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Intersection of harmonically weighted Dirichlet spaces

Intersection d'espaces de Dirichlet à poids harmonique

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ABSTRACT

In 1991, S. Richter introduced harmonically weighted Dirichlet spaces $\mathcal{D}(\mu)$, motivated by his study of cyclic analytic two-isometries. In this paper, we consider $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$, the intersection of $\mathcal{D}(\mu)$ spaces, where $\mathbb{P}$ is the family of Borel probability measures. Several function-theoretic characterizations of the Banach space $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ are given. We also show that $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ is located strictly between some classical analytic Lipschitz spaces and $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ can be regarded as the endpoint case of analytic Morrey spaces in some sense.

RÉSUMÉ

En 1991, S. Richter a introduit les espaces de Dirichlet $\mathcal{D}(\mu)$ à poids harmonique, motivé par l'étude des 2-isométries analytiques. Dans cet article, on considère une intersection $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ d'espaces $\mathcal{D}(\mu)$, où $\mathbb{P}$ est l'espace des mesures de probabilité boréliennes. On donne plusieurs caractérisations de $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ en termes de théorie des fonctions. On montre également que $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ se compare dans les deux sens par des relations d'inclusion strictes avec certains espaces de fonctions analytiques Lipschitz et que $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ peut être considéré comme le cas extrême des espaces de Morrey analytiques en un certain sens.

1. Introduction

Denote by $\mathbb{T}$ the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. In 1991, S. Richter [7] introduced harmonically weighted Dirichlet spaces $\mathcal{D}(\mu)$ when he investigated cyclic analytic two-isometries. Let μ be a finite positive Borel measure on $\mathbb{T}$. The space $\mathcal{D}(\mu)$ consists of functions $f \in H(\mathbb{D})$ with

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$$\|f\|^2_{\mathcal{D}(\mu)} = \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\,\mathrm{d}A(z) < +\infty,$$

where $\mathrm{d}A$ is the Lebesgue measure on $\mathbb{D}$ and

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}\mu(\zeta)$$

is the Poisson integral of μ on $\mathbb{D}$. S. Richter [7] showed that every cyclic analytic two-isometry can be represented as a multiplication by z on a harmonically weighted Dirichlet space $\mathcal{D}(\mu)$. If $\mu = \delta_\zeta$, the Dirac measure at $\zeta \in \mathbb{T}$, then $\mathcal{D}(\delta_\zeta)$ is called the local Dirichlet space at ζ. If μ is the arc-length Lebesgue measure on $\mathbb{T}$, then $\mathcal{D}(\mu)$ coincides with the classical Dirichlet space $\mathcal{D}$. As proved in [7], $\mathcal{D}(\mu)$ spaces are always subsets of the Hardy space H^2. See [8,9] for deep studies of $\mathcal{D}(\mu)$ spaces, which have attracted a lot of attention in recent years. We refer to the recent book [3] for a general exposition on $\mathcal{D}(\mu)$ spaces.

Let $\mathbb{P}$ denote the family of positive Borel probability measures on $\mathbb{T}$. It is well known that the functions in $\mathcal{D}(\mu)$, $\mu \in \mathbb{P}$, behave nicely near the points of the support of μ; in particular, their radial limit exists at μ-almost every point of $\mathbb{T}$ (cf. [3, p. 112]). Based on this fact, one may ask for a description of the smoothness properties of functions that belong to $\mathcal{D}(\mu)$ for every $\mu \in \mathbb{P}$. The purpose of this paper is to consider $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$, the intersection of $\mathcal{D}(\mu)$ spaces with positive Borel probability measures μ. The properties and the applications of the intersection of Poletsky–Stessin Hardy spaces have been studied recently in [5,6].

In section 2, we give some function-theoretic characterizations of $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. By these characterizations, we know that $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ is a Banach function space. We also characterize analytic functions with nonnegative coefficients in the space $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. We answer the question of the previous paragraph in section 3, where we show that $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ is located strictly between some classical analytic Lipschitz spaces. In Section 4, we point out that $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ can be regarded as the endpoint case of analytic Morrey spaces in some sense.

Throughout this paper, we will write $a \lesssim b$ if there exists a constant C such that $a \le Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$.

2. Characterizations of $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$

In this section, we give several characterizations of $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. We also characterize analytic functions with nonnegative coefficients in $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$.

