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Dynamical covering problems on the triadic Cantor set

Problèmes de recouvrement dynamique sur les ensembles de Cantor triadiques

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A R T I C L E I N F O

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A B S T R A C T

In this note, we consider the metric theory of the dynamical covering problems on the triadic Cantor set $\mathcal{K}$. More precisely, let $Tx = 3x \pmod 1$ be the natural map on $\mathcal{K}$, μ the standard Cantor measure and $x_0 \in \mathcal{K}$ a given point. We consider the size of the set of points in $\mathcal{K}$ which can be well approximated by the orbit $\{T^n x_0\}_{n\ge 1}$ of x_0, namely the set

$$\mathcal{D}(x_0, \varphi) := \Big\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N} \Big\},$$

where φ is a positive function defined on $\mathbb{N}$. It is shown that for μ almost all $x_0 \in \mathcal{K}$, the Hausdorff measure of $\mathcal{D}(x_0, \varphi)$ is either zero or full depending upon the convergence or divergence of a certain series. Among the proof, as a byproduct, we obtain an inhomogeneous counterpart of Levesley, Salp and Velani's work on a Mahler's question about the Diophantine approximation on the Cantor set $\mathcal{K}$.

R É S U M É

Nous considérons dans cette Note la théorie métrique des recouvrements dynamiques dans l'ensemble de Cantor triadique $\mathcal{K}$. Plus précisément, soit $Tx = 3x \pmod 1$ l'application naturelle sur $\mathcal{K}$, μ la mesure de Cantor standard et $x_0 \in \mathcal{K}$ un point donné. Nous considérons la mesure de l'ensemble des points de $\mathcal{K}$ qui peuvent être bien approchés par l'orbite $\{T^n x_0\}_{n\ge 1}$ de x_0, c'est-à-dire l'ensemble

$$\mathcal{D}(x_0, \varphi) := \Big\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n) \text{ pour une infinité de } n \in \mathbb{N} \Big\},$$

où φ est une fonction positive définie sur $\mathbb{N}$. Nous montrons que pour μ-presque tout $x_0 \in \mathcal{K}$ la mesure de Hausdorff de $\mathcal{D}(x_0, \varphi)$ est soit zéro, soit pleine, selon la convergence ou la divergence d'une certaine série. Notre démonstration fournit en passant une contre-

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partie inhomogène au travail de Levesley, Salp et Velani sur une question de Mahler relative à l'approximation rationnelle des points de l'ensemble de Cantor.

1. Introduction

Assume that (X, d) is a complete metric space. Let $\{x_n\}_{n\geq 1}$ be a sequence of points in X and $\{\ell_n\}_{n\geq 1}$ be a sequence of positive numbers. The *covering problem* concerns the limsup set

$$\mathcal{D} := \Big\{ y \in X : d(x_n, y) < \ell_n, \text{i.m.}\, n \in \mathbb{N} \Big\},$$

i.e. the set of points that can be covered by the balls $B(x_n, \ell_n)$ for infinitely many times. Here we use *i.m.* for *infinitely many*.

The well-known Borel–Cantelli lemma can be viewed as the first principle on the size of $\mathcal{D}$ in the sense of measure. The size of $\mathcal{D}$ was extensively studied when $\{x_n\}$ is a sequence of independent and identically distributed random variables. One is referred to [7,11,16] for the history and [9,10] for recent progress.

Instead of a sequence of random variables, recently Fan, Schmeling and Troubetzkoy [8] introduced the question when $\{x_n\}_{n\geq 1}$ is the orbit of a given point driven by a dynamical system. More precisely, let (X, T) be a dynamical system and x_0 be a given point, they considered the limsup set

$$\Big\{ y \in X : y \in B(T^n x_0, \ell_n),\ \text{i.m.}\, n \in \mathbb{N} \Big\}. \tag{1.1}$$

This is called *the dynamical covering problem* for its analogy with the classic covering problem. In [8], Fan, Schmeling and Troubetzkoy studied the set (1.1) defined by the doubling map on the unit interval. By using a general principle presented in [1], Liao and Seuret successfully enlarged the setting to finite expanding Markov systems [13].

It should be mentioned that the works [8,13] required that the phase space X is the whole unit interval. So, instead of the whole interval, in this note, we consider the same question in the simplest fractal: the triadic Cantor set $\mathcal{K}$. More precisely, let $Tx = 3x \pmod 1$ and $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function with $\varphi(n) \to 0$ as $n \to \infty$. We consider the size of the set

$$\mathcal{D}(x_0, \varphi) := \Big\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n),\ \text{i.m.}\, n \in \mathbb{N} \Big\},$$

i.e. the set of points in $\mathcal{K}$ which can be well approximated by the orbit of x_0.

