of these papers [10] addresses the upper-convected Maxwellian model. This model has been extensively studied ([12] and references therein).

The Maxwellian model is discussed in [4, Theorem 22.5]. However, they do not state or prove the equivalence Theorem 3.2 established below. That is, they show that a smooth solution to the Maxwellian model would satisfy an associated Navier-Stokes-type system. But they do not establish that, conversely, all solutions of the associated Navier-Stokes-like system yield solutions of the Maxwellian model. Thus the existence of smooth solutions of the Maxwellian model is left open. This feature is common with [10].

There are physical reasons to assume that $\lambda_{1}>0$, but we will allow $\lambda_{1}<0$ as well. The case $\lambda_{1}=0$, which corresponds to the Navier-Stokes equations, has not been considered here, but it can be treated similarly and is essentially trivial by comparison. From now on, we assume that $\lambda_{1} \neq 0$.

## 3. Alternative formulation

The difficulty with the formulation (2.7)-(2.8) is that there is no obvious smoothing for $\mathbf{u}$. Renardy [10] proposed combining (2.7) and (2.8) to obtain (note $\nabla \cdot \mathbf{E}=\Delta \mathbf{u}$ )

$$
\begin{equation*}
-\eta \Delta \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}+\nabla p=\mathbf{f}-\nabla \cdot\left(\lambda_{1}\left(\mathbf{u} \cdot \nabla \mathbf{T}-(\nabla \mathbf{u}) \mathbf{T}-\mathbf{T}\left(\nabla \mathbf{u}^{t}\right)\right)+\left(\lambda_{1}-\mu_{1}\right)(\mathbf{E T}+\mathbf{T E})\right) . \tag{3.9}
\end{equation*}
$$

Renardy [10] further substituted all occurrences of $\nabla \cdot \mathbf{T}$ on the right-hand side of (3.9) using (2.7) written as

$$
\begin{equation*}
\nabla \cdot \mathbf{T}=\mathbf{u} \cdot \nabla \mathbf{u}+\nabla p-\mathbf{f} \tag{3.10}
\end{equation*}
$$

A modified version of the Renardy formulation, introduced in [4], uses this substitution more selectively to obtain

$$
\begin{align*}
-\eta \Delta \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}+\nabla p+\lambda_{1} \mathbf{u} \cdot \nabla(\nabla p) & =\mathbf{f}+\lambda_{1} \mathbf{u} \cdot \nabla \mathbf{f} \\
-\lambda_{1}(\mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla \mathbf{u})-\nabla \cdot((\nabla \mathbf{u}) \mathbf{T})) & -\left(\lambda_{1}-\mu_{1}\right) \nabla \cdot(\mathbf{E T}+\mathbf{T E}) \tag{3.11}
\end{align*}
$$

This formulation is simpler analytically and may be more effective numerically.
Define an auxiliary pressure function $\pi$ by

$$
\begin{equation*}
\pi=p+\lambda_{1} \mathbf{u} \cdot \nabla p \tag{3.12}
\end{equation*}
$$

Then $\nabla \pi=\nabla p+\lambda_{1}\left((\nabla \mathbf{u})^{t} \nabla p+\mathbf{u} \cdot \nabla(\nabla p)\right)$, and substituting this in (3.11) yields

$$
\begin{equation*}
-\eta \Delta \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}+\nabla \pi=\mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}) \tag{3.13}
\end{equation*}
$$

where $\mathcal{F}$ is defined by

$$
\begin{equation*}
\mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T})=\mathbf{f}+\lambda_{1} \mathbf{u} \cdot \nabla \mathbf{f}+\lambda_{1}(\nabla \mathbf{u})^{t} \nabla p-\lambda_{1}(\mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla \mathbf{u})-\nabla \cdot((\nabla \mathbf{u}) \mathbf{T}))-\left(\lambda_{1}-\mu_{1}\right) \nabla \cdot(\mathbf{E T}+\mathbf{T E}) . \tag{3.14}
\end{equation*}
$$

We can think of (3.12) as determining $p$ from $\pi$. This is exactly the problem addressed in [8].

