Dynamical covering problems on the triadic Cantor set

Problèmes de recouvrement dynamique sur les ensembles de Cantor triadiques

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A R T I C L E   I N F O

Article history:
Received 27 November 2016
Accepted after revision 31 May 2017
Available online 28 June 2017

A B S T R A C T

In this note, we consider the metric theory of the dynamical covering problems on the triadic Cantor set $\mathcal{K}$. More precisely, let $T_x = 3x \pmod{1}$ be the natural map on $\mathcal{K}$, $\mu$ the standard Cantor measure and $x_0 \in \mathcal{K}$ a given point. We consider the size of the set of points in $\mathcal{K}$ which can be well approximated by the orbit $\{T^n x_0\}_{n \geq 1}$ of $x_0$, namely the set

$$
\mathcal{D}(x_0, \varphi) := \left\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N} \right\},
$$

where $\varphi$ is a positive function defined on $\mathbb{N}$. It is shown that for $\mu$ almost all $x_0 \in \mathcal{K}$, the Hausdorff measure of $\mathcal{D}(x_0, \varphi)$ is either zero or full depending upon the convergence or divergence of a certain series. Among the proof, as a byproduct, we obtain an inhomogeneous counterpart of Levesley, Salp and Velani’s work on a Mahler’s question about the Diophantine approximation on the Cantor set $\mathcal{K}$.

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R É S U M É

Nous considérons dans cette Note la théorie métrique des recouvrements dynamiques dans l’ensemble de Cantor triadique $\mathcal{K}$. Plus précisément, soit $T_x = 3x \pmod{1}$ l’application naturelle sur $\mathcal{K}$, $\mu$ la mesure de Cantor standard et $x_0 \in \mathcal{K}$ un point donné. Nous considérons la mesure de l’ensemble des points de $\mathcal{K}$ qui peuvent être bien approchés par l’orbite $\{T^n x_0\}_{n \geq 1}$ de $x_0$, c’est-à-dire l’ensemble

$$
\mathcal{D}(x_0, \varphi) := \left\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n) \text{ pour une infinité de } n \in \mathbb{N} \right\},
$$

où $\varphi$ est une fonction positive définie sur $\mathbb{N}$. Nous montrons que pour $\mu$-presque tout $x_0 \in \mathcal{K}$, la mesure de Hausdorff de $\mathcal{D}(x_0, \varphi)$ est soit zéro, soit pleine, selon la convergence ou la divergence d’une certaine série. Notre démonstration fournit en passant une contre-
1. Introduction

Assume that \((X, d)\) is a complete metric space. Let \(\{x_n\}_{n \geq 1}\) be a sequence of points in \(X\) and \(\{\ell_n\}_{n \geq 1}\) be a sequence of positive numbers. The covering problem concerns the limsup set

\[
\mathcal{D} := \left\{ y \in X : d(x_n, y) < \ell_n, \text{ i.m. } n \in \mathbb{N} \right\},
\]

i.e. the set of points that can be covered by the balls \(B(x_n, \ell_n)\) for infinitely many times. Here we use i.m. for infinitely many.

The well-known Borel–Cantelli lemma can be viewed as the first principle on the size of \(\mathcal{D}\) in the sense of measure. The size of \(\mathcal{D}\) was extensively studied when \(\{x_n\}\) is a sequence of independent and identically distributed random variables. One is referred to [7,11,16] for the history and [9,10] for recent progress.

Instead of a sequence of random variables, recently Fan, Schmeling and Troubetzkoy [8] introduced the question when \(\{x_n\}_{n \geq 1}\) is the orbit of a given point driven by a dynamical system. More precisely, let \((X, T)\) be a dynamical system and \(x_0\) be a given point, they considered the limsup set

\[
\left\{ y \in X : y \in B(T^n x_0, \ell_n), \text{ i.m. } n \in \mathbb{N} \right\},
\]

(1.1)

This is called the dynamical covering problem for its analogy with the classic covering problem. In [8], Fan, Schmeling and Troubetzkoy studied the set (1.1) defined by the doubling map on the unit interval. By using a general principle presented in [1], Liao and Seuret successfully enlarged the setting to finite expanding Markov systems [13].

