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A theorem of uniqueness for characteristic functions

Théorème d'unicité pour les fonctions caractéristiques

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ABSTRACT

We discuss a uniqueness property of the characteristic function of an absolutely continuous probability measure. Our study is initiated by the question posed by N.G. Ushakov: is it true that, for any interval $[a,b] \subset \mathbb{R}$, $0 \notin [a,b]$, there exists a characteristic function f such that $f \not\equiv e^{-t^2/2}$, but $f(t) = e^{-t^2/2}$ for all $t \in [a,b]$?

RÉSUMÉ

Nous discutons d'une propriété d'unicité pour : les fonctions caractéristiques des mesures de probabilité. Notre étude tire son origine dans la question de N.G. Ushakov : étant donné $[a,b] \subset \mathbb{R}$, $0 \notin [a,b]$, est-il vrai qu'il existe une fonction caractéristique f telle que $f \not\equiv e^{-t^2/2}$, mais vérifiant $f(t) = e^{-t^2/2}$ pour $t \in [a,b]$?

1. Introduction

Let $M(\mathbb{R})$ denote the Banach algebra of bounded regular complex-valued Borel measures on $\mathbb{R}$ with the total variation norm $\|\mu\|$. In this paper, we define the Fourier–Stieltjes transform of $\mu \in M(\mathbb{R})$ by

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{-itx} d\mu(x), \quad t \in \mathbb{R}.$$

The Lebesgue space $L^1(\mathbb{R})$ can be identified with the closed ideal in $M(\mathbb{R})$ of measures absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}$. Namely, if $\varphi \in L^1(\mathbb{R})$, then φ is associated with the measure

$$\mu_\varphi(E) = \int_E \varphi(x)\, dx$$

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for each Borel subset E of $\mathbb{R}$. Hence, $\widehat{\varphi}(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx}\varphi(x)\,\mathrm{d}x$. We normalize the inverse Fourier transform

$$\check{\varphi}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xt}\varphi(t)\,\mathrm{d}t$$

so that the formula $\widehat{(\check{\varphi})} = \varphi$ is true for suitable $\varphi \in L^1(\mathbb{R})$.

Suppose that $\mu \in M(\mathbb{R})$ is a positive measure such that $\|\mu\| = 1$. Then, in the language of probability theory, μ and $f(t) := \hat{\mu}(t)$, $t \in \mathbb{R}$, are called a probability measure and its characteristic function, respectively. In particular, if $\mu = \varphi(x)\mathrm{d}x$, where $\varphi \in L^1(\mathbb{R})$ and φ is such that $\|\varphi\|_{L^1(\mathbb{R})} = 1$ and $\varphi \geq 0$ on $\mathbb{R}$, then φ is called the probability density function of μ, or the probability density for short.

Next we discuss a uniqueness property of the characteristic function of a probability measures with continuous densities. Our study is initiated by the question posed by Ushakov and Ushakov [7, p. 275]:

$$\text{Is it true that for any interval, which does not contain the origin, there exists a}$$
$$\text{characteristic function } f(t) \text{ such that } f(t) = \mathrm{e}^{-t^2/2} \text{ on this interval but } f(t) \not\equiv \mathrm{e}^{-t^2/2}? \tag{1}$$

Throughout the following, for $U \subset \mathbb{R}$, we shall write $f = g$ on U if $f(t) = g(t)$ for $t \in U$. If there $U = \mathbb{R}$, then we shall write $f \equiv g$.

Assume that f is the characteristic function such that it coincides with $\mathrm{e}^{-t^2/2}$ on (a, b), $a < b$, $(a, b) \neq \mathbb{R}$ but $f \not\equiv \mathrm{e}^{-t^2/2}$. Then (a, b) cannot contain the origin. This is an immediate consequence of the usual uniqueness theorem for an analytic function and the following theorem (see [5, p. 193]): if the characteristic function f can be continued in a complex neighborhood of origin to an analytic function, then f is analytic in a certain horizontal strip

$$H_a = \{\lambda \in \mathbb{C} : |\Im\lambda| < a\}, \quad a > 0. \tag{2}$$

In this paper, we study the question (1) in the more general case with the characteristic function $f = \widehat{\varphi}$ of an arbitrary continuous probability density φ, instead of $\mathrm{e}^{-t^2/2}$.

