Algebraic geometry

Birational geometry of the moduli space of pure sheaves on quadric surface

Géométrie birationnelle de l'espace moduli des faisceaux purs sur une surface quadrique

Kiryong Chung\textsuperscript{a}, Han-Bom Moon\textsuperscript{b}

\textsuperscript{a} Department of Mathematics Education, Kyungpook National University, 80 Daehakro, Bukgu, Daegu 41566, Republic of Korea
\textsuperscript{b} School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, United States

1. Introduction

The geometry of the moduli space of sheaves on a projective plane has been studied from various viewpoints, for instance curve counting, the strange duality conjecture, and birational geometry via Bridgeland stability. For a detailed description of the motivation, see \cite{6} and references therein. Even further, for small degree cases, it was possible to classify all rational contractions ([5, Section 1.3]) and compute the cohomology ring of the moduli space ([5, Theorem 1.2]).

It is natural to extend this result to del Pezzo surfaces. In this paper, we consider the next simplest case of a quadric surface. Here we construct a flip between the moduli space of sheaves and a projective bundle, and show that their common blown-up space is the moduli space of stable pairs ([12]). We expect that this analysis provides some insight into the study...
of a general Bridgeland wall-crossing over the moduli space of sheaves on a del Pezzo surface. To the authors’ knowledge, there is no explicit study of wall-crossings in the case of moduli spaces of torsion sheaves on smaller-degree del Pezzo surfaces.

Let \( Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \) be a smooth quadric surface in \( \mathbb{P}^3 \) with a very ample polarization \( L := O_Q(1, 1) \). For the convenience of the reader, we start with a list of relevant moduli spaces.

**Definition 1.1.**

1. Let \( M := M_1(Q, (2, 3), 5m + 1) \) be the moduli space of stable sheaves \( F \) on \( Q \) with \( c_1(F) = c_1(O_Q(2, 3)) \) and \( \chi(F(m)) = 5m + 1 \).
2. Let \( M^\alpha := M_1(Q, (2, 3), 5m + 1) \) be the moduli space of \( \alpha \)-stable pairs \((s, F)\) with \( c_1(F) = c_1(O_Q(2, 3)) \) and \( \chi(F(m)) = 5m + 1 \) ([112] and [7, Theorem 2.6]). Let \( M^\alpha := M^\alpha \) for \( 0 < \alpha \ll 1 \).
3. Let \( G = Gr(2, 4) \) and let \( G_1 \) be the blow-up of \( G \) along \( \mathbb{P}^1 \) that parametrizes projective lines in \( Q \subset \mathbb{P}^3 \) of type \((1, 0)\) (Section 2.1).
4. Let \( P := P(U) \) and \( P^- := P(U^-) \), where \( U \) (resp. \( U^- \)) is a rank 10 vector bundle over \( G \) (resp. \( G_1 \)) defined in [3] in Section 2.1 (resp. Section 3.3).

The aim of this paper is to explain and justify the following commutative diagram between moduli spaces.

\[ \begin{array}{ccc}
M^\alpha & \longrightarrow & P^- = P(U^-) \\
\downarrow & & \downarrow \mu \\
M & \longrightarrow & G_1 \\
\end{array} \]

We have to explain two flips (dashed arrows) on the diagram.

One of key ingredients is the *elementary modification* of vector bundles ([14]), sheaves ([8, Section 2.B]), and pairs ([3, Section 2.2]). It has been widely used in the study of sheaves on a smooth projective variety. Let \( F \) be a vector bundle on a smooth projective variety \( X \) and \( Q \) be a vector bundle on a smooth divisor \( Z \subset X \) with a surjective map \( F|_Z \twoheadrightarrow Q \). The elementary modification of \( F \) along \( Z \) is the kernel of the composition

\[ \text{elm}_Z(F) := \ker(F \twoheadrightarrow F|_Z \twoheadrightarrow Q). \]

A similar definition is valid for sheaves and pairs, too. Note that the category of pairs is abelian ([7, Theorem 1.3]).

On \( G_1 \), let \( U^- := \text{elm}_{Y_{10}}(u^*U) \) be the elementary transformation of \( u^*U \) along a smooth divisor \( Y_{10} \) (Section 2.1).

**Proposition 1.2.** Let \( P^- = P(U^-) \). The flip \( P^- \dashrightarrow P(u^*U) = G_1 \times_G P(U) \) is a composition of a blow-up and a blow-down. The blow-up center in \( P^- \) (resp. \( P(u^*U) \)) is a \( \mathbb{P}^1 \) (resp. \( \mathbb{P}^7 \))-bundle over \( Y_{10} \).

**Theorem 1.3.** There is a flip between \( M \) and \( P^- \), which is a blow-up followed by a blow-down, and the master space is \( M^\alpha \), the moduli space of \( +\)-stable pairs (Definition 1.1 (2)).

As the referee pointed out, all morphisms in (1) are \( SL_2 \)-equivariant for the natural \( SL_2 \)-action on the second ruling of \( Q \). Thus one may expect an \( SL_2 \)-quotient version of the main result. We did not pursue this direction because we could not find any explicit moduli theoretic interpretation.

As applications, we compute the Poincaré polynomial of \( M \) and show the rationality of \( M \) (Corollary 3.8), which were obtained by Maican by different methods ([13]). Since each step of the birational transformation is described in terms of blow-ups/downs along explicit subvarieties, in principle the cohomology ring and the Chow ring of \( M \) can be obtained from that of \( G \). Also one may aim for the completion of Mori’s program for \( M \). We will carry on these projects in forthcoming papers.

## 2. Relevant moduli spaces

In this section, we give definitions and basic properties of some relevant moduli spaces.

