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Hamming cube and martingales ☆

Cube de Hamming et martingales

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ABSTRACT

In the present paper, we show that a correctly chosen Legendre transform of the Bellman functions of martingale problems give us the right tool to prove isoperimetric inequalities on Hamming cube independent of the dimension. We illustrate the power of this "dual function approach" by proving certain Poincaré inequalities on Hamming cube and by improving a particular inequality of Beckner on the Hamming cube.

RÉSUMÉ

Dans le présent article, nous montrons qu'une transformée de Legendre adéquate des fonctions de Bellman, issues de problèmes de martingale, fournit le bon outil pour démontrer les inégalités isopérimetriques sur le cube de Hamming, indépendamment de la dimension. Nous illustrons la puissance de cette « approche par fonction duale » en démontrant une inégalité de Poincaré et en améliorant une inégalité de Beckner sur le cube de Hamming.

Version française abrégée

Le résultat principal de cet article est le théorème suivant, qui montre une dualité interessante entre les fonctions de Bellman issue de problèmes de martingale et les résultats isopérimetriques sur le cube de Hamming.

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Théorème 0.1. *Soit $U(p,q)$ une fonction continue sur $\mathbb{R}\times\mathbb{R}_+$ concave en p et convexe en q, telle que $\lim_{|p|\to\infty}(px+U(p,0))=-\infty$, $\lim_{q\to\infty}(-qy+U(0,q))=\infty$,*

$$2U(p,|q|)\geq U(p+a,\sqrt{a^2+|q|^2})+U(p-a,\sqrt{a^2+q^2}).$$

Alors $M(x,y):=\inf_{q\leq 0}\sup_p(px+qy+U(p,|q|))$, $(x,y)\in\mathbb{R}\times\mathbb{R}_+$ satisfait l'inégalité à deux points suivante :

$$M(x,\|y\|)\geq\frac{1}{2}\left(M(x+a,\sqrt{a^2+\|y+b\|^2})+M(x-a,\sqrt{a^2+\|y-b\|^2})\right), \tag{1}$$

$x,a\in\mathbb{R}$, $y,b\in\mathbb{R}^N$, $N\geq 1$.

Théorème 0.2. *Soit $\Omega\subset\mathbb{R}$, et soit $M(x,y)$ une fonction satisfaisant (1), alors, pour chaque n et chaque $f:\{-1,1\}^n\to\Omega$, on a l'inégalité suivante*

$$\mathbb{E}_n M(f,|\nabla f|)\leq M(\mathbb{E}_n f,0)\,. \tag{2}$$

En combinant ces deux resultats, on obtient des inégalités nouvelles sur le cube. Bien entendu, on obtient aussi les inégalités isopérimetriques pour la mesure gaussienne. On illustre cette construction en démontrant certaines inégalités de Beckner et Poincaré sur le cube de Hamming.

1. Applications

Consider the Hamming cube $\{-1,1\}^n$ of an arbitrary dimension $n\geq 1$. For any $f:\{-1,1\}^n\to\mathbb{R}$ define the discrete gradient

$$|\nabla f|^2(x)=\sum_{y\sim x}\left(\frac{f(x)-f(y)}{2}\right)^2,$$

where the summation runs over all neighbor vertices of x in $\{-1,1\}^n$. Set $\mathbb{E}_n f=\frac{1}{2^n}\sum_{x\in\{-1,1\}^n}f(x)$.

Theorem 1.1. *For any $1<p\leq 2$, $n\geq 1$, and any $f:\{-1,1\}^n\to\mathbb{R}$, we have*

$$s_{p'}(\mathbb{E}_n|f|^p-|\mathbb{E}_n f|^p)^{1/p}\leq(\mathbb{E}_n|\nabla f|^p)^{1/p}. \tag{3}$$

Here $p'=\frac{p}{p-1}$ is the conjugate exponent of p, and by s_q we denote the smallest positive zero of the confluent hypergeometric function ${}_1F_1(-\frac{q}{2},\frac{1}{2},\frac{x^2}{2})$.