Given a continuous function $\varphi : \mathbb{R} \to \mathbb{C}$, we define its essential support to be

$$S_\varphi = \{x \in \mathbb{R} : |\varphi(x)| > 0\}.$$

Then $N_\varphi = \mathbb{R} \setminus S_\varphi$ is the zero set of φ. Throughout the following, for $\sigma > 0$,

$$U_\sigma = \mathbb{R} \setminus (-\sigma, \sigma),$$

denotes a neighborhood of infinity.

An affirmative answer to (1) comes from the following theorem.

Theorem 1.1. *(see [6, Theorem 1.2]) Let $f : \mathbb{R} \to \mathbb{C}$ be the characteristic function of a continuous probability density φ. Assume that $\alpha \in \mathbb{R}$ and $d > 0$ are such that $(\alpha, \alpha + 2d) \subset S_\varphi$. If*

$$\sigma d \geq 2\sqrt{3},$$

then there is a characteristic function g such that $f = g$ on U_σ but $f \not\equiv g$.

The next corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2. *For any characteristic function $f : \mathbb{R} \to \mathbb{C}$ of a continuous probability density, there exist the characteristic function g and a neighborhood of infinity U_σ such that $f = g$ on U_σ but $f \not\equiv g$.*

Although the answer to (1) is positive, some examples show that is not true in general (see Theorem 1.3 below). It turns out that for certain characteristic functions and some related neighborhoods of infinity, appropriate uniqueness theorems can be true.

Let $\mathbb{Z}$ denote the set of integers. We state the main result of this paper.

Theorem 1.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be the characteristic function of a continuous probability density φ. Suppose that there exist lattices $\Lambda_j = \tau_j + \alpha_j\mathbb{Z}$, $\tau_j \in \mathbb{R}$, $\alpha_j > 0$, $j = 1, 2$, such that*

$$\Lambda_1, \Lambda_2 \subset N_\varphi \tag{3}$$

and

$$\Lambda_1 \cap \Lambda_2 = \emptyset. \tag{4}$$

Assume that $g : \mathbb{R} \to \mathbb{C}$ is the characteristic function of a probability measure such that $f = g$ on U_σ. If

$$\sigma\alpha_j \le 2\pi \tag{5}$$

for $j = 1, 2$, then $f \equiv g$.

Theorem 1.3 gives us an idea that, in such uniqueness theorems, the significant role is not played only by how big both S_φ and U_φ are, but also by an arithmetic structure of S_φ.
The following proposition shows that the statement of Theorem 1.3 is exact (in some sense).

Proposition 1.4. *Let $\alpha_1, \alpha_2 > 0$ be given. Assume that*

$$\sigma\alpha_j > 2\pi \tag{6}$$

for $j = 1, 2$. Then there are $\tau_1, \tau_2 \in \mathbb{R}$ such that:

(i) *lattices $\Lambda_j = \tau_j + \alpha_j\mathbb{Z}$, $j = 1, 2$, satisfy (4);*
(ii) *to any characteristic function f of a continuous probability density φ with $N_\varphi = \Lambda_1 \cup \Lambda_2$, there exists the characteristic function g such that $f = g$ on U_σ, but $f \not\equiv g$.*

2. Preliminaries and proofs

Let Ω be a closed subset of $\mathbb{R}$. A function $\omega \in L^p(\mathbb{R})$, $1 \le p \le \infty$, is called bandlimited to Ω if $\widehat{\omega}$ vanishes outside Ω. If $2 < p \le \infty$, then we understand the Fourier transform of $\omega \in L^p(\mathbb{R})$ in a distributional sense of the Schwartz space $S'(\mathbb{R})$.
For $\sigma > 0$, we denote by B^p_σ the Bernstein space of all $F \in L^p(\mathbb{R})$ such that F is bandlimited to $[-\sigma, \sigma]$. The space B^p_σ is equipped with the norm

$$\|F\|_p = \Big(\int_{\mathbb{R}} |F(x)|^p \,\mathrm{d}x\Big)^{1/p} \quad \text{for} \quad 1 \le p < \infty \qquad \text{and} \qquad \|F\|_\infty = \operatorname{ess\,supp}_{x\in\mathbb{R}} |F(x)|.$$