### 2.1. Grassmannian as a moduli space of Kronecker quiver representations

The moduli space of representations of a Kronecker quiver parametrizes the isomorphism classes of stable sheaf homomorphisms

\[ O_Q(0, 1) \longrightarrow O_Q(1, 2)^{\oplus 2} \]

up to the natural action of the automorphism group \( \mathbb{C}^* \times GL_2/\mathbb{C}^* \cong GL_2 \). For two vector spaces \( E \) and \( F \) of dimensions 1 and 2, respectively, and \( V^* := H^0(Q, L) \), the moduli space is constructed as \( G := \text{Hom}(F, V^* \otimes E)/GL_2 \cong V^* \otimes E \otimes F^*/GL_2 \),
with an appropriate linearization ([9]). We regard \( \mathbf{G} \) as a moduli space of complexes. But also note that the \( \text{GL}_2 \) acts as a row operation on the space of \( 2 \times 4 \) matrices, thus \( \mathbf{G} \cong \text{Gr}(2, V^*) \cong \text{Gr}(2, 4) \).

Let \( \mathbf{H}(n) \) be the Hilbert scheme of \( n \) points on \( \mathbb{Q} \). There is a birational map \( \mathbf{H}(2) \rightarrow \mathbf{G} \) that maps \( Z \) to a resolution of \( I_Z(2, 3) \) of the type (2). For any \( Z \in \mathbf{H}(2) \), let \( \ell_Z \) be the unique line in \( \mathbb{P}^3 \supset \mathbb{Q} \) containing \( Z \). Then either \( \ell_Z \cap \mathbb{Q} = \emptyset \) or \( \ell_Z \subset \mathbb{Q} \). In the second case, the class of \( \ell_Z \) is of the type \((1, 0)\) or \((0, 1)\). Let \( Y_{10} \) (resp. \( Y_{01} \)) be the locus of subschemes such that \( \ell_Z \) is a line of the type \((1, 0)\) (resp. \((0, 1)\)). Then \( Y_{10} \) and \( Y_{01} \) are two disjoint subvarieties that are isomorphic to a \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^1 \).

**Proposition 2.1** ([1, Example 6.1]). There exists a morphism \( t : \mathbf{H}(2) \rightarrow \mathbf{G}_1 \rightarrow \mathbf{G} \). The first (resp. the second) map contracts the divisor \( Y_{01} \) (resp. \( Y_{10} \)) to \( \mathbb{P}^1 \). If \( \ell_Z \cap \mathbb{Q} = \emptyset \), then \( t(Z) \) is a resolution of \( I_Z(2, 3) \). If \( Z \in Y_{10} \), then \( t(Z) \) is a resolution of \( E_{10} \in \mathbb{P}(\mathbb{E}^1(\mathcal{O}(1, 3), \ell_Z(1))) \) (pt). If \( Z \in Y_{01} \), then \( t(Z) \) is a resolution of \( E_{01} \in \mathbb{P}(\mathbb{E}^1(\mathcal{O}(2, 2), \ell_Z(1))) \) (pt).

The morphism \( \wedge^2 V^* \otimes H^0(\mathcal{O}(0, 1)) \rightarrow V^* \otimes H^0(\mathcal{O}(2, 2)) \) induces the universal morphism \( \phi : p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1, 2) \rightarrow p_1^* \mathcal{S} \otimes p_2^* \mathcal{O}(1, 2) \) where \( p_1 : \mathbb{G} \rightarrow \mathcal{G} \) and \( p_2 : \mathbb{G} \times \mathbb{Q} \rightarrow \mathcal{G} \) are two projections ([9, Proposition 5.3]), and \( \mathcal{S} \) is the universal subbundle of \( \mathbf{G} \). Let \( \mathcal{U} \) be the cokernel of \( p_{1*} \phi \). On the stable locus, \( p_{1*} \phi \) is injective. Thus we have an exact sequence

\[
0 \rightarrow \mathcal{O}_G(-1) \otimes H^0(\mathcal{O}(0, 1)) \xrightarrow{p_{1*} \phi} \mathcal{S} \otimes H^0(\mathcal{O}(1, 2)) \rightarrow \mathcal{U} \rightarrow 0
\]

and \( \mathcal{U} \) is a rank-10 vector bundle. Let \( \mathcal{P} := \mathcal{P}(\mathcal{U}) \).

### 2.2. Moduli space \( \mathcal{M} \) of stable sheaves

Recall that \( \mathcal{M} := \mathcal{M}_i(\mathbb{Q}, (2, 2), 5m + 1) \) is the moduli space of stable sheaves \( F \) on \( \mathbb{Q} \) with \( c_1(F) = c_1(\mathcal{O}(2, 3)) \) and \( \chi(F(m)) = 5m + 1 \). There are four types of points in \( \mathcal{M} \) ([13, Theorem 1.1]). Let \( \mathcal{C} \in \mathcal{O}(2, 3) \).

1. \( F = \mathcal{O}_G(p + q) \), where the line \( (p, q) \) is not contained in \( \mathbb{Q} \);
2. \( F = \mathcal{O}_G(p + q) \), where the line \( (p, q) \) in \( \mathbb{Q} \) is of type \((1, 0)\);
3. \( F = \mathcal{O}_G(0, 1) \);
4. \( F \) fits into a non-split extension \( 0 \rightarrow \mathcal{O}_E \rightarrow F \rightarrow \mathcal{O}_G \rightarrow 0 \) where \( E \) is a \((2, 2)\)-curve and \( \ell \) is a \((0, 1)\)-line.

