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Mathematical analysis

Circumventing the lack of dissipation in certain Oldroyd models

Comment contourner le manque de dissipation de certains modèles de Oldroyd

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ABSTRACT

We modify an argument of Renardy proving existence and regularity for a subset of a class of models of non-Newtonian fluids suggested by Oldroyd, including the upper-convected and lower-convected Maxwellian models. We suggest an effective method for solving these models, which can provide a variational formulation suitable for finite element computation.

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RÉSUMÉ

Nous modifions le raisonnement utilisé par Renardy pour prouver l'existence et la régularité de solutions d'une sous-classe de modèles de fluides non newtoniens introduits par Oldroyd, comme les modèles maxwelliens de sur-convection et sous-convection. Nous proposons une méthode itérative variationnelle de calcul de solutions qui s'adapte aux éléments finis.

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1. Introduction

We summarize here results obtained in [7] regarding models for non-Newtonian fluids that are a subset of the Oldroyd models [9], including the upper-convected and lower-convected Maxwellian models. The subset we study involves three parameters, the fluid kinematic viscosity η and two rheological parameters λ_1 and μ_1 . We refer to this subset as the "three-parameter" subset. We modify the existence proof of Renardy [10] and show that it can be the basis for an effective solution algorithm.

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Well-posedness has also been established [4] for a "five-parameter" subset of the Oldroyd models [9] involving two additional rheological parameters λ_2 and μ_2 . The techniques used for these models are quite different from the ones used by Renardy [10] and revisited here. For some reasons explained in [7], we are forced to limit our approach to the threeparameter case. The approaches are complementary, and this potentially reflects significant differences in these models. In [4], $\lambda_2 \neq 0$ is explicitly required, and (as far as we are aware) the bounds obtained would degenerate as $\lambda_2 \rightarrow 0$. The condition $\lambda_2 > 0$ leads to an explicit dissipation term that is used in obtaining bounds. When $\lambda_2 = 0$, such explicit dissipation is missing. Thus there is an open question regarding bounds, when $\lambda_2 > 0$, that hold uniformly for λ_2 small.

1.1. Notation

We assume that the fluid domain $\mathcal{D} \subset \mathbb{R}^d$ is connected and has a Lipschitz boundary $\partial \mathcal{D}$. For simplicity, we assume that the boundary conditions on the fluid velocity are Dirichlet: $\mathbf{u} = \mathbf{0}$ on $\partial \mathcal{D}$. We utilize standard Sobolev spaces $W_q^s(\mathcal{D})$ for nonnegative integers s and $1 \le q \le \infty$, consisting of functions whose derivatives of order s or less are in the Lebesgue space $L_q(\mathcal{D})$ [5,1,3]. For vector-valued functions \mathbf{v} and matrix-valued functions \mathbf{T} , we will write $\mathbf{v} \in W_q^s(\mathcal{D})^d$ or $\mathbf{T} \in W_q^s(\mathcal{D})^{d^2}$ to indicate that each component of \mathbf{v} or \mathbf{T} is $W_q^s(\mathcal{D})$. We will also write the corresponding norms for vector-valued and tensor-valued functions via

$$\|\mathbf{T}\|_{W^{s}_{q}(\mathcal{D})} = \sum_{m=0}^{s} \||\nabla^{m}\mathbf{T}|\|_{L_{q}(\mathcal{D})},$$

where for tensor quantities **T** of any order $r \ge 1$, we denote by $|\mathbf{T}|$ the Euclidean norm of **T** when viewed as a vector of dimension d^r .