By the Paley–Wiener–Schwartz theorem (see [3, p. 68]), any $F \in B^p_\sigma$ is infinitely differentiable on $\mathbb{R}$. Even more is true: this F has an extension onto the complex plane $\mathbb{C}$ to an entire function of exponential type at most σ. That is, for every $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ such that

$$|F(z)| \le M\mathrm{e}^{(\sigma+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$ (see, e.g., [4, p. 149]). Any $F \in B^p_\sigma$ satisfies the Plancherel–Polya inequality [4, p. 150]

$$\int_{\mathbb{R}} |F(x + \mathrm{i}y)|^p \,\mathrm{d}x \le \mathrm{e}^{p\sigma|y|}\|F\|^p_p \quad \text{for} \quad 1 \le p < \infty, \tag{7}$$

where $y \in \mathbb{R}$, and

$$|F(x + \mathrm{i}y)| \le \mathrm{e}^{\sigma|y|}\|F\|_\infty \quad \text{for} \quad p = \infty. \tag{8}$$

Note that $1 \le p \le r \le \infty$ implies ([3, p. 49], Lemma 6.6)

$$B^1_\sigma \subset B^p_\sigma \subset B^r_\sigma \subset B^\infty_\sigma. \tag{9}$$

For $F \in B^1_\sigma$ with $x \in \mathbb{R}$ and $a > 0$, the Poisson summation formula reads (see, e.g., [2, p. 63] and [1, p. 509])

$$\sum_{n\in\mathbb{Z}} F(x + an) = \frac{1}{a}\sum_{k\in\mathbb{Z}} \widehat{F}\Big(\frac{2\pi}{a}k\Big)\mathrm{e}^{\mathrm{i}\frac{2\pi x}{a}k}, \tag{10}$$

where both sums converge absolutely.

Proof of Theorem 1.3. By Bochner's theorem, there exists a probability measure $\mu \in M(\mathbb{R})$ such that $g = \hat{\mu}$. Under the conditions of our theorem, it follows that $f - g$ is supported on $[-\sigma, \sigma]$. Then $(f - g) \in L^1(\mathbb{R})$. This yields that $\varphi - \mu = (\check{f} - \check{g}) \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the usual Banach space of complex-valued continuous functions on $\mathbb{R}$ that vanish at infinity. We see that μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Therefore, we can identify $\check{g}$ with some continuous probability density $\psi \in L^1(\mathbb{R})$. By definition, put

$$F = \varphi - \psi. \tag{11}$$

Then $F \in B^1_\sigma$, since $\widehat{\varphi - \psi}$ is supported on $[-\sigma, \sigma]$. Moreover, (11) implies that

$$F(x) \le \varphi(x) \tag{12}$$

for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} F(x)\,\mathrm{d}x = 0. \tag{13}$$

Our strategy is to prove that if an arbitrary $F \in B^1_\sigma$ satisfies (12) and (13), then $F \equiv 0$. We do this in three steps. First, we show that $F = 0$ on lattices Λ_1 and Λ_2, i.e.

$$F(\tau_j + \alpha_j n) = 0 \tag{14}$$

for $j = 1, 2$ and all $n \in \mathbb{Z}$. Indeed, combining (3) and (12), we see that

$$F(\tau_j + \alpha_j n) \le 0, \tag{15}$$

$j = 1, 2$, $n \in \mathbb{Z}$. On the other hand, using the Poisson formula (10), we get

$$\sum_{n\in\mathbb{Z}} F(\tau_j + \alpha_j n) = \frac{1}{\alpha_j}\sum_{k\in\mathbb{Z}} \hat{F}\left(\frac{2\pi}{\alpha_j}k\right)\mathrm{e}^{\mathrm{i}\frac{2\pi\tau_j}{\alpha_j}k}, \tag{16}$$