Lemma 3.1. Suppose that $q>d, \mathbf{v} \in W_{q}^{2}(\mathcal{D})^{d}, \mathbf{T} \in W_{q}^{1}(\mathcal{D})^{d^{2}}, \mathbf{f} \in W_{q}^{1}(\mathcal{D})^{d}$, and $p \in W_{q}^{1}(\mathcal{D})$. Then

$$
\begin{align*}
\|\mathcal{F}(\mathbf{f}, \mathbf{v}, p, \mathbf{T})\|_{L_{q}(\mathcal{D})} \leq & \|\mathbf{f}\|_{L_{q}(\mathcal{D})}+\sigma_{q}\left|\lambda_{1}\right|\|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})}\left(\|\mathbf{f}\|_{W_{q}^{1}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D})}+2 \sigma_{q}\|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})}^{2}+\|\mathbf{T}\|_{W_{q}^{1}(\mathcal{D})}\right) \\
& +4 \sigma_{q}\left|\lambda_{1}-\mu_{1}\right|\|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})}\|\mathbf{T}\|_{W_{q}^{1}(\mathcal{D})} \tag{3.15}
\end{align*}
$$

where $\sigma_{q}$ is a (Sobolev) constant that satisfies $\|\mathbf{v}\|_{L_{\infty}(\mathcal{D})} \leq \sigma_{q}\|\mathbf{v}\|_{W_{q}^{1}(\mathcal{D})}$ for all $\mathbf{v} \in W_{q}^{1}(\mathcal{D})^{d}$.

### 3.1. The new system

We can now present the alternative system. It involves (2.8) to define $\mathbf{T}$ in terms of $\mathbf{u}$, the Navier-Stokes system (3.13), and the pressure transport equation (3.12):

$$
\begin{gather*}
-\eta \Delta \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}+\nabla \pi=\mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}) \\
\nabla \cdot \mathbf{u}=0 \text { in } \mathcal{D} \text { and } \mathbf{u}=\mathbf{0} \text { on } \partial \mathcal{D} \\
p+\lambda_{1} \mathbf{u} \cdot \nabla p=\pi  \tag{3.16}\\
\mathbf{T}+\lambda_{1}\left(\mathbf{u} \cdot \nabla \mathbf{T}-(\nabla \mathbf{u}) \mathbf{T}-\mathbf{T}\left(\nabla \mathbf{u}^{t}\right)\right)+\left(\lambda_{1}-\mu_{1}\right)(\mathbf{E T}+\mathbf{T E})=2 \eta \mathbf{E}
\end{gather*}
$$

where $\mathcal{F}$ is defined by (3.14) and $\mathbf{E}=\frac{1}{2}\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{t}\right)$.
We have [7] the following equivalence theorem.
Theorem 3.2. The formulations (2.7)-(2.8) and (3.16) are equivalent. More precisely, let $q>d$. If $\mathbf{u} \in W_{q}^{2}(\mathcal{D})^{d}, \mathbf{T} \in W_{q}^{1}(\mathcal{D})^{d^{2}}$, and $p \in W_{q}^{1}(\mathcal{D}) / \mathbb{R}$ satisfy one of them, then they satisfy the other.

In our derivation of (3.16), we assumed that we had a solution of (2.7)-(2.8) with the stated regularity. Thus we have proved one direction of the equivalence. To prove the other direction, we must deal with the issue that we have created a new system by differentiation. Thus we need to be sure that we can go back to the original system and still have a solution. To do so, we use the following result.

Lemma 3.3. Suppose that $\mathbf{v} \in W_{q}^{2}(\mathcal{D})^{d}$ with $\nabla \cdot \mathbf{v}=0$ in $\mathcal{D}$ and $\mathbf{v}=\mathbf{0}$ on $\partial \mathcal{D}$, that $\mathbf{z} \in L_{q}(\mathcal{D})^{m}$, and that

$$
\begin{equation*}
\mathbf{z}+\mathbf{v} \cdot \nabla \mathbf{z}=\mathbf{0} \tag{3.17}
\end{equation*}
$$

where we interpret $\mathbf{v} \cdot \nabla \mathbf{z} \in H^{-1}(\mathcal{D})^{m}$. Then $\mathbf{z}=\mathbf{0}$.
Remark. What makes the uniqueness result of Lemma 3.3 so much simpler than the results of [6] is the extra regularity we are assuming on $\mathbf{v}$. Thus the product of $\mathbf{v} \in W_{q}^{2}(\mathcal{D})^{d}$ and $\nabla \mathbf{z}$ (for $\mathbf{z} \in L_{q}(\mathcal{D})^{m}$ ) is well defined in $H^{-1}(\mathcal{D})^{d m}$, whereas if we assume only that $\mathbf{v} \in H^{1}(\mathcal{D})^{d}$ as in [6], such a product is defined only in a weaker sense.