Regarding the regularity of the domain boundary, we make the following assumptions. Consider the elliptic equations

$$\mathbf{v} - \Delta \mathbf{v} = f \text{ in } \mathcal{D}, \qquad \nabla \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \mathcal{D}, \tag{11}$$

where **n** is the unit outer normal to ∂D , and

$$-\Delta v = f \text{ in } \mathcal{D}, \qquad v = 0 \text{ on } \partial \mathcal{D}. \tag{1.2}$$

We introduce the following condition: suppose that the domain \mathcal{D} has the property that there is a constant C such that each problem (1.1) and (1.2) has a unique solution $v \in H^2(\mathcal{D})$ for all $f \in L_2(\mathcal{D})$ satisfying

$$\|v\|_{H^{2}(\mathcal{D})} \leq C \|f\|_{L_{2}(\mathcal{D})}.$$
(1.3)

Similarly, we consider a Stokes system,

$$-\Delta \mathbf{v} + \nabla p = \mathbf{f} \text{ in } \mathcal{D}, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \mathcal{D}, \quad \mathbf{v} = \mathbf{0} \text{ on } \partial \mathcal{D}.$$

$$\tag{1.4}$$

We introduce the following condition: suppose that, for some q > 1, the domain \mathcal{D} has the property that there is a constant $C_{q,\mathcal{D}}$ such that, for all $\mathbf{f} \in L_q(\mathcal{D})^d$, there is a unique pair $\mathbf{v} \in W_q^2(\mathcal{D})^d$ and $p \in W_q^1(\mathcal{D})/\mathbb{R}$ solving (1.4) such that

$$\|\mathbf{v}\|_{W^2_{q}(\mathcal{D})} + \|p\|_{W^1_{q}(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D}} \|\mathbf{f}\|_{L_{q}(\mathcal{D})} \text{ for all } \mathbf{f} \in L_{q}(\mathcal{D})^d.$$

$$(1.5)$$

We assume this holds for all $q \le q_0$ where $q_0 > 1$. Ultimately, many of the results will be restricted to the case $q_0 > d$, where *d* is the dimension of \mathcal{D} .

2. Rheology models

In all (time-independent) models of fluids, the basic equation can be written as

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbf{T} + \mathbf{f},\tag{2.6}$$

where \mathbf{T} is called the extra (or deviatoric) stress and \mathbf{f} represents externally given data. The models differ only according to the dependence of the stress on the velocity \mathbf{u} .

A three parameter subset of the eight-parameter model of Oldroyd [9] for the extra stress takes the form

 $\mathbf{T} + \lambda_1 (\mathbf{u} \cdot \nabla \mathbf{T} + \mathbf{RT} + \mathbf{TR}^t) - \mu_1 (\mathbf{ET} + \mathbf{TE}) = 2\eta \mathbf{E},$

where the five parameters λ_2 , μ_2 , μ_0 , ν_0 , and ν_1 in [9] are set to zero, and $\mathbf{R} = \frac{1}{2}(\nabla \mathbf{u}^t - \nabla \mathbf{u})$ and $\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$. This can be written equivalently as

$$\mathbf{T} + \lambda_1 (\mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})\mathbf{T} - \mathbf{T}(\nabla \mathbf{u}^{\mathrm{I}})) + (\lambda_1 - \mu_1)(\mathbf{ET} + \mathbf{TE}) = 2\eta \mathbf{E}.$$

We can write the full model in the steady case as

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbf{T} + \mathbf{f} \text{ in } \mathcal{D},$$

$$\nabla \cdot \mathbf{u} = \mathbf{0} \text{ in } \mathcal{D}, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial \mathcal{D},$$
(2.7)

$$\mathbf{T} + \lambda_1 (\mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})\mathbf{T} - \mathbf{T}(\nabla \mathbf{u}^t)) + (\lambda_1 - \mu_1)(\mathbf{E}\mathbf{T} + \mathbf{T}\mathbf{E}) = 2\eta \mathbf{E} \text{ in } \mathcal{D}.$$
(2.8)

When $\lambda_1 = \mu_1$, (2.8) is the upper-convected Maxwellian model [10]. When $\lambda_1 = -\mu_1$, (2.8) is the lower-convected Maxwellian model.

The first mathematical results on solutions for visco-elastic fluid models were presented by Renardy [10,11]. The first of these papers [10] addresses the upper-convected Maxwellian model. This model has been extensively studied ([12] and references therein).

The Maxwellian model is discussed in [4, Theorem 22.5]. However, they do not state or prove the equivalence Theorem 3.2 established below. That is, they show that a smooth solution to the Maxwellian model would satisfy an associated Navier–Stokes-type system. But they do not establish that, conversely, all solutions of the associated Navier–Stokes-like system yield solutions of the Maxwellian model. Thus the existence of smooth solutions of the Maxwellian model is left open. This feature is common with [10].