It should be mentioned that the works [8,13] required that the phase space \(X\) is the whole unit interval. So, instead of the whole interval, in this note, we consider the same question in the simplest fractal: the triadic Cantor set \(\mathcal{K}\). More precisely, let \(Tx = 3x \mod 1\) and \(\varphi : \mathbb{N} \to \mathbb{R}^+\) be a positive function with \(\varphi(n) \to 0\) as \(n \to \infty\). We consider the size of the set

\[
\mathcal{D}(x_0, \varphi) := \left\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n), \text{ i.m. } n \in \mathbb{N} \right\},
\]

i.e. the set of points in \(\mathcal{K}\) which can be well approximated by the orbit of \(x_0\).

This can be viewed as a dynamical counterpart to a Mahler’s question [14] about the Diophantine approximation on the Cantor set \(\mathcal{K}\): how well can the points in \(\mathcal{K}\) be approximated by rationals? See [4,5,12] for partial progress on Mahler’s question.

In this note, we give a complete characterization of the size of \(\mathcal{D}(x_0, \varphi)\) for almost all \(x_0\) with respect to the standard Cantor measure \(\mu\). Before the statement of our main result, we fix some notation:

- dimension function: an increasing continuous function \(f : (0, \infty) \to (0, \infty)\) such that \(\lim_{x \to 0} f(x) = 0\);
- \(\mathcal{H}^f\) : \(f\)-Hausdorff measure for a dimension function \(f\);
- \(\gamma = \log 2/\log 3\) : the Hausdorff dimension of the Cantor set \(\mathcal{K}\).

**Theorem 1.1.** Let \(\varphi : \mathbb{N} \to \mathbb{R}^+\). For \(\mu\)-almost all \(x_0 \in \mathcal{K}\),

\[
\mu(\mathcal{D}(x_0, \varphi)) = \begin{cases} 
0, & \text{if } \sum_{n \geq 1} \varphi'(n) < \infty; \\
1, & \text{if } \sum_{n \geq 1} \varphi'(n) = \infty.
\end{cases}
\]

**Theorem 1.2.** Let \(f\) be a dimension function with \(f(x)/x^\gamma\) being increasing as \(x \to 0\). For \(\mu\)-almost all \(x_0 \in \mathcal{K}\), the Hausdorff measure of the set \(\mathcal{D}(\varphi, x_0)\) satisfies a dichotomy law

\[
\mathcal{H}^f(\mathcal{D}(x_0, \varphi)) = \begin{cases} 
0, & \text{if } \sum_{n \geq 1} f(\varphi(n)) < \infty; \\
\mathcal{H}^f(\mathcal{K}), & \text{if } \sum_{n \geq 1} f(\varphi(n)) = \infty.
\end{cases}
\]

Consequently, the dimension of \(\mathcal{D}(x_0, \varphi)\) is given by the convergence exponent of \(\varphi\), i.e.

\[
\inf \left\{ s \geq 0 : \sum_{n \geq 1} \varphi(n)^s < \infty \right\}.
\]
Among the proof, as a byproduct, we can obtain an inhomogeneous counterpart of Levesley, Salp and Velani’s work [12] on the above-mentioned Mahler’s question about the Diophantine approximation on $K$ (see Theorem 3.3 below).

It should also be mentioned that in [8,13], they considered multifractal measures, while in this work, we only pay attention to a monofractal measure $\mu$.

2. Preliminaries

In this section, we cite two known results, one for the measure of a limsup set, the other for the Hausdorff measure of a shrunk limsup set, which is often referred to as the “mass transference principle”.

2.1. Measure of a limsup set

Let $(X, B, \mu)$ be a measure space. Let $\{A_n\}_{n\geq 1}$ be a sequence of measurable sets. Define the limsup set

$$E = \limsup_{n \to \infty} A_n = \left\{ x \in X : x \in A_n, \text{ i.m. } n \in \mathbb{N} \right\}.$$ 

The following is a widely used result to determine the measure of $E$ from below (see [17, Lemma 5]).