The next sections are devoted to showing that the system (3.16) has a solution $\mathbf{u} \in W_{q}^{2}(\mathcal{D})^{d}, \mathbf{T} \in W_{q}^{1}(\mathcal{D})^{d^{2}}$, and $p \in$ $W_{q}^{1}(\mathcal{D})$ for $q>d$. This will be done in three steps, first establishing in Section 3.2 the regularity of solutions of (2.8) given smooth $\mathbf{u}$. The reversed roles, showing $\mathbf{u}$ is smooth given smooth $\mathbf{T}$ is standard Navier-Stokes theory, which we address in Section 3.3. By an iterative scheme in Section 5, we combine the two together to prove existence.

### 3.2. Regularity for $\mathbf{T}$

We now consider the question of determining the regularity of the solution $\mathbf{T}$ of (2.8) in terms of corresponding regularity of $\mathbf{u}$. We will later return to the Navier-Stokes type equation (3.13) to close the loop, deriving regularity of $\mathbf{u}$ in terms of $\mathbf{T}$.

The tensor $\mathbf{T}$ can be viewed as a type of projection of the symmetric gradient $\mathbf{E}$ of $\mathbf{u}$. We can simplify (2.8) by defining $\mathbf{v}=\lambda_{1} \mathbf{u}$, and it becomes

$$
\mathbf{T}+\left(\mathbf{v} \cdot \nabla \mathbf{T}-(\nabla \mathbf{v}) \mathbf{T}-\mathbf{T}\left(\nabla \mathbf{v}^{t}\right)\right)+\left(1-\mu_{1} / \lambda_{1}\right)(\tilde{\mathbf{E}} \mathbf{T}+\mathbf{T} \tilde{\mathbf{E}})=2 \eta \mathbf{E}
$$

where $\widetilde{\mathbf{E}}=\lambda_{1} \mathbf{E}=\frac{1}{2}\left(\nabla \mathbf{v}+\nabla \mathbf{v}^{t}\right)$.
The following result can be derived from [2,8] and is reviewed in [7].
Lemma 3.4. Suppose that $2 \leq d \leq 4, \tilde{\mu} \in \mathbb{R}, q \geq 2, \mathcal{D} \subset \mathbb{R}^{d}$ is bounded and Lipschitz, and $\mathbf{v} \in W_{\infty}^{1}(\mathcal{D})^{d}$, with $\nabla \cdot \mathbf{v}=0$ in $\mathcal{D}, \mathbf{v} \cdot \mathbf{n}=0$ on $\partial \mathcal{D}$ and

$$
\begin{equation*}
\|\nabla \mathbf{v}\|_{L_{\infty}(\mathcal{D})}=\||\nabla \mathbf{v}|\|_{L_{\infty}(\mathcal{D})} \leq \frac{\left(1-c_{0}\right)}{|1+\tilde{\mu}|+|1-\tilde{\mu}|}, \text { where } 0<c_{0}<1 \tag{3.18}
\end{equation*}
$$

Then for each $\mathbf{g} \in L_{q}(\mathcal{D})^{d^{2}}$, there is a unique solution $\mathbf{T} \in L_{q}(\mathcal{D})^{d^{2}}$ of the equation

$$
\begin{equation*}
\mathbf{T}+\mathbf{v} \cdot \nabla \mathbf{T}+\widetilde{\mathbf{R}} \mathbf{T}+\mathbf{T}^{t}-\tilde{\mu}(\widetilde{\mathbf{E}} \mathbf{T}+\mathbf{T} \tilde{\mathbf{E}})=\mathbf{g} \tag{3.19}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
\|\mathbf{T}\|_{L_{q}(\mathcal{D})} \leq \frac{1}{c_{0}}\|\mathbf{g}\|_{L_{q}(\mathcal{D})} \tag{3.20}
\end{equation*}
$$