Let \( \mathcal{M}_i \) be the locus of sheaves of the form (i). Each \( \mathcal{M}_i \) is a subvariety of codimension \( i \) in \( \mathcal{M} \) and for \( i > 0 \), \( \mathcal{M}_i \) is closed. \( \mathcal{M}_1 \) is a \( \mathbb{P}^3 \)-bundle over \( \mathbb{P}^2 \times \mathbb{P}^1 \). \( \mathcal{M}_2 \) is a singular subvariety that admits a finite birational map from a \( \mathbb{P}^1 \)-bundle over \( \mathcal{O}(2, 2) \) and \( \mathbb{Q}(0, 1) \). \( \mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_3 = \emptyset \) ([13, Theorem 1.1]). \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have been parametrized by \( \mathcal{M}_3 \) and \( \mathbb{P}^2 \) respectively.

### 2.3. Moduli spaces of stable pairs

A pair \((s, F)\) consists of \( F \in \text{Coh}(\mathbb{Q}) \) and a section \( \mathcal{O}_\mathcal{Q} \rightarrow F \). Fix \( \alpha \in \mathbb{Q}_{>0} \). A pair \((s, F)\) is called \( \alpha \)-semistable (resp. \( \alpha \)-stable) if \( F \) is pure and, for any proper subsheaf \( F' \subset F \), the inequality

\[
\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \leq (\leq) \frac{P(F)(m) + \alpha}{r(F)}
\]

holds for \( m \gg 0 \). Here \( \delta = 1 \) if the section \( s \) factors through \( F' \) and \( \delta = 0 \) otherwise. Let \( \mathcal{M}^{s} := \mathcal{M}^{s}(\mathbb{Q}, (2, 3), 5m + 1) \) be the moduli space of \( s \)-equivalence classes of \( \alpha \)-semistable pairs \((s, F)\) such that the support of \( F \) has a class \( c_1(\mathcal{O}(2, 3)) \) ([12, Theorem 4.12] and [7, Theorem 2.6]). The extremal case that \( \alpha \) is sufficiently large (resp. small) is denoted by \( \alpha = \infty \) (resp. \( \alpha = + \)). The deformation theory of pairs is studied in [7, Corollary 1.6 and Corollary 3.6].

**Proposition 2.2.**

1. ([4, Lemma 2.2 (3)]) There exists a natural forgetful map \( r : \mathcal{M}^+ \rightarrow \mathcal{M} \) which maps \((s, F)\) to \( F \).
2. ([7, Section 4.4]) The moduli space \( \mathcal{M}^{\infty} \) of \( \infty \)-stable pairs is isomorphic to the relative Hilbert scheme of two points on the complete linear system \( \mathcal{O}(2, 3) \).

The birational map \( \mathcal{M}^{\infty} \rightarrow \mathcal{M}^+ \) is analyzed in [13, Theorem 5.7]. It turns out that this is a single flip over \( \mathcal{M}^+ \) and is a composition of a smooth blow-up and a smooth blow-down. By identifying the space \( \mathcal{M}^{\infty} \) as the relative Hilbert scheme (Proposition 2.2 (2)), the blow-up center is isomorphic to a \( \mathbb{P}^2 \)-bundle over \( \mathcal{O}(2, 2) \times \mathcal{O}(0, 1) \), where a fiber \( \mathbb{P}^2 \) parameterizes two points lying on a \((0, 1)\)-line. After the fiber, the flipped locus, denoted by \( \mathcal{M}_i^2 \), on \( \mathcal{M}^2 \) is a \( \mathbb{P}^1 \)-bundle over \( \mathcal{O}(2, 2) \times \mathcal{O}(0, 1) \). For the forgetful map \( r : \mathcal{M}^+ \rightarrow \mathcal{M} \), we define \( \mathcal{M}_i^+ := \mathcal{M}_i^{r^{-1}(M_i)} \) if \( i \neq 3 \). Then \( r(\mathcal{M}_i^+) = \mathcal{M}_i \), but \( r : \mathcal{M}_3^+ \rightarrow \mathcal{M}_3 \) is a birational finite map (this implies that \( \mathcal{M}_3 \) is not normal). The map \( r \) contracts \( \mathcal{M}_3^+ \), which is a
3. Decomposition of the birational map between $M$ and $P$

In this section, we prove Proposition 1.2 and Theorem 1.3 by describing the birational map between $M$ and $P$.

3.1. Construction of a birational map $M^+ -\rightarrow P$

Lemma 3.1. There exists a surjective morphism $w : M^+ \rightarrow G$ that maps $(s, O_C(p + q)) \in M_{1+}^+ \rightarrow l_{(p, q)}(2, 3)$, maps $(s, O_C(p + q)) \in M_{1+}^+ \rightarrow (1, 0)$, maps $(s, f) \in M_{2}^+ \rightarrow (0, 1)$ determined by a section, and maps $(s, f) \in M_{2}^+ \rightarrow \ell$ (see Section 2.2 for the notation), a $(0, 1)$-line.

