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Differential geometry

A note on the almost-one-half holomorphic pinching

Une note sur le pincement holomorphe presque un demi

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ABSTRACT

Motivated by a previous work by Zheng and the second-named author, we study pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature. In particular, we prove a gap theorem on Kähler manifolds with almost-one-half pinched holomophic sectional curvature. The proof is motivated by the work of Petersen and Tao on Riemannian manifolds with almost-quarter-pinched sectional curvature.

RÉSUMÉ

Motivés par un travail précédent de Zheng et du second auteur, nous étudions les constantes de pincement des variétés kählériennes compactes avec courbure sectionnelle holomorphe positive. En particulier, nous prouvons un théorème de l'écart sur des variétés kählériennes de courbure sectionnelle holomorphe avec pincement presque un demi. La preuve s'appuie sur le travail de Petersen et de Tao sur les variétés riemanniennes avec une courbure sectionnelle presque $\frac{1}{4}$-pincée.

1. The theorem

Let (M, J, g) be a complex manifold with a Kähler metric g, one can define the *holomorphic sectional curvature* (H) of any J-invariant real 2-plane $\pi = \mathrm{Span}\{X, JX\}$ by

$$H(\pi) = \frac{R(X, JX, JX, X)}{||X||^4}.$$

It is the Riemannian sectional curvature restricted on any J-invariant real 2-plane (p. 165, [18]). In terms of complex coordinates, it is equivalent to write

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$$H(\pi) = \frac{R(V, \overline{V}, V, \overline{V})}{||V||^4}$$

where $V = X - \sqrt{-1} JX \in T^{1,0}(M)$.

In this note, we study the pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature ($H > 0$). The goal is to prove the following rigidity result on a compact Kähler manifold with the almost-one-half pinching.

Theorem 1.1. *For any integer $n \geq 2$, there exists a positive constant $\epsilon(n)$ such that any compact Kähler manifold with $\frac{1}{2} - \epsilon(n) \leq H \leq 1$ of dimension n is biholomorphic to any of the following:*

(i) $\mathbb{CP}^n$,
(ii) $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$,
(iii) *an irreducible rank-*2 *compact Hermitian symmetric space of dimension n.*

Before we discuss the proof, let us review some background on compact Kähler manifolds with $H > 0$. The condition $H > 0$ is less understood and seems mysterious. For example, $H > 0$ does not imply positive Ricci curvature, though it leads to positive scalar curvature. Essentially, one has to work on a fourth-order tensor from the viewpoint of linear algebra, while usually the stronger notion of holomorphic bisectional curvature leads to bilinear forms.

Naturally, one may wonder if there is a characterization of such an interesting class of Kähler manifolds. In particular, Yau ([30] and [31]) asked if the positivity of holomorphic sectional curvature can be used to characterize the rationality of algebraic manifolds. For example, is such a manifold a rational variety? There is much progress on Kähler surfaces with $H > 0$. In 1975, Hitchin [17] proved that any compact Kähler surface with $H > 0$ must be a rational surface, and conversely he constructed examples of such metrics on any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^1})$. It remains an interesting question to find out if Kähler metrics with $H > 0$ exist on other rational surfaces.

In higher dimensions, much less is known on $H > 0$, except recent important works of Heier–Wong (see [16] for example). One of their results states that any projective manifold that admits a Kähler metric with $H > 0$ must be rationally connected. It could be possible that any compact Kähler manifold with $H > 0$ is in fact projective, again it is an open question.[4] We also remark that some generalization of Hitchin's construction of Kähler metrics of $H > 0$ in higher dimensions has been obtained in [2].

If Yau's conjecture is true, then how do we study complexities of rational varieties that admit Kähler metrics with $H > 0$? A naive thought is that the global and local holomorphic pinching constants of H should give a stratification among all such rational varieties. Here the local holomorphic pinching constant of a Kähler manifold (M, J, g) of $H > 0$ is the maximum of all $\lambda \in (0, 1]$ such that $0 < \lambda H(\pi') \leq H(\pi)$ for any J-invariant real 2-planes $\pi, \pi' \subset T_p(M)$ at any $p \in M$, while the global holomorphic pinching constant is the maximum of all $\lambda \in (0, 1]$ such that there exists a positive constant C so that $\lambda C \leq H(p, \pi) \leq C$ holds for any $p \in M$ and any J-invariant real 2-plane $\pi \subset T_p(M)$. Obviously, the global holomorphic pinching constant is no larger than the local one, and there are examples of Kähler metrics with different global and local holomorphic pinching constants on Hirzebruch manifolds ([29]).