There are physical reasons to assume that $\lambda_1 > 0$, but we will allow $\lambda_1 < 0$ as well. The case $\lambda_1 = 0$, which corresponds to the Navier–Stokes equations, has not been considered here, but it can be treated similarly and is essentially trivial by comparison. From now on, we assume that $\lambda_1 \neq 0$.

3. Alternative formulation

The difficulty with the formulation (2.7)–(2.8) is that there is no obvious smoothing for **u**. Renardy [10] proposed combining (2.7) and (2.8) to obtain (note $\nabla \cdot \mathbf{E} = \Delta \mathbf{u}$)

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} - \nabla \cdot \left(\lambda_1 (\mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})\mathbf{T} - \mathbf{T}(\nabla \mathbf{u}^t)) + (\lambda_1 - \mu_1)(\mathbf{E}\mathbf{T} + \mathbf{T}\mathbf{E}) \right).$$
(3.9)

Renardy [10] further substituted all occurrences of $\nabla \cdot \mathbf{T}$ on the right-hand side of (3.9) using (2.7) written as

$$\nabla \cdot \mathbf{T} = \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}. \tag{3.10}$$

A modified version of the Renardy formulation, introduced in [4], uses this substitution more selectively to obtain

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p + \lambda_1 \mathbf{u} \cdot \nabla (\nabla p) = \mathbf{f} + \lambda_1 \mathbf{u} \cdot \nabla \mathbf{f} -\lambda_1 (\mathbf{u} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) - \nabla \cdot ((\nabla \mathbf{u})\mathbf{T})) - (\lambda_1 - \mu_1) \nabla \cdot (\mathbf{ET} + \mathbf{TE}).$$
(3.11)

This formulation is simpler analytically and may be more effective numerically.

Define an auxiliary pressure function π by

$$\pi = p + \lambda_1 \mathbf{u} \cdot \nabla p. \tag{3.12}$$

Then $\nabla \pi = \nabla p + \lambda_1 ((\nabla \mathbf{u})^t \nabla p + \mathbf{u} \cdot \nabla (\nabla p))$, and substituting this in (3.11) yields

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}), \tag{3.13}$$

where ${\mathcal F}$ is defined by

$$\mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}) = \mathbf{f} + \lambda_1 \mathbf{u} \cdot \nabla \mathbf{f} + \lambda_1 (\nabla \mathbf{u})^t \nabla p - \lambda_1 \left(\mathbf{u} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) - \nabla \cdot ((\nabla \mathbf{u}) \mathbf{T}) \right) - (\lambda_1 - \mu_1) \nabla \cdot (\mathbf{ET} + \mathbf{TE}).$$
(3.14)

We can think of (3.12) as determining p from π . This is exactly the problem addressed in [8].

Lemma 3.1. Suppose that q > d, $\mathbf{v} \in W_a^2(\mathcal{D})^d$, $\mathbf{T} \in W_a^1(\mathcal{D})^{d^2}$, $\mathbf{f} \in W_a^1(\mathcal{D})^d$, and $p \in W_a^1(\mathcal{D})$. Then

$$\|\mathcal{F}(\mathbf{f}, \mathbf{v}, p, \mathbf{T})\|_{L_{q}(\mathcal{D})} \leq \|\mathbf{f}\|_{L_{q}(\mathcal{D})} + \sigma_{q}|\lambda_{1}| \|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})} \left(\|\mathbf{f}\|_{W_{q}^{1}(\mathcal{D})} + \|p\|_{W_{q}^{1}(\mathcal{D})} + 2\sigma_{q}\|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})}^{2} + \|\mathbf{T}\|_{W_{q}^{1}(\mathcal{D})}\right) + 4\sigma_{q}|\lambda_{1} - \mu_{1}| \|\mathbf{v}\|_{W_{q}^{2}(\mathcal{D})} \|\mathbf{T}\|_{W_{q}^{1}(\mathcal{D})},$$
(3.15)

where σ_q is a (Sobolev) constant that satisfies $\|\mathbf{v}\|_{L_{\infty}(\mathcal{D})} \leq \sigma_q \|\mathbf{v}\|_{W_a^1(\mathcal{D})}$ for all $\mathbf{v} \in W_q^1(\mathcal{D})^d$.