Proof. By Proposition 2.2, $M^\infty$ is the relative Hilbert scheme of two points on the universal $(2, 3)$-curves, which is a $\mathbb{P}^1$-bundle over $H(2)$ ([3, Lemma 2.3]). By composing with $t : H(2) \to G$ in Proposition 2.1, we have a morphism $M^\infty \rightarrow G$. On the other hand, since the flip $M^\infty \rightarrow M^+$ is the composition of a single blow-up/down, the blown-up space $M^\infty$ admits two morphisms to $M^\infty$ and $M^+$, and the flipped locus is $M_{1+}^+$. Note that each point in $M_{1+}^+$ can be regarded as a collection of data $(\ell, e)$, where $\ell$ is a $(2, 2)$-bundle, $e$ is a $(0, 1)$-line, and $e \in \text{Pic}^1(F, O_{2})$. The fiber $M^\infty \rightarrow M^+$ over the point in the blow-up center $M_{1+}^+$ is a $\mathbb{P}^2$ that parameterizes two points on $\ell$. The composition map $M^\infty \rightarrow M^+ \rightarrow G$ is constant along the $\mathbb{P}^2$, but $G$ does not remember points on the line $\ell \subset Q$. By the rigidity lemma ([10, Lemma 1.6]), $M^\infty \rightarrow G$ factors through $M^+$, and we obtain a map $w : M^+ \rightarrow G$. \[\square\]

Note that $M_{1+}^+ \cong M_1$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2 \times \mathbb{P}$ and $M_{1+}^+ \cong \mathbb{P}^1$-bundle over $|O_{Q}(2, 3)| \cong \mathbb{P}^1$. They are disjoint divisors on $M^+$.

Proposition 3.2. There is a birational morphism $q : M^+ \setminus M_{1+}^+ \rightarrow P = \mathbb{P}(\mathcal{L})$ such that $p \circ q : M^+ \setminus M_{1+}^+ \rightarrow P \rightarrow G$ coincides with $w|_{M^+ \setminus M_1^+}$ in Lemma 3.1. Furthermore, $q$ is the smooth blow-down along $M_{1+}^+$.

The proof consists of several steps. Since $P = \mathbb{P}(\mathcal{L})$ is a projective bundle over $G$, it is sufficient to construct a surjective homomorphism $w^{-1} \mathcal{L} \rightarrow \mathcal{L} \rightarrow 0$ on $M^+ \setminus M_{1+}^+$ for some $\mathcal{L} \in \text{Pic}(M^+ \setminus M_{1+}^+)$, or equivalently, a bundle morphism $0 \rightarrow \mathcal{L}^+ \rightarrow w^*\mathcal{L}$.

Recall that a family $(\mathcal{L}, F)$ of pairs on a scheme $S$ is a collection of data $\mathcal{L} \in \text{Pic}(S), F \in \text{Coh}(S \times Q)$, which is a flat family of pure sheaves, and a surjective morphism $\mathcal{E}xt_{F}^{2}(\mathcal{F}, \omega_{p}) \rightarrow \mathcal{L}$ where $\pi : S \times Q \rightarrow S$ is the projection and $\omega_{p}$ is the totally dualizing sheaf (see [12, Proposition 4.3] for the explanation of why we take the dual). Now let $(\mathcal{L}, F)$ be the universal pair ([7, Theorem 4.8]) on $M^+ \times \mathcal{Q}$.

Applying $\text{Hom}(\mathcal{L}, \mathcal{O}) \rightarrow \mathcal{E}xt_{F}^{2}(\mathcal{F}, \omega_{p}) \rightarrow \mathcal{L}$, we obtain $0 \rightarrow \mathcal{L}^+ \rightarrow \text{Hom}(\mathcal{E}xt_{\mathcal{F}}^{2}(\mathcal{F}, \omega_{p}), \mathcal{O}) \cong \mathcal{E}xt_{\mathcal{F}}^{1}(\mathcal{E}xt_{\mathcal{F}}^{1}(\mathcal{F}, \mathcal{O}), \mathcal{O})$ (see [5, Section 3.2]). So we have a non-zero element $e \in \text{Hom}(\mathcal{L}^+, \mathcal{E}xt_{\mathcal{F}}^{2}(\mathcal{F}, \mathcal{O}), \mathcal{O}) \cong \mathcal{E}xt_{\mathcal{F}}^{1}(\mathcal{E}xt_{\mathcal{F}}^{1}(\mathcal{F}, \mathcal{O}), \mathcal{O})$ (5, Section 3.2), which provides $0 \rightarrow \mathcal{L}^+ \rightarrow \mathcal{E}xt_{\mathcal{F}}^{1}(\mathcal{F}, \mathcal{O}) \rightarrow 0$ on $\mathcal{M}^+ \times Q$. By taking $\text{Hom}(\mathcal{L}, \mathcal{O})$, we have $\mathcal{E}xt_{\mathcal{F}}^{2}(\mathcal{F}, \omega_{p}) \rightarrow \mathcal{E}xt_{\mathcal{F}}^{2}(\mathcal{F}, \mathcal{O}) \cong \mathcal{L}^+ \rightarrow 0$ because $\mathcal{L}$ is a line bundle. This implies the existence of a flat family of pairs $(\mathcal{L}, \mathcal{E})$ on $M^+ \times Q$. We may explicitly describe this construction fiberwisely in the following way. Let $(s, F) \in M^+$. Let $F^D := \mathcal{E}xt_{s}(\mathcal{F}, \omega_{p})$. For a non-zero section $s \in H^1(F) \cong \mathbb{H}^{1}(F^D)^+ \cong \mathcal{E}xt_{s}(F^D, 2, 2), O_{Q})$, we have a pair $(s^*, G)$ given by

\[0 \rightarrow O_{Q} \rightarrow G \rightarrow F^{D}(2, 2) \rightarrow 0.\]

(4)

The first isomorphism comes from [2, Proposition 4.2.8], and the section $s^*$ is the one-dimensional vector space dual to $s$ ([11, Theorem 5.5]).

Lemma 3.3. The map $(s, F) \mapsto (s^*, G)$ defines a dominant rational map $M^+ -\rightarrow P = \mathbb{P}(\mathcal{L})$, which is regular precisely on $M^+ \setminus (M_{1+}^+ \cup M_{2}^+)$. 