In a previous work of Zheng and the second-named author [29], we observed the following result, which follows from some pinching equality on $H > 0$ due to Berger [3] and from recent works on nonnegative orthogonal bisectional curvature ([6], [9], [12], and [28]).

Proposition 1.2 *([29]). Let (M^n, g) be a compact Kähler manifold with $0 < \lambda \leq H \leq 1$ in the local sense for some constant λ, then the following holds:*

(1) *if $\lambda > \frac{1}{2}$, then M^n is bibolomorphic to $\mathbb{CP}^n$;*
(2) *if $\lambda = \frac{1}{2}$, then M^n satisfies one of the following*
   (i) *M^n is biholomorphic to $\mathbb{CP}^n$;*
   (ii) *M^n is holomorphically isometric to $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$ with a product metric of Fubini–Study metrics; moreover, each factor must have the same constant H;*
   (iii) *M^n is holomorphically isometric to an irreducible compact Hermitian symmeric space of rank* 2 *with its canonical Kähler–Einstein metric.*

Let us remark that in the case that Kähler manifold in Proposition 1.2 is projective and endowed with the induced metric from the Fubini–Study metric of the ambient projective space, a complete characterization of such a projective manifold and the corresponding embedding has been proved by Ros [25].

[4] After an earlier version of this preprint was submitted for consideration for publication in June 2017, X. Yang (arXiv:1708.06713) made important progress on these questions and proved that any compact Kähler manifold with $H > 0$ is projective and rationally connected.

Comparing with Proposition 1.2, we may view Theorem 1.1 as a rigidity result on compact Kähler manifolds with almost-one-half-pinched $H > 0$. For example, Hirzebruch manifolds can not admit Kähler metrics whose global pinching constants are arbitrarily close to $\frac{1}{2}$.

It is very interesting to find the next threshold for holomorphic pinching constants and some characterization of Kähler manifolds with such a threshold pinching constant. Before making any reasonable speculation, it is helpful to understand examples on such holomorphic constants of some canonical Kähler metrics. In this regard, the Kähler–Einstein metric on a irreducible compact Hermitian symmetric space has its holomorphic pinching constant exactly the reciprocal of its rank ([8]). The Kähler–Einstein metrics on many simply-connected compact homogeneous Kähler manifolds (Kähler C-spaces) also have $H > 0$, and it seems very tedious to work with the corresponding Lie algebras carefully to determine these holomorphic pinching constants except in lower dimensions. It was observed in [29] that the flag 3-manifold, the only Kähler C-space in dimension 3 that is not Hermitian symmetric, has $\frac{1}{4}$-holomorphic pinching for its canonical Kähler–Einstein metric. Note that Alvarez–Chaturvedi–Heier [1] studied pinching constants of Hitchin's examples of Kähler metrics with $H > 0$ on a Hirzebruch surface. However, it remains unknown what is the best pinching constant among all Kähler metrics with $H > 0$ on such a surface. We refer the interested reader to [1] and [29] for more discussions.

2. The proof

The proof is motivated by the work of Petersen–Tao [23] on Riemannian manifolds with almost-quarter-pinched sectional curvature.

Assume that, for some complex dimension $n \geq 2$, there exists a sequence of compact Kähler manifolds (M_k, J_k, g_k) $(k \geq 1)$ whose holomorphic sectional curvature satisfies $\frac{1}{2} - \frac{1}{4k} \leq H(M_k, g_k) \leq 1$, and none of (M_k, J_k) is biholomorphic to any of the three listed in the conclusion of Theorem 1.1. In the following steps, $c(n)$, maybe different from line to line, are all constants that only depend on n.