$j = 1, 2$. Under the conditions (5), it follows that $1/\alpha_j \ge \sigma/(2\pi)$, $j = 1, 2$. Then $|2\pi k/\alpha_j| \ge |\sigma k|$ for $j = 1, 2$ and all $k \in \mathbb{Z}$. Hence

$$\hat{F}\left(\frac{2\pi}{\alpha_j}k\right) = 0$$

$k \in \mathbb{Z}$, $k \ne 0$, since $\hat{F}$ is a continuous function on $\mathbb{R}$ and it is supported on $[-\sigma, \sigma]$. Now (16) implies that

$$\sum_{n\in\mathbb{Z}} F(\tau_j + \alpha_j n) = \frac{\hat{F}(0)}{\alpha_j} = \frac{1}{\alpha_j}\int_{\mathbb{R}} F(x)\,\mathrm{d}x,$$

$j = 1, 2$. Finally, combining this with (13) and (15), we get (14).

The second step is to show that if $\xi \in B^1_\sigma$ and

$$\xi(\tau + \alpha n) = 0, \tag{17}$$

for all $n \in \mathbb{Z}$ and certain $\alpha, \tau \in \mathbb{R}$ such that $\alpha \ge \pi/\sigma$, then the function

$$U(z) = \frac{\xi(z)}{\sin(\pi(z-\tau)/\alpha)} \tag{18}$$

is entire and belongs to $B^1_{\sigma_1}$, with $\sigma_1 = \sigma - \pi/\alpha > 0$. Indeed, the function (18) is entire, since both ξ and $\sin(\pi(z-\tau)/\alpha)$ are entire functions and (17) implies that ξ vanishes on the zeros set $\{\tau + \alpha n : n \in \mathbb{Z}\}$ of $\sin(\pi(z-\tau)/\alpha)$. By (9), we can consider $\xi \in B^1_\sigma$ also as an element of B^∞_σ. Therefore, if $z = x + \mathrm{i}y$ with $y \ne 0$, then applying (8) to ξ in (18), we get

$$\begin{aligned}|U(z)| &\le \frac{2\|\xi\|_{L^\infty(\mathbb{R})}\mathrm{e}^{\sigma|y|}}{|\mathrm{e}^{(\pi/\alpha)y} - \mathrm{e}^{-(\pi/\alpha)y}\mathrm{e}^{\mathrm{i}2\pi(x-\tau)/\alpha}|} \le \frac{2\|\xi\|_{L^\infty(\mathbb{R})}\mathrm{e}^{\sigma|y|}}{|\mathrm{e}^{(\pi/\alpha)y} - \mathrm{e}^{-(\pi/\alpha)y}|} \\ &= 2\|\xi\|_{L^\infty(\mathbb{R})}\mathrm{e}^{(\sigma-\pi/\alpha)|y|}\frac{1}{|1-\mathrm{e}^{(-2\pi/\alpha)|y|}|}.\end{aligned} \tag{19}$$

This means that U is bounded on the boundary of any strip (2). Then, by the maximum principle for analytic functions, we have that U is bounded by the same constant on the whole strip (2). Combining this with (19), we can conclude that there exists $0 < A < \infty$ such that

$$|U(x + \mathrm{i}y)| \le A\mathrm{e}^{(\sigma-\pi/\alpha)|y|}$$

for all $x, y \in \mathbb{R}$. Therefore, we have that $U \in B^\infty_{\sigma_1}$ with $\sigma_1 = \sigma - \pi/\alpha$.

Now we will show that $U \in B^1_{\sigma_1}$. For $a \in \mathbb{R}$, $a \ne 0$, let us define $U_a(z) = U(z + \mathrm{i}a)$, $z \in \mathbb{C}$. By the definition (18) of U, and keeping in mind that $\xi \in B^1_\sigma$, we can use (7) for ξ and $p = 1$. This gives $U_a \in B^1_\sigma$. Next we apply the estimate (7) to U_a with $p = 1$ and $y = -a$. In this way, we see that $U \in B^1_{\sigma_1}$ as claimed.