3.1. The new system

We can now present the alternative system. It involves (2.8) to define **T** in terms of **u**, the Navier–Stokes system (3.13), and the pressure transport equation (3.12):

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T})$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \mathcal{D} \text{ and } \mathbf{u} = \mathbf{0} \text{ on } \partial \mathcal{D}$$

$$p + \lambda_1 \mathbf{u} \cdot \nabla p = \pi$$

$$\mathbf{T} + \lambda_1 (\mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})\mathbf{T} - \mathbf{T}(\nabla \mathbf{u}^t)) + (\lambda_1 - \mu_1)(\mathbf{ET} + \mathbf{TE}) = 2\eta \mathbf{E},$$
(3.16)

where \mathcal{F} is defined by (3.14) and $\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$.

We have [7] the following equivalence theorem.

Theorem 3.2. The formulations (2.7)–(2.8) and (3.16) are equivalent. More precisely, let q > d. If $\mathbf{u} \in W_q^2(\mathcal{D})^d$, $\mathbf{T} \in W_q^1(\mathcal{D})^{d^2}$, and $p \in W_q^1(\mathcal{D})/\mathbb{R}$ satisfy one of them, then they satisfy the other.

In our derivation of (3.16), we assumed that we had a solution of (2.7)–(2.8) with the stated regularity. Thus we have proved one direction of the equivalence. To prove the other direction, we must deal with the issue that we have created a new system by differentiation. Thus we need to be sure that we can go back to the original system and still have a solution. To do so, we use the following result.

Lemma 3.3. Suppose that $\mathbf{v} \in W_a^2(\mathcal{D})^d$ with $\nabla \cdot \mathbf{v} = 0$ in \mathcal{D} and $\mathbf{v} = \mathbf{0}$ on $\partial \mathcal{D}$, that $\mathbf{z} \in L_a(\mathcal{D})^m$, and that

$$\mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} = \mathbf{0},\tag{3.17}$$

where we interpret $\mathbf{v} \cdot \nabla \mathbf{z} \in H^{-1}(\mathcal{D})^m$. Then $\mathbf{z} = \mathbf{0}$.

Remark. What makes the uniqueness result of Lemma 3.3 so much simpler than the results of [6] is the extra regularity we are assuming on **v**. Thus the product of $\mathbf{v} \in W_q^2(\mathcal{D})^d$ and $\nabla \mathbf{z}$ (for $\mathbf{z} \in L_q(\mathcal{D})^m$) is well defined in $H^{-1}(\mathcal{D})^{dm}$, whereas if we assume only that $\mathbf{v} \in H^1(\mathcal{D})^d$ as in [6], such a product is defined only in a weaker sense.

The next sections are devoted to showing that the system (3.16) has a solution $\mathbf{u} \in W_q^2(\mathcal{D})^d$, $\mathbf{T} \in W_q^1(\mathcal{D})^{d^2}$, and $p \in W_q^1(\mathcal{D})$ for q > d. This will be done in three steps, first establishing in Section 3.2 the regularity of solutions of (2.8) given smooth \mathbf{u} . The reversed roles, showing \mathbf{u} is smooth given smooth \mathbf{T} is standard Navier–Stokes theory, which we address in Section 3.3. By an iterative scheme in Section 5, we combine the two together to prove existence.

3.2. Regularity for T

We now consider the question of determining the regularity of the solution **T** of (2.8) in terms of corresponding regularity of **u**. We will later return to the Navier–Stokes type equation (3.13) to close the loop, deriving regularity of **u** in terms of **T**. The tensor **T** can be viewed as a type of projection of the symmetric gradient **E** of **u**. We can simplify (2.8) by defining $\mathbf{v} = \lambda_1 \mathbf{u}$, and it becomes

 $\mathbf{T} + (\mathbf{v} \cdot \nabla \mathbf{T} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v}^t)) + (1 - \mu_1 / \lambda_1)(\widetilde{\mathbf{E}}\mathbf{T} + \mathbf{T}\widetilde{\mathbf{E}}) = 2\eta \mathbf{E},$

where $\widetilde{\mathbf{E}} = \lambda_1 \mathbf{E} = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^t)$.

