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Correlation between Adomian and partial exponential Bell polynomials

Corrélation des polynômes d'Adomian et des polynômes de Bell exponentiels partiels

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ABSTRACT

We obtain some recurrence relationships among the partition vectors of the partial exponential Bell polynomials. On using such results, the n-th Adomian polynomial for any nonlinear operator can be expressed explicitly in terms of the partial exponential Bell polynomials. Some new identities for the partial exponential Bell polynomials are obtained by solving certain ordinary differential equations using the Adomian decomposition method.

RÉSUMÉ

Nous montrons des relations de récurrence entre les vecteurs partition des polynômes de Bell exponentiels partiels. Utilisant ces relations, le n-ième polynôme d'Adomian, pour n'importe quel opérateur non linéaire, s'exprime explicitement en termes des polynômes de Bell exponentiels partiels. On en déduit des identités nouvelles pour ces derniers, via la solution de certaines équations différentielles ordinaires, en utilisant la méthode de décomposition d'Adomian.

1. Introduction

The Bell polynomials studied by Bell [4,5] are special polynomials in combinatorial analysis, with numerous applications in different areas of mathematics. The incomplete or partial exponential Bell polynomials $B_{n,k}$ (see [7], [10, p. 96]) in $n-k+1$ variables are triangular arrays of polynomials defined by

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$$B_{n,k}(u_1,u_2,\dots,u_{n-k+1})=n!\sum_{\Lambda_n^k}\prod_{j=1}^{n-k+1}\frac{1}{k_j!}\left(\frac{u_j}{j!}\right)^{k_j},\tag{1.1}$$

where the partition set is given by $\Lambda_n^k=\left\{(k_1,k_2,\dots,k_{n-k+1}):\sum_{j=1}^{n-k+1}k_j=k,\sum_{j=1}^{n-k+1}jk_j=n,k_j\in\mathbb{N}_0\right\}$. Here $\mathbb{N}_m=\{x:x\geq m,x\in\mathbb{N}\cup\{0\}\}$ and $\mathbb{N}$ denotes the set of positive integers. Also, the sum

$$B_n(u_1,u_2,\dots,u_n)=\sum_{k=1}^{n}B_{n,k}(u_1,u_2,\dots,u_{n-k+1}),\tag{1.2}$$

is called the n-th complete exponential Bell polynomial. For more details, results and some known identities on Bell polynomials, we refer the reader to [6, pp. 133–137], [7] and [10, pp. 95–98].

Next we briefly explain the Adomian decomposition method (ADM) [2,3], which will be used later to obtain some new identities for Bell polynomials. In ADM, the solution to the functional equation

$$u=f+L(u)+N(u),\tag{1.3}$$

where L and N are linear and nonlinear operators respectively and f is a known function, is expressed in the form of an infinite series

$$u=\sum_{n=0}^{\infty}u_n.\tag{1.4}$$

The nonlinear term $N(u)$ decomposes as

$$N(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\dots,u_n),\tag{1.5}$$

where A_n denotes the n-th Adomian polynomial in $u_0,u_1,\dots,u_n$. Also, the series (1.4) and (1.5) are assumed to be absolutely convergent. So, (1.3) can be rewritten as

$$\sum_{n=0}^{\infty}u_n=f+\sum_{n=0}^{\infty}L(u_n)+\sum_{n=0}^{\infty}A_n.$$

Thus the u_ns are obtained by the following recursive relation

$$u_0=f\quad\text{and}\quad u_n=L(u_{n-1})+A_{n-1}.$$

The crucial step involved in ADM is the calculation of Adomian polynomials. Adomian [2, pp. 19–21] gave a method for determining these polynomials, by parameterizing u as $u_\lambda=\sum_{n=0}^{\infty}u_n\lambda^n$ and assuming $N(u_\lambda)$ to be analytic in λ, which decomposes as $N(u_\lambda)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\dots,u_n)\lambda^n$. Hence, Adomian polynomials are given by

$$A_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\frac{\partial^n N(u_\lambda)}{\partial\lambda^n}\bigg|_{\lambda=0},\ \forall\, n\in\mathbb{N}_0.\tag{1.6}$$