Let $\alpha\geq 2$, and let $N_\alpha(x)$ be the confluent hypergeometric function. $N_\alpha(x)$ satisfies the Hermite differential equation

$$N''_\alpha(x)-xN'_\alpha(x)+\alpha N_\alpha(x)=0\quad\text{for}\quad x\in\mathbb{R} \tag{4}$$

with initial conditions $N_\alpha(0)=1$ and $N'_\alpha(0)=0$. In Theorem 1.1, s_α is the smallest positive zero of $N_\alpha, \alpha=p'$.

Theorem 1.2. *For any $n\geq 1$, and any $f:\{-1,1\}^n\to\mathbb{R}$, we have*

$$\mathbb{E}_n\,\Re\,(f+i|\nabla f|)^{3/2}\leq\Re(\mathbb{E}_n f)^{3/2}, \tag{5}$$

where $z^{3/2}$ is taken in the sense of the principal branch in the upper half-plane.

Inequality (5) improves Beckner's bound for exponent $p=3/2$, see [7]. The function $M(x,y)=\Re\,(x+i\,y)^{3/2}$ is in fact

$$M(x,y)=\frac{1}{\sqrt{2}}(2x-\sqrt{x^2+y^2})\sqrt{\sqrt{x^2+y^2}+x}. \tag{6}$$

Inequality (5) takes the dimension free form

$$\mathbb{E}_n\,M(f,|\nabla f|)\leq M(\mathbb{E}_n\,f,0), \tag{7}$$

where $\mathbb{E}_n$ is the expectation on the Hamming cube $\{-1,1\}^n$. As it is easy to see that pointwisely

$$x^{3/2}-\frac{3}{8}x^{-1/2}y^2\leq\frac{1}{\sqrt{2}}(2x-\sqrt{x^2+y^2})\sqrt{\sqrt{x^2+y^2}+x},\quad x\geq 0,$$

we just obtain the improvement of the Beckner–Sobolev inequality

$$\mathbb{E}_n f^{3/2} - (\mathbb{E}_n f)^{3/2} \leq \frac{3}{8}\mathbb{E}_n(f^{-1/2}|\nabla f|^2)$$

proved by Beckner for any $f : \{-1,1\}^n \to \mathbb{R}_+$.

The proof of (7) is based on the following *two-point inequality* and on an easy result that shows that any M satisfying the following *two-point inequality* generates a dimension-free result on the Hamming cube.

Theorem 1.3. *For all $x, a \in \mathbb{R}$, all $y, b \in \mathbb{R}^N$ and any $N \geq 1$, we have*

$$M(x, \|y\|) \geq \frac{1}{2}\left(M(x+a, \sqrt{a^2 + \|y+b\|^2}) + M(x-a, \sqrt{a^2+\|y-b\|^2})\right). \tag{8}$$

Theorem 1.4. *Let $J \subseteq \mathbb{R}$ be a convex set, and let function $M(x, y)$ satisfy (8) for all $x \pm a \in J$, any $y, b \in \mathbb{R}^N$ and any $N \geq 1$. Then for any $n \geq 1$ and any $f : \{-1,1\}^n \to J$ the following inequality holds*

$$\mathbb{E}_n M(f, |\nabla f|) \leq M(\mathbb{E}_n f, 0)\,. \tag{9}$$

Theorems 1.1, 1.2 are particular cases of Theorem 1.4. But, as we saw, to use Theorem 1.4, one needs to have a certain function satisfying a *two-point inequality* of Theorem 1.3. Our main result below gives a machinery to obtain functions M satisfying the *two-point inequality* of Theorem 1.3. This is done in the next section.

2. Main result. Dualizing the Bellman function $U(p, q)$ and going to the Hamming cube function $M(x, y)$

First we formulate some general minimax theorems. Let P, Q be nonempty closed convex sets in $\mathbb{R}$. We say that a pair $(p^*, q^*) \in P \times Q$ is a saddle point of f on $P \times Q$ if $f(p, q^*) \leq f(p^*, q^*) \leq f(p^*, q)$ for all $(p, q) \in P \times Q$.

