Differential geometry/Mathematical physics

A proof of energy gap for Yang–Mills connections

Une preuve du gap d’énergie pour les connexions de Yang–Mills

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A B S T R A C T

In this note, we prove an $L^2$-energy gap result for Yang–Mills connections on a principal $G$-bundle over a compact manifold without using the Lojasiewicz–Simon gradient inequality ([2] Theorem 1.1). © 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Dans cette note, nous démontrons un résultat concernant le gap d’énergie $L^2$ pour les connexions de Yang–Mills sur un fibré principal de groupe structural $G$ sur une variété compacte, sans utiliser l’inégalité du gradient de Lojasiewicz–Simon. © 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Let $X$ be a compact $n$-dimensional Riemannian manifold endowed with a smooth Riemannian metric $g$. $P \to X$ a principal $G$-bundle over $X$, where $G$ is a compact Lie group. We define the Yang–Mills functional by

$$YM(A) = \int_X |F_A|^2 \text{dvol}_g,$$

where $A$ is a $C^\infty$-connection on $P$ and $F_A$ is the curvature of $A$.

A connection $A$ on $P$ is called a Yang–Mills connection if it is a critical point of $YM$, i.e. it obeys the Yang–Mills equation with respect to the metric $g$:

$$d_A^* F_A = 0. \quad (1.1)$$

In [2], Feehan proved an $L^2$-energy gap result for Yang–Mills connections on the principal $G$-bundle $P$ over an arbitrary closed smooth Riemannian manifold with dimension $n \geq 2$ ([2] Theorem 1.1). Feehan applied the Lojasiewicz–Simon gradient...
inequality (2) Theorem 3.2) to remove a positivity hypothesis on the Riemannian curvature tensors in a previous $L^2$-energy gap result due to Gerhardt [3] (Theorem 1.2).

In this note, we give another proof of this $L^2$-energy gap result of Yang–Mills connection without using the Lojasiewicz–Simon gradient inequality.

**Theorem 11.** (2) Theorem 1.1) Let $X$ be a compact Riemannian manifold without boundary of dimension $n \geq 2$ endowed with a smooth Riemannian metric $g$, $P$ be a $G$-bundle over $X$. Then, either any smooth Yang–Mills connection $A$ over $X$ with compact Lie group $G$ satisfies

$$\int_X |F_A|^2 \, d\text{vol}_g \geq C_0$$

for a constant $C_0 > 0$ depending only on $X, n, G$, or the connection $A$ is flat.

2. Preliminaries and basic estimates

We shall generally adhere to the now standard gauge-theory conventions and notation of Donaldson and Kronheimer [1] and Feehan [2]. Throughout our article, $G$ denotes a compact Lie group and $P$ a smooth principal $G$-bundle over a compact Riemannian manifold $X$ of dimension $n \geq 2$ endowed with a Riemannian metric $g$, $\mathfrak{g}_P$ denote the adjoint bundle of $P$, endowed with a $G$-invariant inner product and $\Omega^p(X, \mathfrak{g}_P)$ denote the smooth $p$-forms with values in $\mathfrak{g}_P$. Given a connection on $P$, we denote by $\nabla_A$ the corresponding covariant derivative on $\Omega^p(X, \mathfrak{g}_P)$ induced by $A$ and the Levi-Civita connection of $X$. Let $d_A$ denote the exterior derivative associated with $\nabla_A$.

For $u \in L^p(X, \mathfrak{g}_P)$, where $1 \leq p < \infty$ and $k$ is an integer, we denote

$$\|u\|_{L^p_k(X)} := \left( \sum_{j=0}^k \int_X |\nabla^j_A u|^p \, d\text{vol}_g \right)^{1/p},$$

where $\nabla^j_A := \nabla \circ \ldots \circ \nabla_A$ (repeated $j$ times for $j \geq 0$). For $p = \infty$, we denote

$$\|u\|_{L^\infty_k(X)} := \sum_{j=0}^k \text{ess sup}_X |\nabla^j_A u|.$$

At first, we review a key result due to Uhlenbeck for the connections with $L^p$-small curvature $(2p > n)$ [5], which provides the existence of a flat connection $\Gamma$ on $P$, of a global gauge transformation $u$ of $A$ to Coulomb gauge with respect to $\Gamma$, and of a Sobolev norm estimate for the distance between $\Gamma$ and $A$.

**Theorem 21.** ([5] Corollary 4.3 and [2] Theorem 5.1) Let $X$ be a closed, smooth manifold of dimension $n \geq 2$ endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group, and $2p > n$. Then there are constants, $\varepsilon = \varepsilon(n, g, G, p) \in (0, 1]$ and $C = C(n, g, G, p) \in [1, \infty)$, with the following property. Let $A$ be a $L^2_1$-connection on a principal $G$-bundle $P$ over $X$. If the curvature $F_A$ obeys

$$\|F_A\|_{L^p(X)} \leq \varepsilon,$$

then there exist a flat connection, $|\Gamma|$, on $P$, and a gauge transformation $u \in L^p_2(X)$ such that

1. $d^*_u (u^*(A) - \Gamma) = 0$ on $X$,
2. $\|u^*(A) - \Gamma\|_{L^p_1} \leq C \|F_A\|_{L^p(X)}$, and
3. $\|u^*(A) - \Gamma\|_{L^p_1} \leq C \|F_A\|_{L^2_1(X)}$.