An improved version of the above result (see Zhu et al. [13]) is given by

$$A_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\frac{\partial^n N(\sum_{k=0}^{n}u_k\lambda^k)}{\partial\lambda^n}\bigg|_{\lambda=0},\ \forall\, n\in\mathbb{N}_0.\tag{1.7}$$

Rach [12] suggested the following formula for these polynomials: $A_0(u_0)=N(u_0)$ and

$$A_n(u_0,u_1,\dots,u_n)=\sum_{k=1}^{n}C(k,n)N^{(k)}(u_0),\ \forall\, n\in\mathbb{N},\tag{1.8}$$

where

$$C(k,n)=\sum_{\Theta_n^k}\prod_{j=1}^{n}\frac{u_j^{k_j}}{k_j!},\tag{1.9}$$

and the summation is taken over the partition set $\Theta_n^k=\left\{(k_1,k_2,\dots,k_n):\sum_{j=1}^{n}k_j=k,\sum_{j=1}^{n}jk_j=n,k_j\in\mathbb{N}_0\right\}$. Also, $N^{(k)}(.)$ denotes the k-th derivative of the nonlinear term. One can easily show the equivalence of (1.6) and (1.8) using the Faà di Bruno's formula. Recently, Kataria and Vellaisamy [11] obtained simple parameterization methods for generating these Adomian polynomials both explicitly and recursively.

In this paper, we first obtain some results related to the partition vectors of the partial exponential Bell polynomials. Then we show that the $C(k,n)$'s, which are homogeneous polynomials of order k, can be represented in terms of the well-known Bell polynomials. Hence, a closed form expression of the n-th order Adomian polynomial for any nonlinear operator is obtained as a finite sum of the partial exponential Bell polynomials. The significance of this result is that any algorithm or identity for the $C(k,n)$'s will give the corresponding results for the Bell polynomials and vice versa. Indeed we use the results of Duan [8,9] to obtain some recursive algorithms for the partial exponential Bell polynomials. Also, we use the Adomian decomposition method to solve certain ordinary differential equations to obtain some new identities for the partial exponential Bell polynomials.

2. The partition vectors of the Bell polynomials

We use the following results by Duan [8] for the partition set Θ_n^k of Adomian polynomials to show some similar results for the partition set Λ_n^k of Bell polynomials.

Lemma 2.1. *For* $1 \le k \le n$, $\Theta_n^k \subset \mathbb{N}_0^n$ *and* $\Theta_1^1 = \{(1)\}$, $\Theta_n^1 = \{(0,0,\ldots,0,1)\}$.

Lemma 2.2. *For every vector* $(k_1,k_2,\ldots,k_n) \in \Theta_n^k$, $2 \le k \le n$, *the last* $(k-1)$ *entries are zero, i.e.* $k_{n-k+2} = k_{n-k+3} = \ldots = k_n = 0$.

Theorem 2.1. *For* $n \in \mathbb{N}_2$, *if* $2 \le k \le \left[\frac{n}{2}\right]$, *then* $\Theta_n^k = \Theta_1 \cup \Theta_2$, *and if* $\left[\frac{n}{2}\right] < k \le n$, *then* $\Theta_n^k = \Theta_1$, *where*

$$\Theta_1 = \{(k_1+1, k_2, \ldots, k_{n-1}, 0) : (k_1, k_2, \ldots, k_{n-1}) \in \Theta_{n-1}^{k-1}\}$$

and

$$\Theta_2 = \{(0, k_1, k_2, \ldots, k_{n-k}, \underbrace{0, 0, \ldots, 0}_{k-1 \text{ times}}) : (k_1, k_2, \ldots, k_{n-k}) \in \Theta_{n-k}^k\}.$$

Next we obtain some results for the partition set Λ_n^k of Bell polynomials. The following lemmas are easy to prove.

Lemma 2.3. *For* $1 \le k \le n$, $\Lambda_n^k \subset \mathbb{N}_0^{n-k+1}$ *with* $\Lambda_n^1 = \Theta_n^1$ *for all* $n \in \mathbb{N}$.

The following result is evident from Lemma 2.2.