**Lemma 2.1.** Assume that $\sum_{n \geq 1} \mu(A_n) = \infty$. Then

$$\mu(E) \geq \limsup_{N \to \infty} \frac{\left( \sum_{n=1}^{N} \mu(A_n) \right)^2}{\sum_{1 \leq m, n \leq N} \mu(A_m \cap A_n)}.$$ 

2.2. Mass transference principle

The mass transference principle, developed by Beresnevich & Velani [3] (see also [2]), is a very powerful tool to determine the Hausdorff measure and dimension of a limsup set. In principle, it says that a full measure statement for a limsup set will imply a full Hausdorff measure theoretic statement for the shrunk limsup set.

Let $(X, d)$ be a locally compact metric space. Let $g$ be a doubling dimension function, i.e. there exists $\lambda \geq 1$ such that, for all small $r$,

$$g(2r) \leq \lambda g(r). \quad (2.1)$$

Suppose that there exist $0 < c_1 \leq c_2 < \infty$ and $r_0 > 0$ such that, for all $x \in X$ and $0 < r < r_0$,

$$c_1 g(r) \leq \mathcal{H}^\delta(B(x, r)) \leq c_2 g(r). \quad (2.2)$$

Write $B^f(x, r)$ for the ball $B(x, g^{-1} \circ f(r))$.

**Theorem 2.2 (Mass transference principle [3]).** Let $(X, d)$ and $g$ be as above and let $\{B_i\}_{i \in \mathbb{N}}$ be a sequence of balls in $X$ with $r(B_i) \to 0$ as $i \to \infty$. Let $f$ be a dimension function such that $f(x)/g(x)$ is monotonically increasing as $x \to 0$. Suppose that, for any ball $B$ in $X$,

$$\mathcal{H}^\delta \left( B \cap \limsup_{i \to \infty} B_i^f \right) = \mathcal{H}^\delta(B). \quad (2.3)$$

Then, for any ball $B$ in $X$,

$$\mathcal{H}^f \left( B \cap \limsup_{i \to \infty} B_i \right) = \mathcal{H}^f(B).$$

In the latter application, we will take $X$ to be the Cantor set $K$ and the doubling dimension function $g(x) = x^\gamma$. The Cantor measure $\mu$ on $K$ is the same as $\mathcal{H}^\delta|_K$. So, the formula (2.2) is fulfilled, i.e.

$$\mu(B(x, r)) \approx r^\gamma. \quad (2.4)$$

The condition (2.3) just says that $\limsup_{i \to \infty} B_i^f$ is full in the sense of $\mathcal{H}^\delta|_K$, so to apply Theorem 2.2, we are required to determine the $\mu$-measure of $\limsup_{i \to \infty} B_i^f$. This is the task of the next section.
3. A dynamical Borel–Cantelli lemma on $\mathcal{K}$

The proof of our result is just a combination of a dynamical Borel–Cantelli lemma, analogous to the one proved by Philipp [15] for $b$-adic expansion, and the mass transference principle (Theorem 2.2). In this section, we show this Borel–Cantelli lemma. For general results about dynamical Borel–Cantelli lemmas, one is referred to Chernov & Kleinbock [6] and references therein.

For each $y \in \mathcal{K}$, let

$$S(y, \varphi) = \left\{ x \in \mathcal{K} : |T^n x - y| < \varphi(n), \text{ i.m. } n \in \mathbb{N} \right\}.$$ 

**Proposition 3.1.** Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ and $y \in \mathcal{K}$. Then

$$\mu(S(y, \varphi)) = \begin{cases} 0, & \text{if } \sum_{n \geq 1} \varphi^{-1}(n) < \infty; \\ 1, & \text{if } \sum_{n \geq 1} \varphi^{-1}(n) = \infty. \end{cases}$$

Before the proof, we begin with a lemma.