This can be viewed as a dynamical counterpart to a Mahler's question [14] about the Diophantine approximation on the Cantor set $\mathcal{K}$: how well can the points in $\mathcal{K}$ be approximated by rationals? See [4,5,12] for partial progress on Mahler's question.

In this note, we give a complete characterization of the size of $\mathcal{D}(x_0, \varphi)$ for almost all x_0 with respect to the standard Cantor measure μ. Before the statement of our main result, we fix some notation:

- dimension function: an increasing continuous function $f : (0, \infty) \to (0, \infty)$ such that $\lim_{x\to 0} f(x) = 0$;
- $\mathcal{H}^f$: f-Hausdorff measure for a dimension function f;
- $\gamma = \log 2/\log 3$: the Hausdorff dimension of the Cantor set $\mathcal{K}$.

Theorem 1.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$. For μ-almost all $x_0 \in \mathcal{K}$,*

$$\mu(\mathcal{D}(x_0, \varphi)) = \begin{cases} 0, & \text{if } \sum_{n\geq 1} \varphi^\gamma(n) < \infty; \\ 1, & \text{if } \sum_{n\geq 1} \varphi^\gamma(n) = \infty. \end{cases}$$

Theorem 1.2. *Let f be a dimension function with $f(x)/x^\gamma$ being increasing as $x \to 0$. For μ-almost all $x_0 \in \mathcal{K}$, the Hausdorff measure of the set $\mathcal{D}(\varphi, x_0)$ satisfies a dichotomy law*

$$\mathcal{H}^f(\mathcal{D}(x_0, \varphi)) = \begin{cases} 0, & \text{if } \sum_{n\geq 1} f(\varphi(n)) < \infty; \\ \mathcal{H}^f(\mathcal{K}), & \text{if } \sum_{n\geq 1} f(\varphi(n)) = \infty. \end{cases}$$

Consequently, the dimension of $\mathcal{D}(x_0, \varphi)$ is given by the convergence exponent of φ, i.e.

$$\inf\Big\{ s \geq 0 : \sum_{n\geq 1} \varphi(n)^s < \infty \Big\}.$$

Among the proof, as a byproduct, we can obtain an inhomogeneous counterpart of Levesley, Salp and Velani's work [12] on the above-mentioned Mahler's question about the Diophantine approximation on $\mathcal{K}$ (see Theorem 3.3 below).

It should also be mentioned that in [8,13], they considered multifractal measures, while in this work, we only pay attention to a monofractal measure μ.

2. Preliminaries

In this section, we cite two known results, one for the measure of a limsup set, the other for the Hausdorff measure of a shrunk limsup set, which is often referred to as the "mass transference principle".

2.1. Measure of a limsup set

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\{A_n\}_{n\geq 1}$ be a sequence of measurable sets. Define the limsup set

$$E = \limsup_{n\to\infty} A_n = \Big\{x \in X : x \in A_n, \ \text{i.m.}\, n \in \mathbb{N}\Big\}.$$

The following is a widely used result to determine the measure of E from below (see [17, Lemma 5]).

Lemma 2.1. *Assume that $\sum_{n\geq 1} \mu(A_n) = \infty$. Then*

$$\mu(E) \geq \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N} \mu(A_n)\right)^2}{\sum_{1\leq m,n\leq N} \mu(A_m \cap A_n)}.$$

2.2. Mass transference principle

The mass transference principle, developed by Beresnevich & Velani [3] (see also [2]), is a very powerful tool to determine the Hausdorff measure and dimension of a limsup set. In principle, it says that a full measure statement for a limsup set will imply a full Hausdorff measure theoretic statement for the *shrunk* limsup set.

Let (X, d) be a locally compact metric space. Let g be a doubling dimension function, i.e. there exists $\lambda \geq 1$ such that, for all small r,

$$g(2r) \leq \lambda g(r). \tag{2.1}$$

Suppose that there exist $0 < c_1 \leq c_2 < \infty$ and $r_0 > 0$ such that, for all $x \in X$ and $0 < r < r_0$,

$$c_1 g(r) \leq \mathcal{H}^g(B(x,r)) \leq c_2 g(r). \tag{2.2}$$

Write $B^f(x,r)$ for the ball $B(x, g^{-1} \circ f(r))$.