Lemma 2.4. *Let* e_j^n *denote the* n*-tuple vector with unity at the* j*-th place and zero elsewhere. Then*

$$\Theta_n^k = \left\{ \sum_{j=1}^{n-k+1} k_j \mathrm{e}_j^n : (k_1, k_2, \ldots, k_{n-k+1}) \in \Lambda_n^k \right\}.$$

Similar recurrence relationships among the partition vectors of the partial exponential Bell polynomials hold.

Theorem 2.2. *For* $n \in \mathbb{N}_2$, *if* $2 \le k \le \left[\frac{n}{2}\right]$, *then* $\Lambda_n^k = \Lambda_1 \cup \Lambda_2$, *and if* $\left[\frac{n}{2}\right] < k \le n$, *then* $\Lambda_n^k = \Lambda_1$, *where*

$$\Lambda_1 = \{(k_1+1, k_2, \ldots, k_{n-k+1}) : (k_1, k_2, \ldots, k_{n-k+1}) \in \Lambda_{n-1}^{k-1}\}$$

and

$$\Lambda_2 = \{(0, k_1, k_2, \ldots, k_{n-2k+1}, \underbrace{0, 0, \ldots, 0}_{k-1 \text{ times}}) : (k_1, k_2, \ldots, k_{n-2k+1}) \in \Lambda_{n-k}^k\}.$$

Proof. The proof is evident on using Lemma 2.2 and Theorem 2.1. □

A relationship between Adomian polynomials and Bell polynomials was first obtained by Abbaoui et al. [1]. Here we establish a relationship between Adomian and partial exponential Bell polynomials, which is different from the one proved in [1].

Theorem 2.3. *Let A_n, $n \geq 1$, be the n-th Adomian polynomial for the nonlinear term $N(u)$. Then*

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \sum_{k=1}^{n} B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) N^{(k)}(u_0). \tag{2.1}$$

Proof. On setting $k_{n-k+j} = 0$ for $j = 2, 3, \ldots, k$ and using Lemma 2.4, we have

$$B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = n! \sum_{\Lambda_n^k} \prod_{j=1}^{n-k+1} \frac{u_j^{k_j}}{k_j!} = n! \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{u_j^{k_j}}{k_j!} = n! C(k, n).$$

Hence,

$$C(k, n) = \frac{1}{n!} B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}). \tag{2.2}$$

The proof completes on using (1.8). □

From (2.2) it is clear that the homogeneous polynomials of order k, *i.e.* the $C(k,n)$'s are the partial exponential Bell polynomials. On using (2.2), the following results follow from the recursive algorithm of $C(k,n)$ given by Corollary 1 of Duan [8].

Corollary 2.1. *For all $n \in \mathbb{N}$, we have $B_{n,1}(1!u_1, 2!u_2, \ldots, n!u_n) = n!u_n$ and $B_{n,n}(u_1) = u_1^n$. When $\left[\frac{n}{2}\right] < k \leq n$ and $n \in \mathbb{N}_2$, we have*

$$B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = n\ B_{n-1,k-1}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1})\big|_{k_1 \to k_1+1}.$$

For $2 \leq k \leq \left[\frac{n}{2}\right]$ and $n \in \mathbb{N}_4$, we have

$$B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = n\ B_{n-1,k-1}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1})\big|_{k_1 \to k_1+1} + (n)_k B_{n-k,k}(1!u_2, 2!u_3, \ldots, (n-2k+1)!u_{n-2k+2}),$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ denotes the falling factorials.

An alternate recursive algorithm for the Bell polynomials follows from Corollary 3 and 4 of Duan [9].

Corollary 2.2. *Let n be any positive integer. Then for $2 \leq k \leq n$, we have*

$$B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = \sum_{j=0}^{n-k} (j+1)(n-1)_j u_{j+1} B_{n-j-1,k-1}(1!u_1, 2!u_2, \ldots, (n-j-k+1)!u_{n-j-k+1}).$$

Alternatively, for $2 \leq k \leq n-1$, we have

$$B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = u_1 B_{n-1,k-1}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) + \sum_{j=1}^{n-k} (j+1)u_{j+1} \frac{\partial}{\partial u_j} B_{n-1,k}(1!u_1, 2!u_2, \ldots, (n-k)!u_{n-k}).$$