The following result can be derived from [2,8] and is reviewed in [7].

Lemma 3.4. Suppose that $2 \le d \le 4$, $\tilde{\mu} \in \mathbb{R}$, $q \ge 2$, $\mathcal{D} \subset \mathbb{R}^d$ is bounded and Lipschitz, and $\mathbf{v} \in W^1_{\infty}(\mathcal{D})^d$, with $\nabla \cdot \mathbf{v} = 0$ in \mathcal{D} , $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial \mathcal{D}$ and

$$\|\nabla \mathbf{v}\|_{L_{\infty}(\mathcal{D})} = \||\nabla \mathbf{v}\|\|_{L_{\infty}(\mathcal{D})} \le \frac{(1-c_0)}{|1+\tilde{\mu}|+|1-\tilde{\mu}|}, \text{ where } 0 < c_0 < 1.$$
(3.18)

Then for each $\mathbf{g} \in L_q(\mathcal{D})^{d^2}$, there is a unique solution $\mathbf{T} \in L_q(\mathcal{D})^{d^2}$ of the equation

$$\mathbf{T} + \mathbf{v} \cdot \nabla \mathbf{T} + \mathbf{\widetilde{R}}\mathbf{T} + \mathbf{T}\mathbf{\widetilde{R}}^{t} - \tilde{\mu}(\mathbf{\widetilde{E}}\mathbf{T} + \mathbf{T}\mathbf{\widetilde{E}}) = \mathbf{g},$$
(3.19)

satisfying

$$\|\mathbf{T}\|_{L_q(\mathcal{D})} \leq \frac{1}{c_0} \|\mathbf{g}\|_{L_q(\mathcal{D})}.$$
(3.20)

Here $\widetilde{\mathbf{R}} = \frac{1}{2}(\nabla \mathbf{v}^t - \nabla \mathbf{v})$ and $\widetilde{\mathbf{E}} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$. *Furthermore,*

$$\|\mathbf{v}\cdot\nabla\mathbf{T}\|_{L_q(\mathcal{D})} \leq \frac{3}{c_0} \|\mathbf{g}\|_{L_q(\mathcal{D})}.$$
(3.21)

The proof of this result assumes $q < \infty$, but once it is proved for arbitrary $q < \infty$, the case $q = \infty$ immediately follows by taking limits on both sides of (3.20) and (3.21) as $q \to \infty$. The following is proved in [7].

Lemma 3.5. Suppose that the conditions of Lemma 3.4 hold, that condition (1.3) holds, and that $\mathbf{g} \in W_q^1(\mathcal{D})^d$. Suppose moreover that $\mathbf{v} \in W_a^2(\mathcal{D})^d$ for some q > d and

$$\|\nabla \mathbf{v}\|_{L_{\infty}(\mathcal{D})} \le \frac{(1-c_1)}{1+|1+\tilde{\mu}|+|1-\tilde{\mu}|},\tag{3.22}$$

where $0 < c_1 < 1$. Then there is a unique solution $\mathbf{T} \in W_a^1(\mathcal{D})^{d^2}$ of (3.19) such that

$$\|\nabla \mathbf{T}\|_{L_q(\mathcal{D})} \leq \frac{1}{c_1} \Big(\|\nabla \mathbf{g}\|_{L_q(\mathcal{D})} + \frac{|1-\tilde{\mu}|+|1+\tilde{\mu}|}{c_0} \|\nabla^2 \mathbf{v}\|_{L_q(\mathcal{D})} \|\mathbf{g}\|_{L_{\infty}(\mathcal{D})} \Big).$$