Step 1: (a uniform lower bound for the maximal existence time of the Kähler–Ricci flow)

It is well known ([18] for example) that bounds on holomorphic sectional curvature lead to bounds on Riemannian sectional curvature and the full curvature tensor. In particular, for any unit orthogonal vectors X and Y, we have:

$$\mathrm{Sec}(X, Y) = R(X, Y, Y, X) = \frac{1}{8}\left[3H\left(\frac{X+JY}{\sqrt{2}}\right) + 3H\left(\frac{X-JY}{\sqrt{2}}\right) - H\left(\frac{X+Y}{\sqrt{2}}\right) - H\left(\frac{X-Y}{\sqrt{2}}\right) - H(X) - H(Y)\right].$$

From the works of Hamilton and Shi ([13], [14], and [26], and Cor 7.7 in [11] for an exposition of these results on compact manifolds), we conclude that, for any $k \geq 1$, there exists a constant $T(n) > 0$ such that the Kähler–Ricci flow $(M_k, J_k, g_k(t))$ with the initial metric g_k is well defined on the time interval $[0, T(n)]$ for any $k \geq 1$. Moreover, we have $|Rm(M_k, g_k(t))|_{g(t)} \leq c(n)$ for some constant $c(n)$ and all $t \in [0, T(n)]$ and $k \geq 1$.

Step 2: (an improved curvature bound on a smaller time interval)

This step is due to Ilmanen, Shi, and Rong (Proposition 2.5 in [24]). Namely, there exists constants $\delta(n) < T(n)$ and $c(n)$ such that for any $t \in [0, \delta(n)]$,

$$\begin{aligned}\min_{p, V \subset T_p(M_k)} \mathrm{Sec}(M_k, g_k(t), p, V) - c(n)t &\leq \mathrm{Sec}(M_k, g_k(t), p, P) \\ &\leq \max_{p, V \subset T_p(M_k)} \mathrm{Sec}(M_k, g_k(t), p, V) + c(n)t.\end{aligned}$$

It is direct to see that a similar estimate holds for holomorphic sectional curvature. Indeed, there exist $\delta(n)$ and $c(n)$ such that for any $t \in [0, \delta(n)]$

$$\frac{1}{2} - c(n)t \leq H(M_k, g_k(t)) \leq 1 + c(n)t.$$

Step 3: (an injective radius bound on $g_k(t_0)$ for some fixed $t_0 \in [0, \delta]$)

We observe that Klingenberg's injectivity radius estimates on even-dimensional Riemannian manifolds with positive sectional curvature (Theorem 5.9 in [7] or p. 178 of [22] for example) can be adapted to show that there exists some constant $\delta_1(n)$ such that for any $t \in [0, \delta_1(n)]$ $\mathrm{inj}(M_k, g_k(t)) \geq c(n)$ for some constant $c(n) > 0$. Indeed, it will follow from the claim below.

Claim 2.1. *Let (M^n, g) be a compact Kähler manifold with positive holomorphic sectional curvature $H \geq \delta > 0$ and $\mathrm{Sec} \leq K$ where the constant $K > 0$, then the injectivity radius $\mathrm{inj}(M^n, g) \geq c(K)$ for some constant $c(K)$.*

The proof of the above claim goes along as the proof of Theorem 5.9 in [7], except that we need to use the variational vector field as $J\gamma'(t)$ where $\gamma(t)$ is a closed geodesic. This is where we use $H > 0$. In fact, such a kind of estimate has been proved in [10] under the assumption of positive bisectional curvature.

Step 4: (a lower bound on orthogonal bisectional curvatures of $(M_k, g_k(t))$

This step is motivated by Peterson–Tao [23], who derived a similar lower bound estimate for isotropic curvatures of almost-quarter-pinched Riemannian manifolds along a Ricci flow.