Next we show that the n-th Adomian polynomial for exponential non-linearity can be expressed in terms of the n-th complete exponential Bell polynomial. By using (2.1), the Adomian polynomials for $N(u) = \mathrm{e}^u$ are $A_0(u_0) = \mathrm{e}^{u_0}$ and

$$A_1(u_0, u_1) = B_{1,1}(1!u_1)\,\mathrm{e}^{u_0} = u_1 \mathrm{e}^{u_0},$$

$$A_2(u_0, u_1, u_2) = \frac{\mathrm{e}^{u_0}}{2!}\left[B_{2,1}(1!u_1, 2!u_2) + B_{2,2}(1!u_1)\right] = \left(u_2 + \frac{u_1^2}{2}\right)\mathrm{e}^{u_0},$$

$$A_3(u_0, u_1, u_2, u_3) = \frac{\mathrm{e}^{u_0}}{3!}\left[B_{3,1}(1!u_1, 2!u_2, 3!u_3) + B_{3,2}(1!u_1, 2!u_2) + B_{3,3}(1!u_1)\right] = \left(u_3 + u_1 u_2 + \frac{u_1^3}{6}\right)\mathrm{e}^{u_0},$$

$$\vdots$$

$$A_n(u_0, u_1, \ldots, u_n) = \frac{\mathrm{e}^{u_0}}{n!} \sum_{k=1}^{n} B_{n,k}(1!u_1, 2!u_2, \ldots, (n-k+1)!u_{n-k+1}) = \frac{\mathrm{e}^{u_0}}{n!} B_n(1!u_1, 2!u_2, \ldots, n!u_n).$$

Now we give two recursive algorithms for the complete exponential Bell polynomials.

Theorem 2.4. *Let n be any positive integer. Then*

$$B_n(1!u_1,\ldots,n!u_n)=\sum_{k=0}^{n-1}(k+1)(n-1)_k u_{k+1}B_{n-k-1}(1!u_1,\ldots,(n-k-1)!u_{n-k-1}),$$

and

$$B_n(1!u_1,\ldots,n!u_n)=u_1B_{n-1}(1!u_1,\ldots,(n-1)!u_{n-1})+\sum_{k=1}^{n-1}(k+1)u_{k+1}\frac{\partial}{\partial u_k}B_{n-1}(1!u_1,\ldots,(n-1)!u_{n-1}).$$

Proof. From Corollary 1 and 2 of Duan [9], we have the following recursive algorithms for Adomian polynomials:

$$A_n(u_0,u_1,\ldots,u_n)=\frac{1}{n}\sum_{k=0}^{n-1}(k+1)u_{k+1}\frac{\partial}{\partial u_0}A_{n-k-1}(u_0,u_1,\ldots,u_{n-k-1}),$$

$$A_n(u_0,u_1,\ldots,u_n)=\frac{1}{n}\sum_{k=0}^{n-1}(k+1)u_{k+1}\frac{\partial}{\partial u_k}A_{n-1}(u_0,u_1,\ldots,u_{n-1}),\quad n\geq 1.$$

The result follows by choosing the nonlinear term $N(u)=\mathrm{e}^u$ in Theorem 2.3 and using (2.1) in the above expressions for Adomian polynomials. □

3. Some identities of the Bell polynomials

We now state and prove some new identities for the partial exponential Bell polynomials. This is achieved by solving certain ordinary differential equations using the Adomian decomposition method.

Theorem 3.1. *Let α, β be any real numbers and n be any positive integer. Then*

$$\sum_{k=1}^{n}(-\beta)^k B_{n,k}(0!\alpha,-1!\alpha^2\beta,\ldots,(-1)^{n-k}(n-k)!\alpha^{n-k+1}\beta^{n-k})=n!(-\alpha\beta)^n. \tag{3.1}$$

Proof. Consider the following ordinary differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\alpha\mathrm{e}^{-\beta u},\quad u(0)=1,\quad |\alpha\beta x|<\mathrm{e}^{\beta}. \tag{3.2}$$