The lemmas are applied with $\mathbf{v} = \lambda_1 \mathbf{u}$ and $\tilde{\mu} = \mu_1 / \lambda_1$. Based on Lemmas 3.4 and 3.5, we can think of (2.8) as defining a mapping $\mathbf{u} \to \mathbf{T}$ such that, for q > d,

$$\|\mathbf{T}(\mathbf{u})\|_{W^{1}_{d}(\mathcal{D})} \le C_{1}\eta \|\mathbf{u}\|_{W^{2}_{d}(\mathcal{D})},$$
(3.23)

provided $\|\mathbf{u}\|_{W^2_q(\mathcal{D})} \leq C_2$, $\eta \geq \eta_0$, $|\lambda_1| \leq \lambda_0 \eta_0$, and $|\mu_1| \leq \mu_0 |\lambda_1|$, where C_1 and C_2 depend only on q, \mathcal{D} , $\eta_0 > 0$, $\lambda_0 < \infty$, and $\mu_0 < \infty$.

3.3. Regularity for u

We consider the system

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \mathcal{D}, \tag{3.24}$$

 $\nabla \cdot \mathbf{u} = 0 \text{ in } \mathcal{D}, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial \mathcal{D}.$

Using the Gagliardo-Nirenberg inequality [3,5], we can prove [7] the following.

Lemma 3.6. Suppose that d = 2, that $2 < q < \infty$, that (1.5) holds, that $\mathbf{f} \in L_q(\mathcal{D})^2$, and that $\mathbf{u} \in H^1(\mathcal{D})^2$ solves (3.24) in the sense of distributions. Then there is a constant $C < \infty$ such that

$$\eta \| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C \Big(\| \mathbf{f} \|_{L_q(\mathcal{D})} + \eta^{-2/\theta} \| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^{1+(1/\theta)} \Big)$$
(3.25)

for any $\theta < \frac{1}{2}q'$, where q' = q/(q-1), and C depends on θ and q, but is independent of **f** and **u**.

Lemma 3.7. Suppose that d = 3, that $3/2 < q < \infty$, that (1.5) holds, that $\mathbf{f} \in L_q(\mathcal{D})^d$, and that $\mathbf{u} \in H^1(\mathcal{D})^d$ solves (3.24) in the sense of distributions. Let $q' = q/(q-1) \in [1, 3[$. Then

$$\eta \| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D}} \Big(\| \mathbf{f} \|_{L_q(\mathcal{D})} + \eta^{2 - (12/q')} \| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^{6/q'} \Big),$$
(3.26)

where $C_{q,\mathcal{D}}$ is independent of **f** and **u**.

As a corollary, we have the following.

Corollary 3.8. Suppose that q > 1 for d = 2 and $q \ge 6/5$ for d = 3, that (1.5) holds, M is any positive real number, and $\eta \ge \eta_0 > 0$. Then for d = 2 and d = 3, there is a constant $C_{q,\mathcal{D},\eta_0,M}$ such that for all $\mathbf{f} \in L_q(\mathcal{D})^d$ satisfying $\|\mathbf{f}\|_{H^{-1}(\mathcal{D})} \le M$ and for all $\mathbf{u} \in H^1(\mathcal{D})^d$ solving (3.24) in the sense of distributions, we have

$$\eta \| \mathbf{u} \|_{W^2_q(\mathcal{D})} + \| p \|_{W^1_q(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D},\eta_0,M} \| \mathbf{f} \|_{L_q(\mathcal{D})}.$$
(3.27)

Corollary 3.9. Suppose that the conditions of Lemma 3.7 hold and that there are two solutions to (3.24), that is,

$$-\eta \Delta \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla \pi_i = \mathbf{f}_i \text{ in } \mathcal{D},$$

$$\nabla \cdot \mathbf{u}_i = 0 \text{ in } \mathcal{D}, \quad \mathbf{u}_i = \mathbf{0} \text{ on } \partial \mathcal{D},$$
(3.28)

for i = 1, 2. Then there is an $\epsilon > 0$ such that, provided $\max_{i=1,2} \| \mathbf{f}_i \|_{H^{-1}(\mathcal{D})} \leq \epsilon \eta^2$,

$$\eta \| \mathbf{u}_1 - \mathbf{u}_2 \|_{H^1(\mathcal{D})} + \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} \le C_{\mathcal{D},\epsilon} \| \mathbf{f}_1 - \mathbf{f}_2 \|_{H^{-1}(\mathcal{D})},$$

for both d = 2 and d = 3.