Proof. Since we have a relative construction of pairs, it suffices to describe the extension $(s^*, G)$ set theoretically. If $(s, F) \in M_{1+}^+ \setminus M_{1+}^+$, then $F \cong \mathcal{O}_{C}(p + q) \cong \mathcal{O}_{C}(0, -1)$ for some curve $C$ and $Z = [p, q] \in H(2)$ such that the line $\ell_{z}$ containing $Z$ is not in $Q$ ([7, Section 4.4]). Then $F^D(2, 2) \cong 1Z_{C}(2, 3)$. Since $\mathcal{E}xt_{1}(F^D, 2), O_{Q}) \cong \mathbb{H}^{1}(F^D)^+ \cong \mathbb{H}^{1}(F), O_{Q}) \cong \mathbb{C},$ from $0 \rightarrow O_{Q}(-2, -3) \rightarrow \mathcal{L}_{C} \rightarrow \mathcal{L}_{C} \rightarrow 0$, we obtain $G = \mathcal{L}_{C}(2, 3)$. If $(s, F) \in M_{1+}^+$, then we have a pair $(s^*, G) \in P$ because $G$ has a resolution of the form $O_{Q}(0, 1) \rightarrow O_{Q}(1, 2)^{\mathbb{P}^2}$. However, if $(s, F) \in M_{1+}^+$, then we have $0 \rightarrow O_{Q}(2, 3) \rightarrow G \cong O_{Q}(2, 3) \rightarrow 1Z_{C}(2, 3) \rightarrow 0$ and $1Z_{C}(2, 3) = O_{Q}(1, 3), 1Z_{C}(2, 3) = O_{Q}(1, 3).$ In particular, $\text{Hom}(O_{Q}(1, 3), G) \neq 0$, and $G$ does not admit a resolution $O_{Q}(0, 1) \rightarrow O_{Q}(1, 2)^{\mathbb{P}^2}$. So $G \notin G$.

Suppose that $(s, F) \in M_{1+}^+ \setminus M_{1+}^+$. Then $F$ fits into a non-split extension $0 \rightarrow O_{E} \rightarrow F \rightarrow \ell \rightarrow 0$. Apply $\text{Hom}(-, \omega_{Q})$, then we have $0 \rightarrow O_{E}(0, 1) \rightarrow F^D(2, 2) \rightarrow O_{E}(2, 2) \rightarrow 0$. By taking the functor $\text{Ext}^1(-, O_{Q})$ in this short exact sequence, one
can see that $\text{Ext}^1(O_E(2, 2), O_Q) \cong \text{Ext}^1(F^D(2, 2), O_Q) \cong H^1(F^D) \cong H^0(F)^* \cong \mathbb{C}$ because of Serre duality and [2, Proposition 4.2.8]. Hence the sheaf $G$ is given by the pull-back:

\[
\begin{array}{cccccc}
0 & \rightarrow & O_Q & \rightarrow & O_Q(2, 2) & \rightarrow & O_E(2, 2) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & O_Q & \rightarrow & G & \rightarrow & F^D(2, 2) & \rightarrow & 0
\end{array}
\]  

(5)

By applying the snake lemma to (5), we conclude that the unique non-split extension $G$ lies on $0 \rightarrow O_Q(0, 1) \rightarrow G \rightarrow O_Q(2, 2) \rightarrow 0$. Hence, $G \in \mathcal{G}$ (Proposition 2.1), and we have an element $(s^*, G) \in \mathcal{P}$.

Now suppose that $(s, F) \in \mathcal{M}_1^+$, so $F = O_C(0, 1)$. Then $F^D(2, 2) = O_C(2, 2)$. So we have $0 \rightarrow O_Q \xrightarrow{s^*} G \rightarrow O_C(2, 2) \rightarrow 0$.

By the snake lemma (consult the proof of [5, Lemma 3.7]), $G$ fits into

\[
0 \rightarrow O_Q(2, 2) \rightarrow G \rightarrow O_\ell \rightarrow 0
\]  

(6)

where $\ell$ is the line of type $(0, 1)$ determined by the section $s$. So $\text{Hom}(O_Q(2, 2), G) \neq \emptyset$ and this implies $G$ does not admit a resolution $O_Q(0, 1) \rightarrow O_Q(1, 2)^{\oplus 2}$. Thus the correspondence is not well defined on $\mathcal{M}_1^+$. □

3.2. The first elementary modification and the extension of the domain

We can extend the morphism in Lemma 3.3 by applying an elementary modification of pairs ([3, Section 2.2]) on $\mathcal{M}_2^+$.

**Lemma 3.4.** There exists an exact sequence of pairs $0 \rightarrow (0, K) \rightarrow ((\mathcal{L}^\ast |_{\mathcal{M}_2^+}, \mathcal{E}|_{\mathcal{M}_2^+} \times Q) \rightarrow (\mathcal{L}''', \mathcal{O}_Z) \rightarrow 0$ where $Z$ is the pull-back of the universal family of $(0, 1)$-lines to $\mathcal{M}_2^+ \times Q$ and $\mathcal{K}_{[m] \times Q} \cong O_Q(2, 2)$ for $m = [s(F)] \in \mathcal{M}_2^+$.