Equivalently,

$$u(x)=u(0)+\alpha\int_0^x \mathrm{e}^{-\beta u(t)}\,\mathrm{d}t. \tag{3.3}$$

In (3.3), the nonlinear term is $N(u)=\mathrm{e}^{-\beta u}$. Substituting $u=\sum_{n=0}^{\infty}u_n$ and $N(u)=\sum_{n=0}^{\infty}A_n$ in the above equation and applying ADM, we get

$$\sum_{n=0}^{\infty}u_n(x)=u(0)+\alpha\sum_{n=0}^{\infty}\int_0^x A_n\left(u_0(t),u_1(t),\ldots,u_n(t)\right)\mathrm{d}t.$$

Therefore, for all $n\geq 0$,

$$u_{n+1}(x)=\alpha\int_0^x A_n\left(u_0(t),u_1(t),\ldots,u_n(t)\right)\mathrm{d}t. \tag{3.4}$$

On comparing $u_0=u(0)=1$, hence u_1, u_2, u_3, u_4 are recursively obtained as

$$A_0\left(u_0(x)\right) = \mathrm{e}^{-\beta u_0} = \frac{1}{\mathrm{e}^{\beta}}, \quad u_1(x) = \alpha \int_0^x A_0\left(u_0(t)\right) \mathrm{d}t = \frac{\alpha}{\mathrm{e}^{\beta}} x,$$

$$A_1\left(u_0(x), u_1(x)\right) = -\beta u_1 \mathrm{e}^{-\beta u_0} = -\frac{\alpha\beta}{\mathrm{e}^{2\beta}} x, \quad u_2(x) = \alpha \int_0^x A_1\left(u_0(t), u_1(t)\right) \mathrm{d}t = -\frac{\alpha^2\beta}{2\mathrm{e}^{2\beta}} x^2,$$

$$A_2\left(u_0(x), u_1(x), u_2(x)\right) = \frac{1}{2!}\left(\beta^2 u_1^2 - 2\beta u_2\right) \mathrm{e}^{-\beta u_0} = \frac{\alpha^2\beta^2}{\mathrm{e}^{3\beta}} x^2,$$

$$u_3(x) = \alpha \int_0^x A_2\left(u_0(t), u_1(t), u_2(t)\right) \mathrm{d}t = \frac{\alpha^3\beta^2}{3\mathrm{e}^{3\beta}} x^3,$$

$$A_3\left(u_0(x), u_1(x), u_2(x), u_3(x)\right) = \frac{1}{3!}\left(-\beta^3 u_1^3 + 6\beta^2 u_1 u_2 - 6\beta u_3\right) \mathrm{e}^{-\beta u_0} = -\frac{\alpha^3\beta^3}{\mathrm{e}^{4\beta}} x^3,$$

$$u_4(x) = \alpha \int_0^x A_3\left(u_0(t), u_1(t), u_2(t), u_3(t)\right) \mathrm{d}t = -\frac{\alpha^4\beta^3}{4\mathrm{e}^{4\beta}} x^4,$$

and so on, where the Adomian polynomials $A_1, A_2, \ldots$ are calculated using (1.7).

By using the method of separation of variables, it is easy to see that $u(x) = 1 + \beta^{-1}\ln(1 + \alpha\beta\mathrm{e}^{-\beta}x)$ is the solution to (3.2). Since $|\alpha\beta\mathrm{e}^{-\beta}x| < 1$, from the Taylor series expansion of $\ln(1 + \alpha\beta\mathrm{e}^{-\beta}x)$, we have

$$u(x) = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\alpha^n \beta^{n-1}}{n\mathrm{e}^{\beta n}} x^n = u(0) + \sum_{n=1}^{\infty} u_n(x). \tag{3.5}$$

By the uniqueness of the solution to the differential equation (3.2), the series solution obtained by ADM is consistent with (3.5). Therefore, the n-th component of $u(x)$ is

$$u_n(x) = (-1)^{n+1} \frac{\alpha^n \beta^{n-1}}{n\mathrm{e}^{\beta n}} x^n, \quad n \geq 1. \tag{3.6}$$