**Lemma 3.2.** Let $E$ be an interval and $F$ a measurable set. For any $n \geq 1$,

$$\mu(E \cap T^{-n}F) = \mu(E) \cdot \mu(F) + \mu(F)O(2^{-n}),$$

where the constant implied in $O$ can be taken to be 2.

**Proof.** It suffices to consider the case when $E = [0, t]$. We define the cylinder set. For each $(\epsilon_1, \cdots, \epsilon_n) \in [0, 2]^n$, call

$$l_n(\epsilon_1, \cdots, \epsilon_n) = \left\{ x \in [0, 1] : x \text{ has a triadic expansion beginning with } (\epsilon_1, \cdots, \epsilon_n) \right\}$$

a cylinder of order $n$ (with respect to the Cantor set $\mathcal{K}$), which is of $\mu$-measure $2^{-n}$.

Let $\{I_n^{(i)} : 1 \leq i \leq \ell + 1\}$ be the cylinders of order $n$ with non-empty intersections with $E$. If we arrange them in increasing order, we have that

$$\bigcup_{i=1}^{\ell} I_n^{(i)} \cap \mathcal{K} \subset (E \cap \mathcal{K}), \text{ and } (E \cap \mathcal{K}) \subset \bigcup_{i=1}^{\ell+1} I_n^{(i)}.$$ 

Then it follows that

$$\frac{\ell}{2^n} \leq \mu(E) \leq \frac{\ell + 1}{2^n},$$

$$\mu(E \cap T^{-n}F) \leq \sum_{i=1}^{\ell+1} \mu(T^{-n}F \cap I_n^{(i)}) \leq \frac{\ell + 1}{2^n} \mu(F),$$

$$\mu(E \cap T^{-n}F) \geq \sum_{i=1}^{\ell} \mu(T^{-n}F \cap I_n^{(i)}) \geq \frac{\ell}{2^n} \mu(F).$$

Substituting the measure of $E$ into the estimation on $\mu(E \cap T^{-n}F)$, we have that

$$|\mu(E \cap T^{-n}F) - \mu(E)\mu(F)| \leq \frac{1}{2^n} \mu(F). \quad \Box$$

**Proof of Proposition 3.1.** Equipped with the fast decaying of correlations (Lemma 3.2), Proposition 3.1 can be inferred from the general principle in [6]. Here we include a proof for its easiness.

For each $n \geq 1$, let $A_n = T^{-n}B(y, \varphi(n)) \cap \mathcal{K}$. Then

$$S(y, \varphi) = \limsup_{n \to \infty} A_n.$$ 

By the invariance of $\mu$ with respect to $T$ and (2.4),

$$\sum_{n=1}^{\infty} \mu(T^{-n}B(y, \varphi(n)) \cap \mathcal{K}) = \sum_{n=1}^{\infty} \mu(B(y, \varphi(n))) \approx \sum_{n=1}^{\infty} \varphi^{1}(n).$$
The convergence case follows from the convergence part of the Borel–Cantelli lemma. For the divergence part, we apply Lemma 2.1. So, we estimate the summation:
\[
\sum_{1 \leq m,n \leq N} \mu(A_m \cap A_n) = 2 \sum_{n=1}^{N} \sum_{m=1}^{n-1} \mu(A_m \cap A_n) + \sum_{n=1}^{N} \mu(A_n).
\]

By Lemma 3.2, it follows that, for \( n > m, \)
\[
\mu(A_m \cap A_n) = \mu \left( B(y, \varphi(m)) \cap T^{-(n-m)} B(y, \varphi(n)) \right)
= \mu(A_m) \mu(A_n) + O(2^{-n+m}) \mu(A_n),
\]
where the invariance of \( \mu \) with respect to \( T \) is used twice. Thus
\[
2 \sum_{n=1}^{N} \sum_{m=1}^{n-1} \mu(A_m \cap A_n) = 2 \sum_{n=1}^{N} \sum_{m=1}^{n-1} \left( \mu(A_m) \mu(A_n) + O \left( \frac{1}{2^{n-m}} \right) \mu(A_n) \right)
\leq \left( \sum_{n=1}^{N} \mu(A_n) \right)^2 + O(1) \sum_{n=1}^{N} \mu(A_n).
\]