Claim 2.2. *There exists some constant $\delta_2(n)$ such that, for any $t \in [0, \delta_2(n)]$, the orthogonal bisectional curvature of $(M_k, g_k(t))$ has a lower bound $-\frac{1}{k}e^{c(n)t}$ for some constant $c(n) > 0$.*

Proof of Claim 2.2. Note that Berger's inequality [3] (see also Lemma 2.5 in [29] for an exposition) implies the orthogonal bisectional curvature of $(M_k, g_k(0))$ is bounded from below by $-\frac{1}{4k}$. Now the proof is based on a maximum principle developed in [14]. In the setup of orthogonal bisectional curvature, it is proved in an unpublished work of Cao–Hamilton [6] dating back to 1992 (the details were written up as Theorem 5.2.14 in [5]), Gu–Zhang [12], and Wilking [28] that the nonnegativity of orthogonal bisectional curvature is preserved along the Kähler–Ricci flow. For the sake of convenience, we simply write $(M, J, g(t))$, where $t \in [0, T(n)]$, instead of the sequence $(M_k, J_k, g_k(t))$.

Following [14], one may use Uhlenbeck's trick. Consider a fixed complex vector bundle $E \to M$ isomorphic to $TM \to M$; with a suitable choice of bundle isomorphims $\iota_t : E \to TM$, one obtains a fixed metric $\iota_t^* g(t)$ on E. Now choose an unitary frame $\{e_\alpha\}$ on $T^{1,0}(E)$ that corresponds to an evolving unitary frame on TM via ι_t, let $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ denote $R(\iota_t^* g(t), e_\alpha, \overline{e_\alpha}, e_\beta, \overline{e_\beta})$, the evolution equation of bisectional curvature reads ([12] for example):

$$\frac{\partial}{\partial t} R_{\alpha\overline{\alpha}\beta\overline{\beta}} = \Delta_{g(t)} R_{\alpha\overline{\alpha}\beta\overline{\beta}} + \sum_{\mu,\nu} (R_{\alpha\overline{\alpha}\mu\overline{\nu}} R_{\beta\overline{\beta}\mu\overline{\nu}} - |R_{\alpha\overline{\mu}\beta\overline{\nu}}|^2 + |R_{\alpha\overline{\beta}\mu\overline{\nu}}|^2). \tag{1}$$

Now assume that $m(t) = \min_{U \perp V \in T^{1,0}(E)} R(U, \overline{U}, V, \overline{V})$, and assume that $m(t_0) = R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ for some t_0 and some point $p \in M$. Considering the first and the second variations of $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ and following the proof of Proposition 2.1 in [12] and the curvature bounds in Step 1, we conclude from (1) that $\frac{\mathrm{d}^- m(t)}{\mathrm{d}t}|_{t=t_0} \geq c(n)m(t_0)$ whenever $m(t_0) < 0$. Therefore there exists some time interval $[0, \delta_2(n)]$ either $m(t) \geq 0$ or $\frac{\mathrm{d}^+(-m(t))}{\mathrm{d}t} \leq c(n)(-m(t))$ if $m(t) < 0$. Recall $m(0) \geq -\frac{1}{4k}$, in either case we end up with $m(t) \geq -\frac{1}{k}e^{c(n)t}$ for all $t \in [0, \delta_2(n)]$. □

Step 5: (a contradiction after taking the limit of $(M_k, J_k, g_k(t))$)

Let us consider $(M_k, J_k, g_k(t))$ where $t \in (0, \delta_2(n)]$ be a sequence of Kähler–Ricci flow; from previous steps, we conclude that there exist $\delta_3(n)$ and $c(n)$ such that

(i) $|Rm|_{g_k(t)}(M_k, g_k(t)) \leq c(n)$ for and $k \geq 1$ and $t \in [0, \delta_3(n)]$.
(ii) $\mathrm{inj}(M_k, g_k(t_0)) \geq \frac{1}{c(n)}$ for some $t_0 \in [0, \delta_3(n)]$.
(iii) $Ric(M_k, g_k(t_0)) \geq \frac{1}{c(n)}$ for any k, this follows from Step 2 and 4.