Now using Rach's formula (1.8) in (3.4), we obtain

$$u_{n+1}(x) = \alpha \int_0^x \sum_{k=1}^{n} N^{(k)}(u_0) \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{u_j^{k_j}(t)}{k_j!} \mathrm{d}t = \frac{\alpha}{\mathrm{e}^{\beta(n+1)}} \int_0^x t^n \mathrm{d}t \sum_{k=1}^{n} (-\beta)^k \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{1}{k_j!} \left(\frac{(-1)^{j+1}\alpha^j \beta^{j-1}}{j}\right)^{k_j}$$

$$= \frac{\alpha x^{n+1}}{(n+1)\mathrm{e}^{\beta(n+1)}} \sum_{k=1}^{n} (-\beta)^k \sum_{\Lambda_n^k} \prod_{j=1}^{n-k+1} \frac{1}{k_j!} \left(\frac{(-1)^{j+1}(j-1)!\alpha^j \beta^{j-1}}{j!}\right)^{k_j},$$

where the last two steps follow from (3.6) and Lemma 2.4, respectively. Finally, by using (3.6) and rearranging the terms, we get

$$\sum_{k=1}^{n} (-\beta)^k \sum_{\Lambda_n^k} \prod_{j=1}^{n-k+1} \frac{n!}{k_j!} \left(\frac{(-1)^{j+1}(j-1)!\alpha^j \beta^{j-1}}{j!}\right)^{k_j} = n!(-\alpha\beta)^n,$$

which is the required identity. □

Substituting $\{\alpha, \beta\} \in \{\{-1, 1\}, \{1, 1\}, \{-1, -1\}, \{1, -1\}\}$ in (3.1), we obtain the following results, respectively.

Corollary 3.1. *For any positive integer n, the following identities hold:*

$$\sum_{k=1}^{n} (-1)^k B_{n,k}(-0!, -1!, \ldots, -(n-k)!) = n!,$$

$$\sum_{k=1}^{n} (-1)^k B_{n,k}(0!, -1!, \ldots, (-1)^{n-k}(n-k)!) = n!(-1)^n,$$

$$\sum_{k=1}^{n} B_{n,k}(-0!, 1!, \dots, (-1)^{n-k+1}(n-k)!) = n!(-1)^n, \tag{3.7}$$

$$\sum_{k=1}^{n} B_{n,k}(0!, 1!, \dots, (n-k)!) = n!. \tag{3.8}$$

Remark 3.1. The identities given by (3.7) and (3.8) above can be expressed in terms of the n-th complete exponential Bell polynomials as follows:

$$B_n(-0!, 1!, \dots, (-1)^n(n-1)!) = n!(-1)^n \quad \text{and} \quad B_n(0!, 1!, \dots, (n-1)!) = n!.$$

Note that (3.8) is a known result, *i.e.* $B_{n,k}(0!, 1!, \dots, (n-k)!) = s(n,k)$, where the $s(n,k)$'s are Stirling numbers of the first kind.

Next we obtain an identity of the partial exponential Bell polynomials in terms of the falling factorials.

Theorem 3.2. *Let α be any non-zero real number and n be any positive integer. Then*

$$\sum_{k=1}^{n} \left(1-\frac{1}{\alpha}\right)_k B_{n,k}((\alpha)_1, (\alpha)_2, \dots, (\alpha)_{n-k+1}) = (\alpha-1)_n,$$

where $(\alpha)_k = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)$ denotes the falling factorial.

Proof. Consider the following ordinary differential equation

$$\frac{du}{dx} = \alpha u^{1-1/\alpha}, \quad u(0)=1, \quad |x|<1. \tag{3.9}$$

Equivalently,

$$u(x) = u(0) + \alpha \int_0^x u^{1-1/\alpha}(t)\,dt. \tag{3.10}$$

In (3.10), there is no linear term, but the nonlinear term is $N(u) = u^{1-1/\alpha}$. Substituting $u = \sum_{n=0}^{\infty} u_n$ and $N(u) = \sum_{n=0}^{\infty} A_n$ in the above equation and applying ADM, we get

$$\sum_{n=0}^{\infty} u_n(x) = u(0) + \alpha \sum_{n=0}^{\infty} \int_0^x A_n\left(u_0(t), u_1(t), \dots, u_n(t)\right) dt.$$