For any z, $w \in \mathbb{D}$, denote by

$$\sigma_z(w) = \frac{z-w}{1-\bar{z}w}$$

the Möbius transformation of the unit disk $\mathbb{D}$ interchanging z and 0. The following theorem is the main result in this section.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} b_n z^n \in H(\mathbb{D})$. Then the following conditions are equivalent.*

(i) $f \in \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$.
(ii) $\sup_{a\in\mathbb{D}} \left(\frac{1}{1-|a|^2} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2)\,\mathrm{d}A(z) \right) < +\infty$.
(iii) $\sup_{\zeta\in\mathbb{T}} \left(\int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\,\mathrm{d}A(z) \right) < +\infty$.
(iv) $\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left| \sum_{k=0}^{n} (k+1) b_{k+1} a^{n-k} \right|^2 < \infty$.
(v) $\sup_{\zeta\in\mathbb{T}} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left| \sum_{k=1}^{n} k b_k \zeta^k \right|^2 < \infty$.

Let μ be a positive Borel measure on $\mathbb{D}$. Note that the function

$$a \to \frac{1}{1-|a|^2} \int_{\mathbb{D}} (1-|\sigma_a(z)|^2)\,\mathrm{d}\mu(z)$$

is subharmonic on $\mathbb{D}$. Bearing in mind the maximum principle for subharmonic functions, we are led to prove the following lemma. We will use it to prove Theorem 2.1.

Lemma 2.2. *Let μ be a positive Borel measure on $\mathbb{D}$. Then*

$$\sup_{a\in\mathbb{D}} \frac{1}{1-|a|^2}\int_{\mathbb{D}} (1-|\sigma_a(z)|^2)\,\mathrm{d}\mu(z) = \sup_{\zeta\in\mathbb{T}}\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mu(z).$$

Proof. Fix $\zeta\in\mathbb{T}$ and let $\{w_n\}\subseteq\mathbb{D}$ satisfying $\lim_{n\to\infty} w_n=\zeta$. From Fatou's lemma,

$$\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mu(z) \le \liminf_{n\to\infty}\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{w}_n z|^2}\,\mathrm{d}\mu(z) = \liminf_{n\to\infty}\left(\frac{1}{1-|w_n|^2}\int_{\mathbb{D}}(1-|\sigma_{w_n}(z)|^2)\,\mathrm{d}\mu(z)\right)$$
$$\le \sup_{a\in\mathbb{D}}\left(\frac{1}{1-|a|^2}\int_{\mathbb{D}}(1-|\sigma_a(z)|^2)\,\mathrm{d}\mu(z)\right).$$

For the other direction, fix $w\in\mathbb{D}$. Note that $z\to(1-|\bar{w}z|^2)/(|1-\bar{w}z|^2)$ is a positive harmonic function on $\mathbb{D}$ as the composition of the Poisson kernel with the holomorphic function $z\to\bar{w}z$. In particular,

$$\frac{1-|\bar{w}z|^2}{|1-\bar{w}z|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2}\mathrm{d}\nu_w(\zeta),$$

where $\mathrm{d}\nu_w(\zeta)=(1-|w|^2)/(2\pi|\zeta-w|^2)|\mathrm{d}\zeta|$ and $\nu_w(\mathbb{T})=1$. We also have

$$\frac{1}{1-|w|^2}\int_{\mathbb{D}}(1-|\sigma_w(z)|^2)\,\mathrm{d}\mu(z) = \int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{w}z|^2}\,\mathrm{d}\mu(z) \le \int_{\mathbb{D}}\frac{1-|\bar{w}z|^2}{|1-\bar{w}z|^2}\,\mathrm{d}\mu(z).$$

Combining the above facts and Fubini's theorem, we deduce that

$$\frac{1}{1-|w|^2}\int_{\mathbb{D}}(1-|\sigma_w(z)|^2)\,\mathrm{d}\mu(z) \le \int_{\mathbb{T}}\int_{\mathbb{D}}\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mu(z)\,\mathrm{d}\nu_w(\zeta)$$
$$\le \nu_w(\mathbb{T})\sup_{\zeta\in\mathbb{T}}\int_{\mathbb{D}}\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mu(z)$$
$$= \sup_{\zeta\in\mathbb{T}}\int_{\mathbb{D}}\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mu(z).$$