It follows from Hamilton's compactness theorem of Ricci flows [15] that $(M_k, J_k, g_k(t))$ converges to a compact limiting Kähler–Ricci flow $(M_\infty, J_\infty, g_\infty(t))$ where $t \in (0, \delta_2(n)]$. It follows from Step 4 that $g_\infty(t)$ has nonnegative orthogonal bisectional curvature and $H(g_\infty(t)) > 0$ for any $0 < t \leq t_0$. Note that all M_k and M_∞ are simply connected ([27]); it follows from [9], [12], and [28] that $(M_\infty, g_\infty(t_0))$ must be of the following form.

$$(\mathbb{CP}^{k_1}, g_{k_1}) \times \cdots \times (\mathbb{CP}^{k_r}, g_{k_r}) \times (N^{l_1}, h_{l_1}) \times \cdots (N^{k_r}, h_{l_s}). \tag{2}$$

Where each of $(\mathbb{CP}^{k_i}, g_{k_i})$ has nonnegative bisectional curvature and each of (N^{l_i}, h_{l_i}) is a compact irreducible Hermitian symmetric spaces of rank ≥ 2 with its canonical Kähler–Einstein metric. Now consider a time $t_1 < t_0$ close to $t = 0$; it follows from Step 2 that $g_\infty(t_1)$ is close to $\frac{1}{2}$-holomorphic pinching and also has the same decomposition as in (2). Indeed the decomposition (2) is reduced to exactly the list in the conclusion of Proposition 1.2. To see it, one may also apply the formula of pinching constants of a product of metrics with $H > 0$ ([1]). However, (M_k, J_k) is not biholomorphic to any of the three listed in the conclusion of Theorem 1.1. Now we have a sequence of Kähler manifolds $(M_\infty, \phi_k^* J_k, \phi_k^* g_k(t_1))$ converging to $(M_\infty, J_\infty, g_\infty(t_1))$, where $\phi_k : M_\infty \to M_k$ are the diffeomorphisms from Hamilton's compactness theorem. The desired contradiction would be found if we can show that $(M_\infty, \phi_k^* J_k)$ is biholomorphic to (M_∞, J_∞) when k is large enough.

Indeed, this will follow from Kuranishi's construction of the Kuranishi family (see [20], [21], and an exposition of these results on pp. 165–173 of [19]). Consider a complex manifold (X, J) endowed with a fixed Hermitian metric g, the space of almost-complex structures has a natural structure of a Fréchet manifold. Any complex structure J' sufficiently close to J in the sense of the Fréchet topology is identified as some $T^{1,0}X$-valued $(0, 1)$ form on X that solves a Maurer–Cartan equation. Considering the action of the diffeomorphism group of X on the space of complex structures, Kuranishi constructed a family of solutions to such an equation whose parameters lie in the finite-dimensional space $H^1(X, \Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X. In more details, Kuranishi showed the existence of *the Kuranishi family*, which is a holomorphic submersion $\pi : (\mathcal{X}, X) \to (B, 0)$, where $\pi^{-1}(0) = (X, J)$ and B is an analytic subset in $H^1(X, \Theta_X) = \mathbb{C}^m$

with $m = \dim H^1(X, \Theta_X)$. For our purpose here, it suffices to note that it was remarked in [21] (p. 154) that any complex structure J' sufficiently close to J can be realized as some $\pi^{-1}(b')$ where $b' \in B$ in the Kuranishi family.

Now we consider the Kuranishi family associated with $(M_\infty, J_\infty, g_\infty)$. For k large enough, $\phi_k^* J_k$ is exactly sufficiently close to J_∞ in the Fréchet topology mentioned above, therefore these $(M_\infty, \phi_k^* J_k)$ could be parameterized by points in $B \subset H^1(M_\infty, \Theta_{M_\infty})$. However, since any compact Hermitian symmetric space has $H^1(M_\infty, \Theta_{M_\infty}) = 0$ (Bott [4]), this forces $B = \{0\}$ and $(M_\infty, \phi_k^* J_k)$ to be isomorphic to (M_∞, J_∞). This finishes the proof of Theorem 1.1.

3. The remark

Note that Proposition 1.2 works in the case of the local one-half pinching, therefore it seems natural to ask:

Question 3.1. *Does Theorem 1.1 hold if we replace the global almost-one-half pinching to the local one?*

Another optimistic hope is that $H > 0$ is preserved along Kähler–Ricci flow as long as the initial metric has a suitable large holomorphic pinching constant. We refer the interested reader to [29] for more discussions. In particular, some conjectures on compact Kähler surfaces and 3-folds with large holomorphic pinching constants were proposed in [29].

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