Here $\widetilde{\mathbf{R}}=\frac{1}{2}\left(\nabla \mathbf{v}^{t}-\nabla \mathbf{v}\right)$ and $\widetilde{\mathbf{E}}=\frac{1}{2}\left(\nabla \mathbf{v}+\nabla \mathbf{v}^{t}\right)$. Furthermore,

$$
\begin{equation*}
\|\mathbf{v} \cdot \nabla \mathbf{T}\|_{L_{q}(\mathcal{D})} \leq \frac{3}{c_{0}}\|\mathbf{g}\|_{L_{q}(\mathcal{D})} \tag{3.21}
\end{equation*}
$$

The proof of this result assumes $q<\infty$, but once it is proved for arbitrary $q<\infty$, the case $q=\infty$ immediately follows by taking limits on both sides of (3.20) and (3.21) as $q \rightarrow \infty$. The following is proved in [7].

Lemma 3.5. Suppose that the conditions of Lemma 3.4 hold, that condition (1.3) holds, and that $\mathbf{g} \in W_{q}^{1}(\mathcal{D})^{d}$. Suppose moreover that $\mathbf{v} \in W_{q}^{2}(\mathcal{D})^{d}$ for some $q>d$ and

$$
\begin{equation*}
\|\nabla \mathbf{v}\|_{L_{\infty}(\mathcal{D})} \leq \frac{\left(1-c_{1}\right)}{1+|1+\tilde{\mu}|+|1-\tilde{\mu}|} \tag{3.22}
\end{equation*}
$$

where $0<c_{1}<1$. Then there is a unique solution $\mathbf{T} \in W_{q}^{1}(\mathcal{D})^{d^{2}}$ of (3.19) such that

$$
\|\nabla \mathbf{T}\|_{L_{q}(\mathcal{D})} \leq \frac{1}{c_{1}}\left(\|\nabla \mathbf{g}\|_{L_{q}(\mathcal{D})}+\frac{|1-\tilde{\mu}|+|1+\tilde{\mu}|}{c_{0}}\left\|\nabla^{2} \mathbf{v}\right\|_{L_{q}(\mathcal{D})}\|\mathbf{g}\|_{L_{\infty}(\mathcal{D})}\right)
$$

The lemmas are applied with $\mathbf{v}=\lambda_{1} \mathbf{u}$ and $\tilde{\mu}=\mu_{1} / \lambda_{1}$. Based on Lemmas 3.4 and 3.5 , we can think of (2.8) as defining a mapping $\mathbf{u} \rightarrow \mathbf{T}$ such that, for $q>d$,

$$
\begin{equation*}
\|\mathbf{T}(\mathbf{u})\|_{W_{q}^{1}(\mathcal{D})} \leq C_{1} \eta\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})} \tag{3.23}
\end{equation*}
$$

provided $\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})} \leq C_{2}, \eta \geq \eta_{0},\left|\lambda_{1}\right| \leq \lambda_{0} \eta_{0}$, and $\left|\mu_{1}\right| \leq \mu_{0}\left|\lambda_{1}\right|$, where $C_{1}$ and $C_{2}$ depend only on $q, \mathcal{D}, \eta_{0}>0, \lambda_{0}<\infty$, and $\mu_{0}<\infty$.

### 3.3. Regularity for $\mathbf{u}$

We consider the system

$$
\begin{align*}
-\eta \Delta \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}+\nabla p & =\mathbf{f} \text { in } \mathcal{D}  \tag{3.24}\\
\nabla \cdot \mathbf{u}=0 \text { in } \mathcal{D}, \quad \mathbf{u} & =\mathbf{0} \text { on } \partial \mathcal{D}
\end{align*}
$$