4. The 3-parameter Oldroyd model

The equations (3.13), (3.12), and (2.8) provide an alternative formulation of the 3-parameter Oldroyd model (2.7)-(2.8). Using this formulation, we can prove [7] the following, which is the main result of the paper.

Theorem 4.1. Suppose that q > d, that (1.3) and (1.5) hold, that the coefficients λ_1 and μ_1 satisfy

$$|\lambda_1| \le \lambda_0 \eta, \quad |\mu_1| \le \mu_0 |\lambda_1|, \quad \text{and} \quad \eta \ge \eta_0. \tag{4.29}$$

Then there are constants $C < \infty$ and $\tilde{C} > 0$, depending only on q, D, λ_0 , μ_0 , and η_0 , such that the 3-parameter Oldroyd system (2.7)–(2.8) has solutions satisfying

$$\eta \| \mathbf{u} \|_{W^2_q(\mathcal{D})} + \| \mathbf{T} \|_{W^1_q(\mathcal{D})} + \| p \|_{W^1_q(\mathcal{D})/\mathbb{R}} \le C \| \mathbf{f} \|_{W^1_q(\mathcal{D})},$$
(4.30)

provided that $\|\mathbf{f}\|_{W^1_a(\mathcal{D})} \leq \widetilde{C}$.

Note that this is suboptimal in terms of the relation between the regularity of **f** and **u**, but the term $\mathbf{u} \cdot \nabla \mathbf{f}$ in (3.14) appears to require this in the case of the estimate (4.30).

The parameter λ in [10] corresponds to λ_1^{-1} here, and thus the auxiliary pressure function q in [10] corresponds to $\lambda_1^{-1}\pi$. However, there appears to be a discrepancy with equations (2.5–6) in [10] with regard to the scaling of the pressure function q.

5. Existence via solution algorithm

The following algorithm is a modification of the iteration proposed by Renardy to demonstrate existence. Given \mathbf{u}^{n-1} , \mathbf{r}^{n-1} , p^{n-1} , we define \mathbf{u}^n , \mathbf{T}^n , p^n as follows. First we solve

$$-\eta \Delta \mathbf{u}^{n} + \mathbf{u}^{n} \cdot \nabla \mathbf{u}^{n} + \nabla \pi^{n} = \mathcal{F}(\mathbf{f}, \mathbf{u}^{n-1}, p^{n-1}, \mathbf{T}^{n-1}) \text{ in } \mathcal{D},$$

$$\nabla \cdot \mathbf{u}^{n} = 0 \text{ in } \mathcal{D}, \quad \mathbf{u}^{n} = \mathbf{0} \text{ on } \partial \mathcal{D}$$
(5.31)

to determine \mathbf{u}^n and π^n , where \mathcal{F} was defined in (3.14). Then we solve

$$p^n + \lambda_1 \mathbf{u}^n \cdot \nabla p^n = \pi^n \tag{5.32}$$

to determine p^n , and we solve

$$\mathbf{T}^{n} + \lambda_{1} \left(\mathbf{u}^{n} \cdot \nabla \mathbf{T}^{n} - (\nabla \mathbf{u}^{n}) \mathbf{T}^{n} - \mathbf{T}^{n} (\nabla \mathbf{u}^{n})^{t} \right) + \frac{1}{2} (\lambda_{1} - \mu_{1}) \left(\left(\nabla \mathbf{u}^{n} + (\nabla \mathbf{u}^{n})^{t} \right) \mathbf{T}^{n} + \mathbf{T}^{n} \left(\nabla \mathbf{u}^{n} + (\nabla \mathbf{u}^{n})^{t} \right) \right) = \eta \left(\nabla \mathbf{u}^{n} + (\nabla \mathbf{u}^{n})^{t} \right)$$
(5.33)

for \mathbf{T}^n . Under the conditions of Theorem 4.1, we prove bounds for these iterates and show that they form a Cauchy sequence [7]. This iteration could be the basis of an effective numerical method.

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