Theorem 2.1 *(Proposition 2.2 in [5], pp. 173). Suppose that $f : P \times Q \to \mathbb{R}$ is continuous, concave in p, convex in q, and there exists $(p_0, q_0) \in P \times Q$ such that*

$$\lim_{p\in P,\ |p|\to\infty} f(p, q_0) = -\infty \quad \textit{and} \quad \lim_{q\in Q,\ |q|\to\infty} f(p_0, q) = +\infty. \tag{10}$$

Then f possesses at least one saddle point on $P \times Q$ and

$$f(p^*, q^*) = \min_{q\in Q}\sup_{p\in P} f(p,q) = \max_{p\in P}\inf_{q\in Q} f(p,q).$$

Let $U(p, q)$ be *any* continuous function defined on $I \times \mathbb{R}_+$, where $I \subseteq \mathbb{R}$ is nonempty closed convex set, and satisfying the *main inequality*:

$$2U(p, |q|) \geq U(p+a, \sqrt{q^2+a^2}) + U(p-a, \sqrt{q^2+a^2})\,, \tag{11}$$

for all $p \pm a \in I$, and all $q \in \mathbb{R}$. Such functions play an important part in [3], [4], [11], [10]. In what follows, we set $\Psi(p, q, x, y) := px + qy + U(p, |q|)$ for $x \in \mathbb{R}$ and $y \geq 0$. Let us assume that the function U from (11) is also *convex in q* and *concave* in p, and $(p, q) \mapsto \Psi(p, q, x, y)$ satisfies (10) with $(p_0, q_0) = (0, 0)$, where $Q = \mathbb{R}_-$ and P is $\mathbb{R}$. In what follows, we define

$$M(x, y) := \min_{q\leq 0}\sup_{p\in\mathbb{R}} \Psi(p, q, x, y) \quad \text{for} \quad x \in \mathbb{R},\ y \geq 0. \tag{12}$$

Lemma 2.2. *For each $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, there exists $(p^*, q^*) = (p^*(x, y), q^*(x, y))$ such that $\min_{q\leq 0}\sup_{p\in\mathbb{R}} \Psi(p, q, x, y) = \max_{p\in\mathbb{R}}\inf_{q\leq 0} \Psi(p, q, x, y) = \Psi(p^*, q^*, x, y)$, and we have*

$$\Psi(p, q^*, x, y) \leq \Psi(p^*, q^*, x, y) \leq \Psi(p^*, q, x, y) \quad \textit{for all} \quad (p, q) \in \mathbb{R} \times \mathbb{R}_-. \tag{13}$$

Theorem 2.3. *Under the assumptions above for M from (12), we have*

$$2M(x, \|y\|) \geq M(x+a, \sqrt{a^2+\|y+b\|^2}) + M(x-a, \sqrt{a^2+\|y-b\|^2}), \tag{14}$$

for all $x, a \in \mathbb{R}$, all $y, b \in \mathbb{R}^N$ and any $N \geq 1$.

This is our main dualization theorem. The reader should notice that dualization produces the inequality (14), which is different from (11).

To prove Theorem 2.3, first we consider the case $N = 1$. Without loss of generality, assume $y \geq 0$. Set $(x_\pm, y_\pm) := (x \pm a, \sqrt{a^2 + (y \pm b)^2})$.

Lemma 2.2 gives the saddle points (p^*, q^*) and $(p^\pm, q^\pm)$ corresponding to (x, y) and $(x_\pm, y_\pm)$. It follows from (13) that to prove (14), it would be enough to find numbers $p \in \mathbb{R}$ and $q_1, q_2 \leq 0$ such that $2\Psi(p, q^*, x, y) \geq \Psi(p^+, q_1, x_+, y_+) + \Psi(p^-, q_2, x_-, y_-)$. The right choice will be $p = \frac{p^+ + p^-}{2}$ and $q_1 = q_2 = -\sqrt{\left(\frac{p^+ - p^-}{2}\right)^2 + (q^*)^2}$. The reader is referred to Lemma 2.2 of [8] for calculations and for checking the validity for an arbitrary $N > 1$.

3. Back to applications

To show Theorems 1.1, 1.2, one needs to choose the corresponding functions U satisfying (11) and *dualize* them according to the previous Section (Theorem 2.3). First we deal with Theorem 1.1.