Using the Gagliardo-Nirenberg inequality [3,5], we can prove [7] the following.
Lemma 3.6. Suppose that $d=2$, that $2<q<\infty$, that (1.5) holds, that $\mathbf{f} \in L_{q}(\mathcal{D})^{2}$, and that $\mathbf{u} \in H^{1}(\mathcal{D})^{2}$ solves (3.24) in the sense of distributions. Then there is a constant $C<\infty$ such that

$$
\begin{equation*}
\eta\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D}) / \mathbb{R}} \leq C\left(\|\mathbf{f}\|_{L_{q}(\mathcal{D})}+\eta^{-2 / \theta}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}^{1+(1 / \theta)}\right) \tag{3.25}
\end{equation*}
$$

for any $\theta<\frac{1}{2} q^{\prime}$, where $q^{\prime}=q /(q-1)$, and $C$ depends on $\theta$ and $q$, but is independent of $\mathbf{f}$ and $\mathbf{u}$.
Lemma 3.7. Suppose that $d=3$, that $3 / 2<q<\infty$, that (1.5) holds, that $\mathbf{f} \in L_{q}(\mathcal{D})^{d}$, and that $\mathbf{u} \in H^{1}(\mathcal{D})^{d}$ solves (3.24) in the sense of distributions. Let $\left.q^{\prime}=q /(q-1) \in\right] 1,3[$. Then

$$
\begin{equation*}
\eta\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D}) / \mathbb{R}} \leq C_{q, \mathcal{D}}\left(\|\mathbf{f}\|_{L_{q}(\mathcal{D})}+\eta^{2-\left(12 / q^{\prime}\right)}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}^{6 / q^{\prime}}\right) \tag{3.26}
\end{equation*}
$$

where $C_{q, \mathcal{D}}$ is independent of $\mathbf{f}$ and $\mathbf{u}$.
As a corollary, we have the following.

Corollary 3.8. Suppose that $q>1$ for $d=2$ and $q \geq 6 / 5$ for $d=3$, that (1.5) holds, $M$ is any positive real number, and $\eta \geq \eta_{0}>0$. Then for $d=2$ and $d=3$, there is a constant $C_{q, \mathcal{D}, \eta_{0}, M}$ such that for all $\mathbf{f} \in L_{q}(\mathcal{D})^{d}$ satisfying $\|\mathbf{f}\|_{H^{-1}(\mathcal{D})} \leq M$ and for all $\mathbf{u} \in H^{1}(\mathcal{D})^{d}$ solving (3.24) in the sense of distributions, we have

$$
\begin{equation*}
\eta\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D}) / \mathbb{R}} \leq C_{q, \mathcal{D}, \eta_{0}, M}\|\mathbf{f}\|_{L_{q}(\mathcal{D})} \tag{3.27}
\end{equation*}
$$

Corollary 3.9. Suppose that the conditions of Lemma 3.7 hold and that there are two solutions to (3.24), that is,

$$
\begin{align*}
-\eta \Delta \mathbf{u}_{i}+\mathbf{u}_{i} \cdot \nabla \mathbf{u}_{i}+\nabla \pi_{i} & =\mathbf{f}_{i} \text { in } \mathcal{D} \\
\nabla \cdot \mathbf{u}_{i}=0 \text { in } \mathcal{D}, \quad \mathbf{u}_{i} & =\mathbf{0} \text { on } \partial \mathcal{D} \tag{3.28}
\end{align*}
$$

for $i=1,2$. Then there is an $\epsilon>0$ such that, provided $\max _{i=1,2}\left\|\mathbf{f}_{i}\right\|_{H^{-1}(\mathcal{D})} \leq \epsilon \eta^{2}$,

$$
\eta\left\|\mathbf{u}_{1}-\mathbf{u}_{2}\right\|_{H^{1}(\mathcal{D})}+\left\|\pi_{1}-\pi_{2}\right\|_{L_{2}(\mathcal{D})} \leq C_{\mathcal{D}, \epsilon}\left\|\mathbf{f}_{1}-\mathbf{f}_{2}\right\|_{H^{-1}(\mathcal{D})}
$$

for both $d=2$ and $d=3$.

## 4. The 3-parameter Oldroyd model

The equations (3.13), (3.12), and (2.8) provide an alternative formulation of the 3-parameter Oldroyd model (2.7)-(2.8). Using this formulation, we can prove [7] the following, which is the main result of the paper.