The proof is complete. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. $(i)\Rightarrow(iii)$. Note that the condition (iii) is equivalent to

$$\sup_{\zeta\in\mathbb{T}}||f||_{\mathcal{D}(\delta_\zeta)} < +\infty.$$

Let $f\in\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ and suppose that $\sup_{\zeta\in\mathbb{T}}||f||_{\mathcal{D}(\delta_\zeta)}=+\infty$. Then, for every $n\in\mathbb{N}$, there exists $\zeta_n\in\mathbb{T}$ such that $||f||_{\mathcal{D}(\delta_{\zeta_n})}\ge 2^n$. Consider the measure $\nu=\sum_{n=1}^{+\infty}2^{-n}\delta_{\zeta_n}$. Then $\nu\in\mathbb{P}$ and

$$||f||^2_{\mathcal{D}(\nu)} = \sum_{n=1}^{+\infty}2^{-n}||f||^2_{\mathcal{D}(\delta_{\zeta_n})} \ge \sum_{n=1}^{+\infty}1 = +\infty.$$

Therefore $f\notin\mathcal{D}(\nu)$, contradiction. Thus the condition (iii) holds.

$(iii)\Rightarrow(i)$. Let $\mu\in\mathbb{P}$. Applying Fubini's theorem, one gets that

$$||f||^2_{\mathcal{D}(\mu)} = \int_{\mathbb{T}}\int_{\mathbb{D}}|f'(z)|^2\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}A(z)\,\mathrm{d}\mu(\zeta) = \int_{\mathbb{T}}||f||^2_{\mathcal{D}(\delta_\zeta)}\,\mathrm{d}\mu(\zeta) \le \sup_{\zeta\in\mathbb{T}}||f||^2_{\mathcal{D}(\delta_\zeta)}\mu(\mathbb{T}) = \sup_{\zeta\in\mathbb{T}}||f||^2_{\mathcal{D}(\delta_\zeta)},$$

which implies the desired result.

$(ii)\Leftrightarrow(iii)$. Note that

$$1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2},\quad a,\ z\in\mathbb{D}.$$

Take $\mathrm{d}\mu(z)=|f'(z)|^2\,\mathrm{d}A(z)$ in Lemma 2.2. Then the equivalence between (ii) and (iii) is from Lemma 2.2 directly.

$(iii) \Leftrightarrow (v)$. For $\zeta \in \mathbb{T}$, the following characterization of the local Dirichlet space $\mathcal{D}(\delta_\zeta)$ via Taylor coefficients can be founded in [3, Theorem 7.2.6].

$$\frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-\overline{\zeta}z|^2}\,\mathrm{d}A(z) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\left|\sum_{k=1}^{n} kb_k\zeta^k\right|^2.$$

The above formula gives that conditions (iii) and (v) are equivalent.

$(ii) \Leftrightarrow (iv)$. Let $a \in \mathbb{D}$. Note that

$$\frac{f'(z)}{1-\overline{a}z} = \sum_{n=0}^{\infty}(n+1)b_{n+1}z^n \sum_{n=0}^{\infty}\overline{a}^n z^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)b_{k+1}\overline{a}^{n-k}\right)z^n.$$

Bear in mind that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-\overline{a}z|^2}\,\mathrm{d}A(z) < \infty$$

if and only if the functions $z \to \int_0^z \frac{f'(w)}{1-\overline{a}w}\mathrm{d}w$ belong to the Hardy space H^2 uniformly for $a \in \mathbb{D}$. Thus the condition (ii) is equivalent to

$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\left|\sum_{k=0}^{n}(k+1)b_{k+1}\overline{a}^{n-k}\right|^2 < \infty.$$

The proof of Theorem 2.1 is finished. □

Remark 1. It follows from the proof of Theorem 2.1 and the Local Douglas formula [3, Theorem 7.2.5, p. 113] that, if $f \in \bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$, then

$$\sup_{\zeta\in\mathbb{T}}\int_{\mathbb{T}}\frac{|f(\xi)-f(\zeta)|^2}{|\xi-\zeta|^2}|\mathrm{d}\xi| = \sup_{\mu\in\mathbb{P}}||f||^2_{\mathcal{D}(\mu)} = \sup_{a\in\mathbb{D}}\frac{1}{(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^2(1-|\sigma_a(z)|^2)\,\mathrm{d}A(z).$$

It is also easy to check that $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ is a Banach space with the norm

$$\|f\| = |f(0)| + \left(\sup_{a\in\mathbb{D}}\frac{1}{1-|a|^2}\int_{\mathbb{D}}|f'(z)|^2(1-|\sigma_a(z)|^2)\,\mathrm{d}A(z)\right)^{1/2}.$$

Recall that for $\alpha > 0$, the Bloch type space $\mathcal{B}^\alpha$ is the class of functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

In [11], H. Wulan obtained many results associated with the coefficients of Bloch type spaces. In particular, he characterized analytic functions with nonnegative coefficients in $\mathcal{B}^\alpha$ as follows.