**Proof.** By relativizing the short exact sequence (6) in the proof of Lemma 3.3, there is an exact sequence of sheaves $0 \rightarrow K \rightarrow \mathcal{E}|_{\mathcal{M}_2^+ \times Q} \rightarrow \mathcal{O}_Z \rightarrow 0$. To obtain the short exact sequence of pairs in the statement of the lemma, it is sufficient to show that, for each fiber $G = \mathcal{E}|_{[(s, F)] \times Q}$, the section $s^*$ of $G$ does not come from $H^0(O_Q(2, 2))$. If it is, we have an injection $O_Q \subset O_Q(2, 2)$ whose cokernel is $O_E(2, 2)$ for some curve $E$ of arithmetic genus one. By the snake lemma once again, we obtain $0 \rightarrow \mathcal{O}_E(2, 2) \rightarrow F^D(2, 2) = O_C(2, 2) \rightarrow O_\ell \rightarrow 0$. It violates the stability of $F^D(2, 2)$. □

Let $(\mathcal{L}', \mathcal{E}')$ be the elementary modification of $(\mathcal{L}''', \mathcal{O}_Z)$ along $\mathcal{M}_2^+$, i.e.

\[
\text{Ker}((\mathcal{L}^\ast, \mathcal{E}) \rightarrow (\mathcal{L}' |_{\mathcal{M}_2^+}, \mathcal{E}' |_{\mathcal{M}_2^+} \times Q) \rightarrow (\mathcal{L}''', \mathcal{O}_Z)).
\]

**Lemma 3.5.** For a point $m = [(s, F = O_C(0, 1))] \in \mathcal{M}_2^+$, the modified pair $(\mathcal{L}', \mathcal{E}')|_{[m] \times Q}$ fits into a non-split exact sequence $0 \rightarrow (s', \mathcal{O}_\ell) \rightarrow (s', \mathcal{E}'|_{[m] \times Q}) \rightarrow (0, \mathcal{O}_Q(2, 2)) \rightarrow 0$ where $\ell$ is a $(0, 1)$-line.

**Proof.** An elementary modification of pairs interchanges the sub pair with the quotient pair ([7, Lemma 4.24]). Thus we obtain the sequence. It remains to show that the sequence is non-split. We will show that the normal bundle $N_{\mathcal{M}_2^+ / \mathcal{M}_2^+}$ at $m$ is canonically isomorphic to $H^0(O_C)^\ast$. Then the element $m$ corresponds to the projective equivalence class of nonzero elements in $H^0(O_C)^\ast \cong \text{Ext}^1((0, O_Q(2, 2)), (s', O_C))$, so it is non-split ([3, Theorem 3.3]).

The $+\text{-stable pair} (s, F)$ fits into $0 \rightarrow (0, O_Q(2, 2)) \rightarrow (s, O_Q(0, 1)) \rightarrow (s, F) \rightarrow 0$. Since

\[
\text{Ext}^0((s, F), (s, F)) \equiv \text{Ext}^0((s, O_Q(0, 1)), (s, F)) \equiv \text{Ext}^0(O_Q(0, 1), F) \equiv H^0(O_C) = \mathbb{C}
\]  

([7, Corollary 1.6]), we have

\[
0 \rightarrow \text{Ext}^0((0, O_Q(2, 2)), (s, F)) \rightarrow \text{Ext}^1((s, F), (s, F)) \rightarrow \text{Ext}^1((s, O_Q(0, 1)), (s, F)) \rightarrow \cdots
\]

The first term $\text{Ext}^0((0, O_Q(2, 2)), (s, F)) \equiv H^0(O_C(2, 2)) \cong \mathbb{C}^{11}$ is the deformation space of the curve $C$ on $Q$. The second term $\text{Ext}^1((s, F), (s, F))$ is $\text{TM}^{+1}$ ([7, Theorem 3.12]). For the third term, by [7, Theorem 3.12] again, we have

\[
0 \rightarrow \text{Hom}(s, H^0(F)/\langle s \rangle) \rightarrow \text{Ext}^1((s, O_Q(0, 1)), (s, F)) \rightarrow \text{Ext}^1(O_Q(0, 1), F) \rightarrow \text{Hom}(s, H^1(F)).
\]

The first term $\text{Hom}(s, H^0(F)/\langle s \rangle) = \mathbb{C}$ is the deformation space of the line $\ell$ in $Q$ determined by the section $s$. By Serre’s duality, $\phi : H^0(O_Q(0, 1))^\ast \rightarrow H^1(O_Q)^\ast$ and the kernel is $H^0(O_C(0, 1))^\ast \equiv H^0(O_C)^\ast$. This proves our assertion. □

Recall that the modified pair $(\mathcal{L}', \mathcal{E}')$ provides a natural surjection $\text{Ext}^2_{\mathcal{M}^+}(\mathcal{E}', \omega_{\mathcal{X}}) \rightarrow \mathcal{L}'$ on $\mathcal{M}^+ \times Q$. By Lemmas 3.3 and 3.5, it is straightforward to check that $\text{Ext}^2_{\mathcal{M}^+}(\mathcal{E}', \omega_{\mathcal{X}})$ has rank 10 at each fiber, thus it is locally free.
Proof of Proposition 3.2. We claim that there exists a surjection \( w^*\mathcal{U} \to L' \to 0 \) up to a twisting by a line bundle on \( M^+ \setminus M^+_1 \). Then there is a morphism \( M^+ \setminus M^+_1 \to P \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
(M^+ \setminus M^+_1) \times Q & \xrightarrow{w := w \times \text{id}} & G \times Q \\
\pi \downarrow & & \downarrow \\
M^+ \setminus M^+_1 & \xrightarrow{w} & G.
\end{array}
\]

Note that \( \mathcal{U} = \pi_\ast (\mathcal{W}) \) where \( \mathcal{W} = \text{coker}(\phi) \) is the universal quotient on \( G \times Q \) (Section 2.1). One can check that \( \mathcal{W} \) is flat over \( G \). By construction of \( w \), \( \mathcal{W} \mid_{M^+ \setminus M^+_1} \cong w^*\mathcal{W} \mid_{M^+ \setminus M^+_1} \) restricted to each point \( m \in M^+ \setminus M^+_1 \). The universal property of \( G \) (as a quiver representation space [9, Proposition 5.6]) tells us that \( w^*\mathcal{W} \cong \mathcal{E}' \) up to a twisting by a line bundle on \( M^+ \setminus M^+_1 \). The base change property implies that there exists a natural isomorphism (up to a twisting by a line bundle) \( w^*\mathcal{U} = w^*(\pi_\ast \mathcal{W}) \cong \pi_\ast (w^*\mathcal{W}) = \pi_\ast \mathcal{E}' \cong \mathcal{E} \otimes \omega_{\mathcal{E}}^* \) by [12, Corollary 8.19]. Hence we have \( w^*\mathcal{U} \cong (w^*\mathcal{U})^* \cong (\pi_\ast (\mathcal{E}))^* \cong \mathcal{E} \otimes \omega_{\mathcal{E}} \to L' \).