In the final step, we show that F defined in (11) is the zero function on $\mathbb{R}$. To this end, let us define the function

$$T(z) = \frac{F(z)}{\sin\Big(\pi(z-\tau_1)/\alpha_1\Big)\sin\Big(\pi(z-\tau_2)/\alpha_2\Big)}.$$

Now if we recall (14) and what was proved in the previous step for the function (18), we can state that $T \in B^1_\theta$ with

$$\theta = \sigma - \left(\frac{\pi}{\alpha_1} + \frac{\pi}{\alpha_2}\right).$$

On the other hand, the conditions (5) imply that T is an entire function of order 1 and no more than minimal type. Then T is a constant function (see [4, p. 39]). Furthermore, $T \equiv 0$, since $T \in B^1_\theta$ and so $T \in L^1(\mathbb{R})$. This proves Theorem 1.3. □

Proof of Proposition 1.4. Let $I := \left[0, 2\pi/\sigma\right]$. Having in mind (6), it is easy to see that there exist $\tau_1, \tau_2 < 0$ such that $\Lambda_1 = \tau_1 + \alpha_1\mathbb{Z}$ and $\Lambda_2 = \tau_2 + \alpha_2\mathbb{Z}$ satisfy (4) and the additional condition

$$\Lambda_j \cap I = \emptyset. \tag{20}$$

Suppose that φ is an arbitrary continuous probability density such that $N_\varphi = \Lambda_1 \cup \Lambda_2$. By f denote the characteristic function of φ. Using (20), we get

$$a = \min_{x\in I} \varphi(x) > 0. \tag{21}$$

For this number a, we define two functions

$$F_1(x) = \frac{a\pi^2}{\sigma}\frac{\sin\left(\frac{\sigma x}{2}\right)}{x(2\pi - \sigma x)} \quad \text{and} \quad F(x) = F_1(x)\sin\frac{\sigma x}{2}.$$

It is clear that $F_1 \in B^1_{\sigma/2}$ and $F \in B^1_\sigma$. Next, it is easy to see that F is nonpositive on $\mathbb{R} \setminus I$ and F attains its maximum value on I at $x = \pi/\sigma$. Since $F(\pi/\sigma) = F_1(\pi/\sigma) = a$, we may conclude from (21) that F satisfies (12).

We claim that F satisfies also (13). Indeed, $\widehat{F_1}$ and $\widehat{F}$ are continuous functions on $\mathbb{R}$, since $F_1 \in B^1_{\sigma/2}$ and $F \in B^1_\sigma$. In addition, they are supported on $[-\sigma/2, \sigma/2]$ and on $[-\sigma, \sigma]$, respectively. Therefore,

$$\widehat{F_1}\left(-\frac{\sigma}{2}\right) = \widehat{F_1}\left(\frac{\sigma}{2}\right) = 0. \tag{22}$$

Next, a direct computation shows that

$$\widehat{F}(t) = \left(\widehat{\sin\frac{\sigma x}{2}}\right) * \widehat{F_1}(t) = \frac{\mathrm{i}}{2}\left(\delta_{\sigma/2} - \delta_{-\sigma/2}\right) * \widehat{F_1}(t) = \frac{\mathrm{i}}{2}\left(\widehat{F_1}\left(t - \frac{\sigma}{2}\right) - \widehat{F_1}\left(t + \frac{\sigma}{2}\right)\right).$$

Combining this and (22), we get

$$\int_{\mathbb{R}} F(x)\,\mathrm{d}x = \widehat{F}(0) = 0$$

as claimed.

If we now define $\psi := \varphi - F$, then (12) and (13) imply that ψ is a continuous probability density. Finally, if we denote $\widehat{\psi}$ by g, then for any $t \in U_\sigma$,

$$g(t) = \widehat{\psi}(t) = \widehat{\varphi}(t) - \widehat{F}(t) = \widehat{\varphi}(t) = f(t),$$

since $F \in B^1_\sigma$. On the other hand, $f - g = \widehat{F} \not\equiv 0$. This proves Proposition 1.4. □

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