Theorem A. Let $\alpha > 0$ and let $f(z) = \sum_{n=0}^{\infty} b_n z^n \in H(\mathbb{D})$ with $b_n \geq 0$ for every n. Then $f \in \mathcal{B}^\alpha$ if and only if

$$\limsup_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n}kb_k < \infty.$$

Next, we describe analytic functions with nonnegative coefficients in the space $\bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$. Unlike the characterizations in Theorem 2.1, the following characterization is without using the supremum.

Theorem 2.3. *Let $f(z) = \sum_{n=0}^{\infty} b_n z^n \in H(\mathbb{D})$ with $b_n \geq 0$ for every n. Then $f \in \bigcap_{\mu\in\mathbb{P}}\mathcal{D}(\mu)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\sum_{k=1}^{n}kb_k\right)^2 < \infty.$$

Proof. Let $f \in \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. By Theorem 2.1, one gets that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=1}^{n} kb_k\right)^2 \le \sup_{\zeta\in\mathbb{T}} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left|\sum_{k=1}^{n} kb_k\zeta^k\right|^2 < \infty.$$

Conversely, since $b_n \ge 0$ for every n, we deduce that

$$\sup_{\zeta\in\mathbb{T}} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left|\sum_{k=1}^{n} kb_k\zeta^k\right|^2 \le \sup_{\zeta\in\mathbb{T}} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=1}^{n} |kb_k\zeta^k|\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=1}^{n} kb_k\right)^2 < \infty. \quad \square$$

3. $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ and analytic Lipschitz spaces

In this section, we show that the space $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ is located strictly between some classical analytic Lipschitz spaces. For $0 < \alpha < 1$, let Λ^α be the analytic Lipschitz space consisting of functions $f \in H(\mathbb{D})$ with

$$|f(z) - f(w)| \le C|z - w|^\alpha$$

for some positive constant C and for all $z, w \in \mathbb{D}$. G. Hardy and J. Littlewood [4] showed that the space Λ^α is equal to the Bloch type space $\mathcal{B}^{1-\alpha}$.

The following theorem reveals the location of the space $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$.

Theorem 3.1.

$$\bigcup_{\alpha\in(0,\frac{1}{2})} \Lambda^{1-\alpha} \subsetneqq \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu) \subsetneqq \Lambda^{\frac{1}{2}}.$$

Proof. Let $f \in \bigcup_{\alpha\in(0,\frac{1}{2})} \Lambda^{1-\alpha}$. Then $f \in \mathcal{B}^\beta$ for some $\beta \in (0, \frac{1}{2})$. Namely

$$\sup_{z\in\mathbb{D}} |f'(z)|(1 - |z|^2)^\beta < \infty.$$

Combining this with the well-known estimate in Zhu's book ([14, Lemma 3.10]), we have

$$\begin{aligned}\sup_{a\in\mathbb{D}} \frac{1}{1-|a|^2} \int_{\mathbb{D}} |f'(z)|^2(1 - |\sigma_a(z)|^2)\,\mathrm{d}A(z) &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-\bar{a}z|^2}\,\mathrm{d}A(z) \\ &\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{1-2\beta}}{|1-\bar{a}z|^2}\,\mathrm{d}A(z) \approx 1.\end{aligned}$$

From Theorem 2.1, we obtain $f \in \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. Hence $\bigcup_{\alpha\in(0,\frac{1}{2})} \Lambda^{1-\alpha} \subseteq \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. Let $h(z) = \sum_{n=1}^{\infty} a_n z^n$ satisfying $a_n = 0$ for $1 \le n \le 8$, $9a_9 = \sqrt{9}(\log 9)^{-1}$ and

$$a_n = \frac{\sqrt{n}(\log n)^{-1} - \sqrt{n-1}[\log(n-1)]^{-1}}{n}, \quad n > 9.$$

Then $a_n \ge 0$ for all n. Moreover,

$$\sum_{n=9}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=1}^{n} ka_k\right)^2 = \sum_{n=9}^{\infty} \frac{1}{(n+1)(\log n)^2} < \infty,$$

and

$$\limsup_{n\to\infty} \frac{1}{n^{2\alpha}} \left(\sum_{k=1}^{n} ka_k\right)^2 = \limsup_{n\to\infty} \frac{n^{1-2\alpha}}{(\log n)^2} = \infty.$$

Note that $\Lambda^{1-\alpha} = \mathcal{B}^\alpha$. From Theorem A and Theorem 2.3, we get that $h \in \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ and $h \notin \bigcup_{\alpha\in(0,\frac{1}{2})} \Lambda^{1-\alpha}$. Thus $\bigcup_{\alpha\in(0,\frac{1}{2})} \Lambda^{1-\alpha} \subsetneqq \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$.