Theorem 4.1. Suppose that $q>d$, that (1.3) and (1.5) hold, that the coefficients $\lambda_{1}$ and $\mu_{1}$ satisfy

$$
\begin{equation*}
\left|\lambda_{1}\right| \leq \lambda_{0} \eta, \quad\left|\mu_{1}\right| \leq \mu_{0}\left|\lambda_{1}\right|, \quad \text { and } \quad \eta \geq \eta_{0} . \tag{4.29}
\end{equation*}
$$

Then there are constants $C<\infty$ and $\widetilde{C}>0$, depending only on $q, \mathcal{D}, \lambda_{0}, \mu_{0}$, and $\eta_{0}$, such that the 3-parameter Oldroyd system (2.7)-(2.8) has solutions satisfying

$$
\begin{equation*}
\eta\|\mathbf{u}\|_{W_{q}^{2}(\mathcal{D})}+\|\mathbf{T}\|_{W_{q}^{1}(\mathcal{D})}+\|p\|_{W_{q}^{1}(\mathcal{D}) / \mathbb{R}} \leq C\|\mathbf{f}\|_{W_{q}^{1}(\mathcal{D})}, \tag{4.30}
\end{equation*}
$$

provided that $\|\mathbf{f}\|_{W_{q}^{1}(\mathcal{D})} \leq \widetilde{\mathcal{C}}$.

Note that this is suboptimal in terms of the relation between the regularity of $\mathbf{f}$ and $\mathbf{u}$, but the term $\mathbf{u} \cdot \nabla \mathbf{f}$ in (3.14) appears to require this in the case of the estimate (4.30).

The parameter $\lambda$ in [10] corresponds to $\lambda_{1}^{-1}$ here, and thus the auxiliary pressure function $q$ in [10] corresponds to $\lambda_{1}^{-1} \pi$. However, there appears to be a discrepancy with equations (2.5-6) in [10] with regard to the scaling of the pressure function $q$.

## 5. Existence via solution algorithm

The following algorithm is a modification of the iteration proposed by Renardy to demonstrate existence. Given $\mathbf{u}^{n-1}$, $\mathbf{T}^{n-1}, p^{n-1}$, we define $\mathbf{u}^{n}, \mathbf{T}^{n}, p^{n}$ as follows. First we solve

$$
\begin{gather*}
-\eta \Delta \mathbf{u}^{n}+\mathbf{u}^{n} \cdot \nabla \mathbf{u}^{n}+\nabla \pi^{n}=\mathcal{F}\left(\mathbf{f}, \mathbf{u}^{n-1}, p^{n-1}, \mathbf{T}^{n-1}\right) \text { in } \mathcal{D}, \\
\nabla \cdot \mathbf{u}^{n}=0 \text { in } \mathcal{D}, \quad \mathbf{u}^{n}=\mathbf{0} \text { on } \partial \mathcal{D} \tag{5.31}
\end{gather*}
$$

to determine $\mathbf{u}^{n}$ and $\pi^{n}$, where $\mathcal{F}$ was defined in (3.14). Then we solve

$$
\begin{equation*}
p^{n}+\lambda_{1} \mathbf{u}^{n} \cdot \nabla p^{n}=\pi^{n} \tag{5.32}
\end{equation*}
$$

to determine $p^{n}$, and we solve

$$
\begin{gather*}
\mathbf{T}^{n}+\lambda_{1}\left(\mathbf{u}^{n} \cdot \nabla \mathbf{T}^{n}-\left(\nabla \mathbf{u}^{n}\right) \mathbf{T}^{n}-\mathbf{T}^{n}\left(\nabla \mathbf{u}^{n}\right)^{t}\right) \\
+\frac{1}{2}\left(\lambda_{1}-\mu_{1}\right)\left(\left(\nabla \mathbf{u}^{n}+\left(\nabla \mathbf{u}^{n}\right)^{t}\right) \mathbf{T}^{n}+\mathbf{T}^{n}\left(\nabla \mathbf{u}^{n}+\left(\nabla \mathbf{u}^{n}\right)^{t}\right)\right)=\eta\left(\nabla \mathbf{u}^{n}+\left(\nabla \mathbf{u}^{n}\right)^{t}\right) \tag{5.33}
\end{gather*}
$$

for $\mathbf{T}^{n}$. Under the conditions of Theorem 4.1, we prove bounds for these iterates and show that they form a Cauchy sequence [7]. This iteration could be the basis of an effective numerical method.

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