Theorem 2.2 *(Mass transference principle [3]). Let (X, d) and g be as above and let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in X with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $f(x)/g(x)$ is monotonically increasing as $x \to 0$. Suppose that, for any ball B in X,*

$$\mathcal{H}^g\Big(B \cap \limsup_{i\to\infty} B_i^f\Big) = \mathcal{H}^g(B). \tag{2.3}$$

Then, for any ball B in X,

$$\mathcal{H}^f\Big(B \cap \limsup_{i\to\infty} B_i\Big) = \mathcal{H}^f(B).$$

In the latter application, we will take X to be the Cantor set $\mathcal{K}$ and the doubling dimension function $g(x) = x^\gamma$. The Cantor measure μ on $\mathcal{K}$ is the same as $\mathcal{H}^g|_{\mathcal{K}}$. So, the formula (2.2) is fulfilled, i.e.

$$\mu(B(x,r)) \approx r^\gamma. \tag{2.4}$$

The condition (2.3) just says that $\limsup_{i\to\infty} B_i^f$ is full in the sense of $\mathcal{H}^g|_{\mathcal{K}}$, so to apply Theorem 2.2, we are required to determine the μ-measure of $\limsup_{i\to\infty} B_i^f$. This is the task of the next section.

3. A dynamical Borel–Cantelli lemma on $\mathcal{K}$

The proof of our result is just a combination of a dynamical Borel–Cantelli lemma, analogous to the one proved by Philipp [15] for b-adic expansion, and the mass transference principle (Theorem 2.2). In this section, we show this Borel–Cantelli lemma. For general results about dynamical Borel–Cantelli lemmas, one is referred to Chernov & Kleinbock [6] and references therein.

For each $y \in \mathcal{K}$, let

$$\mathcal{S}(y,\varphi) = \Big\{x \in \mathcal{K} : |T^n x - y| < \varphi(n), \ \text{i.m.}\ n \in \mathbb{N}\Big\}.$$

Proposition 3.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ and $y \in \mathcal{K}$. Then*

$$\mu(\mathcal{S}(y,\varphi)) = \begin{cases} 0, & \text{if } \sum_{n\geq 1} \varphi^{\gamma}(n) < \infty; \\ 1, & \text{if } \sum_{n\geq 1} \varphi^{\gamma}(n) = \infty. \end{cases}$$

Before the proof, we begin with a lemma.

Lemma 3.2. *Let E be an interval and F a measurable set. For any $n \geq 1$,*

$$\mu(E \cap T^{-n}F) = \mu(E)\cdot\mu(F) + \mu(F)O(2^{-n}),$$

where the constant implied in O can be taken to be 2.

Proof. It suffices to consider the case when $E = [0,t]$. We define the cylinder set. For each $(\epsilon_1, \cdots, \epsilon_n) \in \{0,2\}^n$, call

$$I_n(\epsilon_1,\cdots,\epsilon_n) = \Big\{x \in [0,1] : x \text{ has a triadic expansion beginning with } (\epsilon_1,\cdots,\epsilon_n)\Big\}$$

a cylinder of order n (with respect to the Cantor set $\mathcal{K}$), which is of μ-measure 2^{-n}.

Let $\{I_n^{(i)} : 1 \leq i \leq \ell+1\}$ be the cylinders of order n with non-empty intersections with E. If we arrange them in increasing order, we have that

$$\bigcup_{i=1}^{\ell} I_n^{(i)} \cap \mathcal{K} \subset (E\cap\mathcal{K}), \ \text{and} \ (E\cap\mathcal{K}) \subset \bigcup_{i=1}^{\ell+1} I_n^{(i)}.$$

Then it follows that

$$\frac{\ell}{2^n} \leq \mu(E) \leq \frac{\ell+1}{2^n},$$
$$\mu(E\cap T^{-n}F) \leq \sum_{i=1}^{\ell+1} \mu(T^{-n}F \cap I_n^{(i)}) = \frac{\ell+1}{2^n}\mu(F),$$
$$\mu(E\cap T^{-n}F) \geq \sum_{i=1}^{\ell} \mu(T^{-n}F \cap I_n^{(i)}) = \frac{\ell}{2^n}\mu(F).$$

Substituting the measure of E into the estimation on $\mu(E\cap T^{-n}F)$, we have that

$$|\mu(E\cap T^{-n}F) - \mu(E)\mu(F)| \leq \frac{1}{2^n}\mu(F). \quad \square$$

Proof of Proposition 3.1. Equipped with the fast decaying of correlations (Lemma 3.2), Proposition 3.1 can be inferred from the general principle in [6]. Here we include a proof for its easiness.