Therefore we obtain a morphism \( q : M^+ \setminus M^+_1 \to P \).

By the proof of Lemma 3.5, the modified pair does not depend on the choice of a (2, 3)-curve, so \( q : M^+ \setminus M^+_1 \to P \setminus p^{-1}(\{z(\ell)\}) \) is indeed a contraction of \( M^+_{(\ell)} \) and the image of \( M^+_{(\ell)} \) is isomorphic to \( \mathbb{P}^1 \). Recall that the exceptional divisor \( M^+_{(\ell)} \) is \( \{\mathcal{O}_Q(2, 3) \otimes [\mathcal{O}_Q(0, 1)] \cong \mathbb{P}^1 \times \mathbb{P}^1 \}. \) Note that the sheaf \( F \) in the pair \( (s, F) \in M^+_{(\ell)} \) is parametrized by \( \mathbb{P}^1 = [\mathcal{O}_Q(2, 3)] = \mathbb{P}^\times(\mathcal{O}_Q(2, 3) \otimes [\mathcal{O}_Q(0, 1)] , \mathcal{O}_Q(0, 1)) \). By analyzing \( T_F M = \mathcal{E} \mathcal{W} \mathcal{F} \) (which is similar to [3, Lemma 3.4]), one can see that \( \mathcal{N}_{M^+_{(\ell)}} / \mathcal{O}^p_{11} \cong \mathcal{E} \mathcal{W} \mathcal{F} (\mathcal{O}_Q(0, 1), \mathcal{O}_Q(2, 3) \otimes [\mathcal{O}_Q(0, 1)] \otimes \mathcal{O}^p_{11} \otimes (\mathcal{O}_Q(0, 1)) = \mathbb{P}^1 \). Thus \( \mathcal{N}_{M^+_{(\ell)}} / \mathcal{O}^p_{11} \mathcal{W}^1 \mathcal{F} \mathcal{W}^1 \mathcal{F} \mathcal{W}^1 \mathcal{F} \mathcal{W}^1 \mathcal{F} = \mathbb{P}^1 \) and \( q \) is a smooth blow-down by Fujiki–Nakano criterion [6].

Thus we have two different contractions of \( M^+ \), one is \( M \) obtained by contracting all \( \mathbb{P}^1 \)-fibers on \( M^+_{(\ell)} \), and the other one is defined just below.

**Definition 3.6.** Let \( M^- \) be the contraction of \( M^+ \) which is obtained by contracting all \( \mathbb{P}^1 \)-fibers on \( M^+_{(\ell)} \). We define \( M^-_{(\ell)} \) as the image of \( M^+_{(\ell)} \) for the contraction \( M^+ \to M^- \).

### 3.3. The second elementary modification and \( M^- \)

Recall that \( u : G_1 \to G \) is the blow-up of \( G \) along the \( \mathbb{P}^1 \) parameterizing \((1, 0)\)-lines in \( Q \), and \( Y_{10} \) is the exceptional divisor. \( V \) be the kernel of the universal morphism \( \phi \) on \( G \times Q \) in Section 2.1. Let \( V := (u \times \text{id})^* \mathcal{W} \) be the pull-back of \( \mathcal{W} \) along the map \( u \times \text{id} : G_1 \times Q \to G \times Q \). Then for \( ((\ell), t) \in Y_{10} \), \( V_{((\ell), t)} \times Q \) fits into a non-split exact sequence \( 0 \to \mathcal{O}_Q(1) \to V_{((\ell), t)} \times Q \to \mathcal{O}_Q((1, 3)) \to 0 \). By relativizing it over \( Y_{10} \), we obtain \( 0 \to S \to V_{Y_{10} \times Q} \to Q \to 0 \). Let \( V^- \) be the elementary modification \( V_{Y_{10} \times Q} \). We can show that \( V^- \) is isomorphic to \( \mathbb{P}^1 \) with \( (1, 0) \)-lines in \( Q \). Note that over \( ((\ell), t) \in G_1 \), \( V^-_{((\ell), t)} \times Q \) fits into a non-split exact sequence \( 0 \to \mathcal{O}_Q((1, 3)) \to V^-_{((\ell), t)} \times Q \to \mathcal{O}_Q((1)) \to 0 \). Because the elementary modification interchanges the subquotient sheaves. Let \( \pi_1 : G_1 \times Q \to G_1 \) be the projection into the first factor. Then \( \mathcal{U} := \pi_1 \mathcal{V}^- \) is a rank-10 bundle over \( G_1 \). Then \( P^- := \mathcal{U} \).

The following proposition completes the proof of Theorem 1.3.