Next, we also review another key result due to Uhlenbeck concerning an a priori estimate for the curvature of a Yang–Mills connection over a closed Riemannian manifold.

**Theorem 22.** ([4] Theorem 3.5 and [2] Corollary 4.6) Let $X$ be a compact manifold of dimension $n \geq 2$ endowed with a Riemannian metric $g$, let $A$ be a smooth Yang–Mills connection with respect to the metric $g$ on a smooth $G$-bundle $P$ over $X$. Then there exist constants $\varepsilon = \varepsilon(X, n, g) > 0$ and $C = C(X, n, g)$ with the following property. If the curvature $F_A$ obeys

$$\|F_A\|_{L^2_1(X)} \leq \varepsilon,$$

then

$$\|F_A\|_{L^\infty(X)} \leq C \|F_A\|_{L^2_1(X)}.$$
3. Proof of Theorem 1.1

For any $p \geq 1$, the estimate in Theorem 2.2 yields
\[ \|F_A\|_{L^p(X)} \leq C \|F_A\|_{L^\infty(X)} \leq C \|F_A\|_{L^2(X)}, \tag{3.1} \]
for $C = C(g, n)$.

If $n \geq 4$, using Hölder inequality, we have
\[ \|F_A\|_{L^2(X)} \leq C \|F_A\|_{L^4(X)}, \tag{3.2} \]
and thus
\[ \|F_A\|_{L^2(X)} \leq C \|F_A\|_{L^2(X)}. \tag{3.3} \]

Therefore, by combining (3.1)–(3.3), we obtain
\[ \|F_A\|_{L^p(X)} \leq C \|F_A\|_{L^2(X)}, \quad \forall 2p \geq n \text{ and } n \geq 2. \]

Hence, if we suppose $\|F_A\|_{L^2(X)}$ sufficiently small so that $\|F_A\|_{L^q(X)}$ ($2q > n$ and $n \geq 2$) satisfies the hypothesis of Theorem 2.1, then Theorem 2.1 provides a flat connection $\Gamma$ on $P$, a gauge transformation $u \in \mathcal{G}_P$, and the estimate
\[ \|u^*(A) - \Gamma\|_{L^q(X)} \leq C(q)\|F_A\|_{L^q(X)}, \]
and
\[ d^*_\Gamma(u^*(A) - \Gamma) = 0. \]

We denote $\tilde{A} := u^*(A)$ and $a := u^*(A) - \Gamma$, then the curvature of $\tilde{A}$ is
\[ F_{\tilde{A}} = d_\Gamma a + a \wedge a. \]

The connection $\tilde{A}$ also satisfies Yang–Mills equation
\[ 0 = d^*_\tilde{A} F_{\tilde{A}}. \tag{3.4} \]

Hence, taking the $L^2$-inner product of (3.4) with $a$, we obtain
\[ 0 = (d^*_\tilde{A} F_{\tilde{A}}, a)_{L^2(X)} \]
\[ = (F_{\tilde{A}}, d_\tilde{A} a)_{L^2(X)} \]
\[ = (F_{\tilde{A}}, d_\Gamma a + 2a \wedge a)_{L^2(X)} \]
\[ = (F_{\tilde{A}}, F_{\tilde{A}} + a \wedge a)_{L^2(X)}. \]

Then we get
\[ \|F_A\|^2_{L^2(X)} = \|F_{\tilde{A}}\|^2_{L^2(X)} \]
\[ = -(F_{\tilde{A}}, a \wedge a)_{L^2(X)} \]
\[ \leq \|F_{\tilde{A}}\|_{L^2(X)} \|a \wedge a\|_{L^2(X)} \]
\[ = \|F_A\|_{L^2(X)} \|a \wedge a\|_{L^2(X)} \]

here we use the fact $|F_{u^*(A)}| = |F_A|$ since $F_{u^*(A)} = u \circ F_A \circ u^{-1}$. 

If \( n \geq 4 \):
\[
\|a \wedge a\|_{L^2(X)} \leq C \|a\|^2_{L^4(X)} \\
\leq C \|a\|^2_{L^n(X)} \\
\leq C \|a\|^2_{L^2(X)} \\
\leq C \|F_A\|^2_{L^2(X)} \\
\leq C \|F_A\|^2_{L^\infty(X)} \\
\leq C \|F_A\|^2_{L^2(X)},
\]
where we apply the Sobolev embedding \( L^2 \hookrightarrow L^n \).

If \( n = 2, 3 \),
\[
\|a \wedge a\|_{L^2(X)} \leq C \|a\|^2_{L^4(X)} \\
\leq C \|a\|^2_{L^2(X)} \\
\leq C \|F_A\|^2_{L^2(X)},
\]
where we apply the Sobolev embedding \( L^2 \hookrightarrow L^4 \).

Combining the preceding inequalities, we have
\[
\|F_A\|^2_{L^2(X)} \leq C \|F_A\|^3_{L^2(X)}.
\]
We can choose \( \|F_A\|_{L^2(X)} \) sufficiently small so that \( C \|F_A\|_{L^2(X)} < 1 \), hence \( \|F_A\|_{L^2(X)} \equiv 0 \) and thus \( A \) must be a flat connection. This completes the proof.

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