So, it follows by Lemma 2.1 that
\[
\mu(\mathcal{S}(y, \varphi)) \geq 1. \quad \Box
\]

In [12], as an attempt on Mahler’s question on \( \mathcal{K}, \) Levesley, Salp and Velani proved a 0–1 law on the size of the set
\[
W(\varphi) := \left\{ x \in \mathbb{K} : |x - p/3^n| < \varphi(n), \text{ i.m. } n \in \mathbb{N} \right\}.
\]
This set is the same as \( \mathcal{S}(0, \varphi), \) So, Proposition 3.1 gives an inhomogeneous counterpart of Levesley, Salp and Velani’s work. Combining Proposition 3.1 and the mass transference principle, we obtain the following theorem.

Theorem 3.3. Let \( f \) be a dimension function with \( f(x)/x^\gamma \) being increasing as \( x \to 0. \) For any \( y \in \mathbb{K}, \) we have that
\[
\mathcal{H}^f(\mathcal{S}(y, \varphi)) = \begin{cases} 0, & \text{if } \sum_{n \geq 1} f(\varphi(n)) < \infty; \\ \mathcal{H}^f(\mathcal{K}), & \text{if } \sum_{n \geq 1} f(\varphi(n)) = \infty. \end{cases}
\]

4. Proofs of the main results

Proof of Theorem 1.1. The convergence part is clear since
\[
\mathcal{D}(x_0, \varphi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B(T^n x_0, \varphi(n)) \cap \mathcal{K}
\quad (4.1)
\]
and
\[
\sum_{n \geq 1} \mu(B(T^n x_0, \varphi(n))) \approx \sum_{n \geq 1} \varphi^\gamma(n). \quad (4.2)
\]

For the divergence case, we apply Fubini’s theorem. At first \( \mathcal{S}(y, \varphi) \) is of full \( \mu \)-measure for any \( y \in \mathcal{K}. \) Thus
\[
\left\{ (x, y) \in \mathbb{K}^2 : |T^n x - y| < \varphi(n), \text{ i.m. } n \in \mathbb{N} \right\}
\]
is full for the measure \( \mu \times \mu. \) Thus Fubini’s theorem is applied to conclude that, for almost all \( x_0 \in \mathcal{K}, \)
\[
\mathcal{D}(x_0, \varphi) = \left\{ y \in \mathcal{K} : |T^n x_0 - y| < \varphi(n), \text{ i.m. } n \in \mathbb{N} \right\}
\]
is of full \( \mu \)-measure. \( \Box \)

Proof of Theorem 1.2. For the convergence part, from (4.1), the \( f \)-Hausdorff measure of \( \mathcal{D}(x_0, \varphi) \) can be estimated as
\[
\mathcal{H}^f(\mathcal{D}(x_0, \varphi)) \leq \liminf_{N \to \infty} \sum_{n \geq N} f(\varphi(n)) = 0.
\]
For the divergence part, let $B_n = B(T^n x_0, \varphi(n))$. Then
\[ B_n^f = B(T^n x_0, g^{-1} \circ f(\varphi(n))), \]
where $g(x) = x^\nu$. By Theorem 1.1, we have shown that for almost all $x_0 \in \mathcal{K}$,
\[ \mu(\limsup_{n \to \infty} B_n^f) = 1, \]
since
\[ \sum_{n \geq 1} \left( g^{-1} \circ f(\varphi(n)) \right)^\nu = \sum_{n \geq 1} f(\varphi(n)) = \infty. \]

Then an application of the mass transference principle (Theorem 2.2) yields that
\[ \mathcal{H}^f(D(x_0, \varphi)) = \mathcal{H}^f(\mathcal{K}). \]

Acknowledgements

This work is supported by NSFC (grant Nos. 11225101, 11471130, 11571127) and NCET-13-0236.

References