**Proposition 3.7.** The projective bundle \( P^- \) is isomorphic to \( M^- \) in Definition 3.6.

**Proof.** Since the elementary modification has been done locally around \( Y_{10} \times Q \), \( P^0 \mathcal{U} \) and \( P^- \) are isomorphic over \( G_1 \). On the other hand, set theoretically, it is straightforward to see that the image of \( q \) is \( P \setminus p^{-1}(t(Y_{10})) \), where \( p : P \to G \) is the structure morphism. So we have a birational morphism \( M^+ \setminus M^+_1 \to P \setminus p^{-1}(t(Y_{10})) \cong P(u^* \mathcal{U}) \setminus p^{-1}(Y_{10}) \cong P^{-} \setminus p^{-1}(Y_{10}) \) where we used the same notation \( p \) for the projections \( P(u^* \mathcal{U}) \to G_1 \) and \( P^- \to G_1 \). By Proposition 3.2, this map is a blow-down along \( M^+_{(\ell)} \), thus we have an isomorphism \( \tau : P^- \setminus p^{-1}(Y_{10}) \to M^- \setminus M^+_1 \). So we have a birational map \( \tau : P^- \to M^- \), where its undefined locus is \( p^{-1}(Y_{10}) \).

On the other hand, since the flipped locus for \( M^+ \to M^- \) is \( M^+_1 \), we have an isomorphism \( M^- \setminus ( M^+_2 \cup M^+_3 ) \cong M^+ \setminus ( M^+_2 \cup M^+_3 ) \) (here \( M^+ \) is defined in a known way). Also \( r^{-1}(M^+_2 \cup M^+_3 ) = p^{-1}(Y_{10}) \). Hence if we restrict the domain of \( r \), then we have \( P^- \setminus p^{-1}(Y_{10}) \to M^- \setminus ( M^+_2 \cup M^+_3 ) \cong M^+ \setminus ( M^+_2 \cup M^+_3 ) \). The domain of \( r \) is \( p^{-1}(Y_{10}) \). Therefore we can be regarded as a map into a relative Hilbert scheme. Note that \( M^+_{2, 3} \) is the locus of \((2, 3)\)-curves passing through two points lying on \((0, 1)\)-type lines. Note that \( V^- \) fits into a non-split extension

\[
0 \to \mathcal{O}_Q((1, 3)) \to V^-_{((\ell), t)} \times Q \to \mathcal{O}_Q((1)) \to 0.
\]
By a diagram chasing similar to the second paragraph of the proof of Lemma 3.3, one can check that $V^{-}|_{(t, t) \times Q} \cong I_{Z, Q}(2, 3)$, where $Z \subset \ell$ and $\ell$ is a $(1, 0)$-line.

Now two maps $\tau$ and $\check{\sigma}$ coincide over the intersection $P^{-} \setminus P^{-1}(Y_{10} \cup Y_{01})$ of domains, so we have a birational morphism $P^{-} \to M^{-}$. Since $\rho(P^{-}) = 3 = \rho(M^{-})$ and both of them are smooth, this map is an isomorphism. □

The modification on $G_{1} \times Q$ descends to $G_{1}$. Proposition 1.2 follows from a general result of Maruyama ([14]).

Proof of Proposition 1.2. Let $\pi_{1} : G_{1} \times Q \to G_{1}$ be the projection. We claim that $U^{-} = \text{elm}_{Y_{10}}(u^{*}U, \pi_{1, *}) \cong \pi_{1,*}\text{elm}_{Y_{10} \times Q}(V, Q)$. Indeed, from $0 \to V^{-} \to V \to Q \to 0$, we have $0 \to \pi_{1,*}V^{-} \to \pi_{1,*}V = u^{*}U \to \pi_{1, *}(Q) \to R^{1}\pi_{1,*}V^{-} \to R^{1}\pi_{1,*}V$. It is sufficient to show that $R^{1}\pi_{1,*}V^{-} = 0$. By using the resolution of $\mathcal{V}$ given by the universal morphism $\phi$, we have $R^{1}\pi_{1,*}\mathcal{V} = 0$ and this implies $R^{1}\pi_{1,*}V^{-} = 0$. Over $G_{1} \setminus Y_{10}$, $R^{1}\pi_{1,*}V^{-}$ and $R^{1}\pi_{1,*}V$ are isomorphic. For each point $([t], t) \in Y_{10}$, $H^{1}(V^{-}|_{([t], t) \times Q}) = 0$ by the exact sequence (7). Therefore we obtain $R^{1}\pi_{1,*}V^{-} = 0$.

Note that $u^{*}U|_{Y_{10}}$ fits into a vector bundle sequence $0 \to \pi_{1,*}S \to u^{*}U|_{Y_{10}} \to \pi_{1,*}Q \to 0$ and rank $\pi_{1,*}S = 2$ and rank $\pi_{1,*}Q = 8$. The result follows from [14, Theorem 1.3]. □

As a direct application of Theorem 1.3, we compute the Poincaré polynomial of $M$, which matches with the result in [13, Theorem 1.2]. We denote the Poincaré polynomial of a smooth projective variety $X$ by $P(X) = \sum b_{i}(X)q^{i/2}$ where $b_{i}(X)$ is the $i$-th Betti number of $X$.

Corollary 3.8.

(1) The moduli space $M$ is rational;
(2) The Poincaré polynomial of $M$ is

$$P(M) = q^{13} + 3q^{12} + 8q^{11} + 10q^{10} + 11q^{9} + 11q^{8} + 11q^{7} + 11q^{6} + 11q^{5} + 11q^{4} + 10q^{3} + 8q^{2} + 3q + 1.$$  

Proof. Now $M$ is birational to a $P^{9}$-bundle over $G$, so we obtain Item (1). Item (2) is a straightforward calculation using

$$P(M) = P(P^{11}) - P(P^{9}) + P(M^{-}) = P(P^{11}) - P(P^{9}) - P(P^{2}) + P(P^{-}) + (P(P^{3}) - 1)P(P^{1}).$$ □

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References