For the second inclusion, let Δ_a be the disk with center $a \in \mathbb{D}$ and radius $(1-|a|)/2$. It is well known that

$$1 - |a| \approx |1 - \bar{z}a| \approx 1 - |z|$$

for all $z \in \Delta_a$. Let $f \in \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. From the subharmonicity of $|f|^2$ on $\mathbb{D}$, we obtain that for every $a \in \mathbb{D}$,

$$(1-|a|^2)|f'(a)|^2 \le \frac{(1-|a|^2)}{A(\Delta_a)} \int\limits_{\Delta_a} |f'(z)|^2 \, dA(z) \approx \frac{1}{(1-|a|^2)} \int\limits_{\Delta_a} |f'(z)|^2 \, dA(z)$$
$$\approx \frac{1}{(1-|a|^2)} \int\limits_{\Delta_a} |f'(z)|^2 (1-|\sigma_a(z)|^2) \, dA(z)$$
$$\lesssim \frac{1}{(1-|a|^2)} \int\limits_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2) \, dA(z).$$

This together with Theorem 2.1 gives that $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu) \subseteq \Lambda^{\frac{1}{2}}$. Set $g(z) = \sum_{n=1}^{\infty} b_n z^n$ satisfying $b_1 = b_2 = 0$, $3b_3 = 3^{1/2}(\log 3)^{-1/2}$ and

$$b_n = \frac{n^{1/2}(\log n)^{-1/2} - (n-1)^{1/2}[\log(n-1)]^{-1/2}}{n}, \quad n \ge 4.$$

Then $b_n \ge 0$ for all n. Furthermore,

$$\limsup_{n\to\infty} \frac{1}{n} \left(\sum_{k=1}^{n} k b_k \right)^2 = \limsup_{n\to\infty} \frac{1}{\log n} = 0,$$

and

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=1}^{n} k b_k \right)^2 = \sum_{n=3}^{\infty} \frac{1}{(n+1)\log n} = +\infty.$$

By Theorem A and Theorem 2.3, one gets that $g \in \Lambda^{1/2}$ and $g \notin \bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. Thus $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu) \subsetneqq \Lambda^{\frac{1}{2}}$. The proof is complete. □

4. $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ and analytic Morrey spaces

For $\lambda \in (-\infty, +\infty)$, denote by $\mathcal{L}^{2,\lambda}(\mathbb{T})$ the Morrey space of all Lebesgue measurable functions f on $\mathbb{T}$ that satisfy

$$\sup_{I\subseteq\mathbb{T}} |I|^{-\lambda} \int\limits_I |f(\zeta) - f_I|^2 |d\zeta| < \infty,$$

where $|I|$ denotes the length of the arc I and $f_I = \frac{1}{|I|}\int_I f(\zeta)|d\zeta|$. Clearly, $\mathcal{L}^{2,1}(\mathbb{T})$ coincides with $BMO(\mathbb{T})$, the space of functions with bounded mean oscillation on $\mathbb{T}$ (cf. [1]). Moreover,

$$BMO(\mathbb{T}) \subseteq \mathcal{L}^{2,\lambda_1}(\mathbb{T}) \subseteq \mathcal{L}^{2,\lambda_2}(\mathbb{T}) \subseteq L^2(\mathbb{T}), \quad 0 < \lambda_2 < \lambda_1 < 1.$$

Note that every function in the Hardy space H^2 has non-tangential limit almost everywhere on $\mathbb{T}$ (cf. [2]). Following [12, p. 54], for $\lambda \in (0, 2)$, the analytic Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ is defined as $H^2 \cap \mathcal{L}^{2,\lambda}(\mathbb{T})$. See [10,13] for analytic Morrey spaces. From [12, Corollary 3.2.2.], $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if

$$\sup_{a\in\mathbb{D}} \left((1-|a|^2)^{1-\lambda} \int\limits_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2) \, dA(z) \right) < +\infty.$$

By Theorem 2.1, the above condition with $\lambda = 2$ gives a characterization of $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$. In this sense, $\bigcap_{\mu\in\mathbb{P}} \mathcal{D}(\mu)$ can be regarded as the endpoint case of analytic Morrey spaces.

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