Let $p \in (1, 2)$. Set $\alpha = p' = \frac{p}{p-1}$.

$$u_\alpha(x) := \begin{cases} -\dfrac{\alpha s_\alpha^{\alpha-1}}{N'_\alpha(s_\alpha)} N_\alpha(x), & 0 \leq |x| \leq s_\alpha; \\ s_\alpha^\alpha - |x|^\alpha, & s_\alpha \leq |x|. \end{cases}$$

Clearly $u_\alpha(x)$ is $C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{s_\alpha\})$ smooth even concave function. Finally, we define $U(p, q) := q^\alpha u_\alpha\left(\frac{p}{q}\right)$ $q \geq 0$ with $U(p, 0) = -|p|^\alpha$. For the first time, the function $U(p, q)$ appeared in [4]. Later it was also used in [11] [10] in the form $\widetilde{u}(p, t) = U(p, \sqrt{t})$, $t \geq 0$. It was explained in [4,11,10] that $U(p, q)$ satisfies the *main inequality* (11) and the following *obstacle* property:

$$U(p, |q|) \geq |q|^\alpha s_\alpha^\alpha - |p|^\alpha \text{ for all } (p, q) \in \mathbb{R}^2. \tag{15}$$

One easily checks that this U is convex in q and concave in p. Also notice that $(p, q) \mapsto px + qy + U(p, |q|)$ satisfies (10) with $(p_0, q_0) = (0, 0)$. Then Theorem 2.3 can be applied. We obtain a *dual* function M that satisfies the two-point inequality (14). So Theorem 1.4 is applicable. Then notice that because of (15) function $M(x, y) = \min_{q \leq 0} \sup_{p \in I} (px + qy + U(p, |q|))$ also satisfies $M(x, y) \geq \left(\frac{\alpha-1}{\alpha^\beta}\right)\left(|x|^\beta - \frac{y^\beta}{s_\alpha^\beta}\right)$. Here $\beta := \frac{\alpha}{\alpha-1} \leq 2$ is the conjugate exponent of α. Applying Theorem 1.4, we obtain

$$\left(\frac{\alpha-1}{\alpha^\beta}\right)\left(\mathbb{E}_n|f|^p - \frac{\mathbb{E}_n|\nabla f|^p}{s_{p'}^p}\right) \leq \mathbb{E}_n M(f, |\nabla f|) \leq M(\mathbb{E}_n f, 0) = \left(\frac{\alpha-1}{\alpha^\beta}\right)|\mathbb{E}_n f|^p,$$

which is the claim of Theorem 1.1.

To show Theorem 1.2, we consider another function $U(p, q)$. It is $U(p, q) = -\frac{4}{27}(p^3 - 3pq^2)$, which we wish to consider on $\mathbb{R}_- \times \mathbb{R}_+$. It satisfies (11), it is concave in q and convex in p if $p \leq 0$, $px + qy + U(p, |q|)$ also satisfies (10). By a direct calculation, one can see that for every $x \in \mathbb{R}$, $y \in \mathbb{R}_+$

$$\frac{1}{\sqrt{2}}(2x - \sqrt{x^2 + y^2})\sqrt{\sqrt{x^2 + y^2} + x} = \min_{q \leq 0} \sup_{p \geq 0} \left(px + qy - \frac{4}{27}(p^3 - 3pq^2)\right). \tag{16}$$

Then Theorem 2.3 can be applied. We obtain a *dual* function M that satisfies the two-point inequality (14). So Theorem 1.4 is applicable. This proves Theorem 1.2.

4. Questions

In [6], we improved the Beckner inequality, not only for $p = 3/2$, but for all exponents $p \in (1, 2)$. However, we managed to do this not on the Hamming cube, but only in the situation when all $\mathbb{E}_n$ averages are understood as integrals with respect to the Gaussian measure γ_n on $\mathbb{R}^n$. In other words, in [6] we treated the *continuous case* of the improved Beckner inequalities. We still do not know how to extend this to the *discrete case* of functions on the Hamming cube (except for the special case of $p = 3/2$, where formula (16) saves the day).

Our other question concerns the sharpness of the constant $s_{p'}$ in Theorem 1.1. We definitely know that when $p \to 1$, it is not sharp. This can be understood via Talagrand's inequality, see [9] or from [2], [1]. But when $p \to 2$ our constant seems to be pretty good.

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