Therefore, for all $n \geq 0$,

$$u_{n+1}(x) = \alpha \int_0^x A_n\left(u_0(t), u_1(t), \dots, u_n(t)\right) dt. \tag{3.11}$$

On comparing $u_0 = u(0) = 1$ and hence u_1, u_2, u_3, u_4 are recursively obtained as

$$\begin{aligned}
A_0\left(u_0(x)\right) &= u_0^{1-1/\alpha} = 1, \quad u_1(x) = (\alpha)_1 x, \\
A_1\left(u_0(x), u_1(x)\right) &= \left(1-\frac{1}{\alpha}\right) u_0^{-1/\alpha} u_1 = \frac{(\alpha)_2}{\alpha} x, \quad u_2(x) = \frac{(\alpha)_2}{2!} x^2, \\
A_2\left(u_0(x), u_1(x), u_2(x)\right) &= \frac{1}{2!}\left(1-\frac{1}{\alpha}\right)\left(-\frac{1}{\alpha} u_0^{-1/\alpha-1} u_1^2 + 2u_0^{-1/\alpha} u_2\right) = \frac{(\alpha)_3}{2!\alpha} x^2, \quad u_3(x) = \frac{(\alpha)_3}{3!} x^3, \\
A_3\left(u_0(x), u_1(x), u_2(x), u_3(x)\right) &= \frac{1}{3!}\left(1-\frac{1}{\alpha}\right)\left(\left(\frac{1}{\alpha}+\frac{1}{\alpha^2}\right) u_0^{-1/\alpha-2} u_1^3 - \frac{4}{\alpha} u_0^{-1/\alpha-1} u_1 u_2 + 6u_0^{-1/\alpha} u_3 \right. \\
&\quad \left. - \frac{2}{\alpha} u_0^{-1/\alpha-1} u_1 u_2\right) = \frac{(\alpha)_4}{3!\alpha} x^3, \\
u_4(x) &= \frac{(\alpha)_4}{4!} x^4,
\end{aligned}$$

and so on.

Also, using the method of separation of variables, it is easy to see that $u(x) = (1+x)^{\alpha}$ is the solution to (3.9). Since $|x| < 1$, expand $(1+x)^{\alpha}$ by using generalized binomial theorem

$$u(x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!} x^n = u(0) + \sum_{n=1}^{\infty} u_n(x). \tag{3.12}$$

By the uniqueness of the solution to the differential equation (3.9), the series solution obtained by ADM is consistent with (3.12). Therefore, the n-th component of $u(x)$ is

$$u_n(x) = \frac{(\alpha)_n}{n!} x^n, \ \ n \geq 0. \tag{3.13}$$

Now, from (3.11) and (3.13), we have

$$\begin{aligned} \frac{(\alpha)_{n+1}}{(n+1)!} x^{n+1} = \alpha \int_0^x A_n\left(u_0(t), u_1(t), \dots, u_n(t)\right) \mathrm{d}t &= \alpha \int_0^x \sum_{k=1}^{n} N^{(k)}(u_0) \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{u_j^{k_j}(t)}{k_j!} \mathrm{d}t \qquad \text{(using (1.8) and (1.9))} \\ &= \alpha \int_0^x t^n \, \mathrm{d}t \sum_{k=1}^{n} \left(1 - \frac{1}{\alpha}\right)_k \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{1}{k_j!} \left(\frac{(\alpha)_j}{j!}\right)^{k_j} \qquad \text{(using (3.13))} \\ &= \frac{\alpha}{n+1} x^{n+1} \sum_{k=1}^{n} \left(1 - \frac{1}{\alpha}\right)_k \sum_{\Lambda_n^k} \prod_{j=1}^{n-k+1} \frac{1}{k_j!} \left(\frac{(\alpha)_j}{j!}\right)^{k_j}, \end{aligned}$$

where the last step follows on using Lemma 2.4. On rearranging the terms, we get

$$\sum_{k=1}^{n} \left(1 - \frac{1}{\alpha}\right)_k \sum_{\Lambda_n^k} n! \prod_{j=1}^{n-k+1} \frac{1}{k_j!} \left(\frac{(\alpha)_j}{j!}\right)^{k_j} = (\alpha - 1)_n,$$

which is the required identity. □

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