For each $n \geq 1$, let $A_n = T^{-n}B(y,\varphi(n)) \cap \mathcal{K}$. Then

$$\mathcal{S}(y,\varphi) = \limsup_{n\to\infty} A_n.$$

By the invariance of μ with respect to T and (2.4),

$$\sum_{n=1}^{\infty} \mu(T^{-n}B(y,\varphi(n))\cap\mathcal{K}) = \sum_{n=1}^{\infty} \mu(B(y,\varphi(n))) \approx \sum_{n=1}^{\infty} \varphi^{\gamma}(n).$$

The convergence case follows from the convergence part of the Borel–Cantelli lemma. For the divergence part, we apply Lemma 2.1. So, we estimate the summation:

$$\sum_{1\le m,n\le N} \mu(A_m\cap A_n) = 2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\mu(A_m\cap A_n) + \sum_{n=1}^{N}\mu(A_n).$$

By Lemma 3.2, it follows that, for $n>m$,

$$\begin{aligned}\mu(A_m\cap A_n) &= \mu\Big(B(y,\varphi(m))\cap T^{-(n-m)}B(y,\varphi(n))\Big)\\ &= \mu(A_m)\mu(A_n) + O(2^{-n+m})\mu(A_n),\end{aligned}$$

where the invariance of μ with respect to T is used twice. Thus

$$\begin{aligned}2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\mu(A_m\cap A_n) &= 2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\Big(\mu(A_m)\mu(A_n) + O(\frac{1}{2^{n-m}})\mu(A_n)\Big)\\ &\le \Big(\sum_{n=1}^{N}\mu(A_n)\Big)^2 + O(1)\sum_{n=1}^{N}\mu(A_n).\end{aligned}$$

So, it follows by Lemma 2.1 that

$$\mu(\mathcal{S}(y,\varphi)) \ge 1. \quad \square$$

In [12], as an attempt on Mahler's question on $\mathcal{K}$, Levesley, Salp and Velani proved a 0–1 law on the size of the set

$$W(\varphi) := \Big\{x\in\mathcal{K} : |x - p/3^n| < \varphi(n),\ \text{i.m. } n\in\mathbb{N}\Big\}.$$

This set is the same as $\mathcal{S}(0,\varphi)$. So, Proposition 3.1 gives an inhomogeneous counterpart of Levesley, Salp and Velani's work. Combining Proposition 3.1 and the mass transference principle, we obtain the following theorem.

Theorem 3.3. *Let f be a dimension function with $f(x)/x^\gamma$ being increasing as $x\to 0$. For any $y\in\mathcal{K}$, we have that*

$$\mathcal{H}^f(\mathcal{S}(y,\varphi)) = \begin{cases} 0, & \text{if } \sum_{n\ge1} f(\varphi(n)) < \infty;\\ \mathcal{H}^f(\mathcal{K}), & \text{if } \sum_{n\ge1} f(\varphi(n)) = \infty.\end{cases}$$

4. Proofs of the main results

Proof of Theorem 1.1. The convergence part is clear since

$$\mathcal{D}(x_0,\varphi) = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} B(T^n x_0,\varphi(n))\cap K \tag{4.1}$$

and

$$\sum_{n\ge1}\mu(B(T^n x_0,\varphi(n))) \approx \sum_{n\ge1}\varphi^\gamma(n). \tag{4.2}$$

For the divergence case, we apply Fubini's theorem. At first $\mathcal{S}(y,\varphi)$ is of full μ-measure for any $y\in\mathcal{K}$. Thus

$$\Big\{(x,y)\in\mathcal{K}^2 : |T^n x - y| < \varphi(n),\ \text{i.m. } n\in\mathbb{N}\Big\}$$

is full for the measure $\mu\times\mu$. Thus Fubini's theorem is applied to conclude that, for almost all $x_0\in\mathcal{K}$,

$$\mathcal{D}(x_0,\varphi) = \Big\{y\in\mathcal{K} : |T^n x_0 - y| < \varphi(n),\ \text{i.m. } n\in\mathbb{N}\Big\}$$

is of full μ-measure. $\square$

Proof of Theorem 1.2. For the convergence part, from (4.1), the f-Hausdorff measure of $\mathcal{D}(x_0,\varphi)$ can be estimated as

$$\mathcal{H}^f(\mathcal{D}(x_0,\varphi)) \le \liminf_{N\to\infty}\sum_{n\ge N} f(\varphi(n)) = 0.$$

For the divergence part, let $B_n = B(T^n x_0, \varphi(n))$. Then

$$B_n^f = B(T^n x_0, g^{-1} \circ f(\varphi(n))),$$

where $g(x) = x^\gamma$. By Theorem 1.1, we have shown that for almost all $x_0 \in \mathcal{K}$,

$$\mu(\limsup_{n\to\infty} B_n^f) = 1,$$

since

$$\sum_{n\geq 1} \left(g^{-1} \circ f(\varphi(n))\right)^\gamma = \sum_{n\geq 1} f(\varphi(n)) = \infty.$$

Then an application of the mass transference principle (Theorem 2.2) yields that

$$\mathcal{H}^f(\mathcal{D}(x_0, \varphi)) = \mathcal{H}^f(\mathcal{K}). \quad \square$$

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